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ON THE GAUSS MAP OF B-SCROLLS

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Abstract. B-scrolls over null curves in the 3-dimensional Lorentz-Minkowski space L^3 are characterized as the only ruled surfaces with null rulings whose Gauss map satisfies the condition $\Delta G = \Lambda G$, Λ being an endomorphism of L^3. This note completes the classification of such surfaces given by S. M. Choi in Tsukuba J. Math. **19** (1995), 285–304.

1. Introduction

Let M be a connected surface in Euclidean 3-space R^3 and let $G: M \to S^2 \subset R^3$ be its Gauss map. It is well known (see [9]) that M has constant mean curvature if and only if $\Delta G = \|dG\|^2 G$, Δ being the Laplace operator on M corresponding to the induced metric on M from R^3. As a special case one can consider Euclidean surfaces whose Gauss map is an eigenfunction of the Laplacian, i.e., $\Delta G = \lambda G$, $\lambda \in R$. In [3], C. Baikoussis and D. E. Blair asked for ruled surfaces in R^3 whose Gauss map satisfies $\Delta G = \Lambda G$, where Λ stands for an endomorphism of R^3. They showed that the only ones are planes and circular cylinders. Recently, S. M. Choi in [5], investigates the Lorentz version of the above result and she essentially obtains the same result. Namely, the only ruled surfaces in L^3 whose Gauss map satisfies $\Delta G = \Lambda G$ are the planes R^2 and L^2, as well as the cylinders $S_1^1 \times R^1$, $R_1^1 \times S^1$ and $H^1 \times R^1$.

It should be pointed out that all surfaces obtained above have diagonalizable shape operator. However, it is well known that a self-adjoint linear operator on a 2-dimensional Lorentz vector space has a matrix of exactly three types, two of them being non-diagonalizable. This makes a chief difference with regard to the

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Riemannian submanifolds that has been greatly exploited (see, for example, [1], [2] and [7]). To illustrate the current situation, we bring here the famous example of L. K. Graves (see [8]), the so called *B-scroll*. This is a surface which can be parametrized as a "ruled surface" in L^3 with null directrix curve and null rulings, i.e., $X(s,t) = x(s) + tB(s)$, $x(s)$ being a null curve and $B(s)$ a null vector field along $x(s)$ satisfying $\langle x', B\rangle = -1$.

The main purpose of this short note is to complete Choi's classification of ruled surfaces in L^3 whose Gauss map satisfies the condition $\Delta G = \Lambda G$. Actually, we will show that B-scrolls over null curves are the only ruled surfaces in L^3 with null rulings satisfying the above condition.

We would like to thank to the referee for bringing to our attention the preprint [6], where some related topics are considered.

2. Setup

Let $x : I \subset \boldsymbol{R} \to L^3$ be a regular curve in L^3 and $B : I \subset \boldsymbol{R} \to L^3$ a vector field along x. Consider the ruled surface parametrized by $X(s,t) = x(s) + tB(s)$. Let us write down, as usually, $X_s := \partial X/\partial s = x' + tB'$ and $X_t := \partial X/\partial t = B$. Observe that, at $t = 0$, $X_s(s,0) = x'(s)$ and $X_t(s,0) = B(s)$. Then $X(s,t)$ is a regular surface in L^3 provided that the plane $\Pi = \operatorname{span}\{x', B\}$ is non degenerate in L^3. In fact, the matrix of the metric of $X(s,t)$ is given by

$$\mathbf{g}(s,t) = \begin{pmatrix} \langle x', x'\rangle + 2t\langle x', B'\rangle + t^2\langle B', B'\rangle & \langle x', B\rangle + t\langle B', B\rangle \\ \langle x', B\rangle + t\langle B', B\rangle & \langle B, B\rangle \end{pmatrix},$$

so that when the plane Π is spacelike (respectively, timelike) $X(s,t)$ parametrizes a spacelike surface (respectively, timelike surface) on the domain

$$\{(s,t) \in I \times \boldsymbol{R} : \det \mathbf{g}(s,t) > 0 \quad (\text{respectively, } \det \mathbf{g}(s,t) < 0)\}.$$

According to the causal character of x' and B, there are four possibilities:

(1) x' and B are non-null and linearly independent.

(2) x' is null and B is non-null with $\langle x', B\rangle \neq 0$.

(3) x' is non-null and B is null with $\langle x', B\rangle \neq 0$.

(4) x' and B are null with $\langle x', B\rangle \neq 0$.

Let us first see that, with an appropiate change of the curve x, cases (2) and (3) can be locally reduced to (1) and (4), respectively. Let $X(s,t)$ be in case (2). Reparametrizing the null curve x and normalizing the rulings B if necessary, we may assume that

$$\langle B, B\rangle = \varepsilon = \pm 1, \qquad \text{and} \quad \langle x', B\rangle = -1,$$

so that

$$g(s,t) = \det \mathbf{g}(s,t) = \varepsilon(2t\langle x', B'\rangle + t^2\langle B', B'\rangle) - 1 < 0. \tag{2.1}$$

We are looking for a curve $\gamma(s) = x(s) + t(s)B(s)$ in the surface with $\langle \gamma', \gamma'\rangle = \varepsilon$ and such that γ' and B are linearly independent. Writing $\gamma' = x' + t'B + tB'$, the condition $\langle \gamma', \gamma'\rangle = \varepsilon$ is equivalent to the following differential equation for $t = t(s)$

$$(t')^2 - 2\varepsilon t' + g(s,t) = 0. \tag{2.2}$$

From (2.1) the discriminant of (2.2) is positive and we can locally integrate (2.2) to obtain t. Besides, γ' and B are linearly independent because $\langle \gamma', \gamma'\rangle = \langle B, B\rangle = \varepsilon$ and $\langle \gamma', B\rangle = -1 + t'\varepsilon \neq \pm\varepsilon$ due to (2.2). This shows that $X(s,t)$ can be reparametrized as in case (1) taking γ as the directrix curve. On the other hand, if $X(s,t)$ is in case (3), reparametrizing the null curve x and normalizing the rulings B if necessary, we may assume that

$$\langle x', x'\rangle = \varepsilon = \pm 1, \quad \text{and} \quad \langle x', B\rangle = -1.$$

We are now looking for a curve $\gamma(s) = x(s) + t(s)B(s)$ in the surface with $\langle \gamma', \gamma'\rangle = 0$ and $\langle \gamma', B\rangle \neq 0$. Writing $\gamma' = x' + t'B + tB'$, the condition $\langle \gamma', \gamma'\rangle = 0$ now becomes

$$2t' = \varepsilon + 2t\langle x', B'\rangle + t^2\langle B', B'\rangle. \tag{2.3}$$

Equation (2.3) can be locally integrated to obtain t. Moreover, $\langle \gamma', B\rangle = \langle x', B\rangle \neq 0$. Thus, using the curve γ as the directrix, $X(s,t)$ can be reparametrized as in case (4).

Since case (1) has been discussed in [5], we will pay attention to the latter one which we aim to characterize in terms of the Laplacian of its Gauss map. Therefore, let M be a ruled surface in L^3 parametrized by $X(s,t) = x(s) + tB(s)$, where the directrix $x(s)$, as well as the rulings $B(s)$, are null. Furthermore, and without loss of generality, we may assume $\langle x', B\rangle = -1$. First of all, we will do a detailed study of this kind of surfaces.

The matrix of the metric on M writes, with respect to coordinates (s,t), as follows

$$\begin{pmatrix} 2t\langle x', B'\rangle + t^2\langle B', B'\rangle & -1 \\ -1 & 0 \end{pmatrix}.$$

In terms of local coordinates $(y_1, \dots, y_n)$, the Laplacian Δ of a manifold is defined by (see [4, p. 100])

$$\Delta = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial y_i} \left(g g^{ij} \frac{\partial}{\partial y_j} \right),$$

where $g = det(g_{ij})$ and (g_{ij}) denotes the components of the metric with respect to $(y_1, \dots, y_n)$. Then the Laplacian on the surface M is nothing but

$$\Delta = -2\frac{\partial^2}{\partial s \partial t} - 2\{\langle x', B' \rangle + t\langle B', B' \rangle\}\frac{\partial}{\partial t} - \{2t\langle x', B' \rangle + t^2\langle B', B' \rangle\}\frac{\partial^2}{\partial t^2}.$$

Now we will recall the notion of cross product in L^3. There is a natural orientation in L^3 defined as follows: an ordered basis $\{X, Y, Z\}$ in L^3 is positively oriented if $\det[XYZ] > 0$, where $[XYZ]$ is the matrix with X, Y, Z as row vectors. Now let ω be the volume element on L^3 defined by $\omega(X, Y, Z) = \det[XYZ]$. Then given X, $Y \in L^3$, the cross product $X \times Y$ is the unique vector in L^3 such that $\langle X \times Y, Z \rangle = \omega(X, Y, Z)$, for any $Z \in L^3$.

Then the Gauss map can be directly obtained from $X_s \times X_t$ getting

$$G(s, t) = x'(s) \times B(s) + tB'(s) \times B(s).$$

By putting $C = x' \times B$, then $\{x', B, C\}$ is a frame field along x of L^3. In this frame, we easily see that $B' \times B = -fB$, f being the function defined by $f = \langle x', B' \times B \rangle$. Thus

$$G(s, t) = -tf(s)B(s) + C(s). \tag{2.4}$$

Also, and for later use, we find out that

$$B' = -\langle x', B' \rangle B - fC \tag{2.5}$$

and

$$C' = -fx' - \langle x', x'' \times B \rangle B. \tag{2.6}$$

As for the shape operator S we have that

$$G_t := \frac{\partial G}{\partial t} = B' \times B = -fB = -fX_t \tag{2.7}$$

and

$$G_s := \frac{\partial G}{\partial s} = -(\langle x', x'' \times B \rangle + tf')X_t - fX_s. \tag{2.8}$$

So S writes down as

$$\begin{pmatrix} f & 0 \\ tf' + \langle x', x'' \times B \rangle & f \end{pmatrix}.$$

A straightforward computation yields

$$\Delta G = 2\{f' + tf\langle B', B' \rangle\}B - 2f^2 C. \tag{2.9}$$

We now present a very typical example.

EXAMPLE. Let $x(s)$ be a null curve in L^3 with Cartan frame $\{A, B, C\}$, i.e., A, B, C are vector fields along x in L^3 satisfying the following conditions:

$$\langle A, A \rangle = \langle B, B \rangle = 0, \quad \langle A, B \rangle = -1,$$
$$\langle A, C \rangle = \langle B, C \rangle = 0, \quad \langle C, C \rangle = 1,$$

and

$$x' = A,$$
$$C' = -aA - \kappa(s)B,$$

a being a constant and $\kappa(s)$ a function vanishing nowhere. Then the map

$$X : L^2 \to L^3$$
$$(s, t) \to x(s) + tB(s)$$

defines a Lorentz surface M in L^3 that L. K. Graves [8] called a *B-scroll*. It is not difficult to see that a unit normal vector field is given by

$$G(s, t) = -atB(s) + C(s),$$

and the shape operator writes down, relative to the usual frame $\{\partial X/\partial s, \partial X/\partial t\}$, as

$$S = \begin{pmatrix} a & 0 \\ k(s) & a \end{pmatrix}.$$

Thus the *B*-scroll has non-diagonalizable shape operator with minimal polynomial $P_S(u) = (u - a)^2$. It has constant mean curvature $\alpha = a$ and constant Gaussian curvature $K = a^2$ and satisfies $\Delta G = \lambda G$, where $\lambda = 2a^2$.

3. Main results

It seems natural to state the following problem: *is a B-scroll the only ruled surface in L^3 with null rulings satisfying the equation $\Delta G = \Lambda G$?*

Then our major result states as follows.

THEOREM 1. *B-scrolls over null curves are the only ruled surfaces in L^3 with null rulings satisfying the equation $\Delta G = \Lambda G$.*

From here and Choi's result we have got the complete classification of ruled surfaces in the 3-dimensional Lorentz-Minkowski space whose Gauss map satisfies $\Delta G = \Lambda G$.

COROLLARY 2. *A ruled surface M in L^3 satisfies the equation $\Delta G = \Lambda G$ if and only if M is one of the following surfaces*:

(1) R^2, L^2 *and the cylinders* $S_1^1 \times R^1$, $R_1^1 \times S^1$ *and* $H^1 \times R^1$;

(2) *a B-scroll over a null curve.*

PROOF OF THE THEOREM. Suppose that the Gauss map of M satisfies the equation $\Delta G = \Lambda G$. From Choi's result we may suppose that M has null rulings, so we only have to study the case (4). We are going to show that the function $f = \langle x', B' \times B \rangle$ is constant or, equivalently, that the open set $\mathscr{U} = \{s \in I : f(s)f'(s) \neq 0\}$ is empty. Otherwise, for $s \in \mathscr{U}$, differentiating with respect to t in $\Delta G = \Lambda G$, we have

$$2f\langle B', B' \rangle B = -f\Lambda B, \tag{3.10}$$

where we have used equations (2.4), (2.7) and (2.9). By (2.5) we obtain $\langle B', B' \rangle = f^2$, so that from (3.10) we see that $-2f^2$ is an eigenvalue of Λ, unless $f = 0$. Then f is a constant function, which is a contradiction that finishes the proof.

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