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# RECOGNIZING SPECIAL METRICS BY TOPOLOGICAL PROPERTIES OF THE "METRIC"-PROXIMAL HYPERSPACE

By

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**Abstract.** In this paper, we first characterize those compatible metrics $d$ on a metrizable space $X$ which give rise to a connected $d$-proximal hyperspace. We show that the space of irrational numbers, in particular, admits a complete metric with this property and, as a consequence, we get a negative answer to a question of [11] about selections for hyperspace topologies. Next, we characterize the compatible metrics on $X$ which are uniformly equivalent to ultrametrics showing that this is equivalent to the zero-dimensionality of the corresponding proximal hyperspaces. Applications and related results about other disconnectedness-like properties of proximal hyperspaces are obtained.

## 1. Introduction

Let $X$ be a $T_1$-space, and let $\mathscr{F}(X)$ be the family of all non-empty closed subsets of $X$. Identifying the points of $X$ with the corresponding singletons, we may consider $\mathscr{F}(X)$ as a *set-theoretical* extension of the set $X$. From this point of view, a topology $\tau$ on $\mathscr{F}(X)$ is *admissible* [14] if $(\mathscr{F}(X), \tau)$ is also a *topological extension* of the topological space $X$. Here, in effect, "admissible" means admissible with respect to the topological structure on $X$ which is the terminology we will adopt for this particular paper. It should be said that "admissible" may regard also some additional structures on $X$ (see [14]).

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The two best known examples of admissible topologies on $\mathscr{F}(X)$ are the Vietoris and Hausdorff topologies. The *Vietoris topology* $\tau_V$ depends only on the topology of $X$, and a base for this topology is given by all collections of the form

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{F}(X) : S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \varnothing, \text{ whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of $X$.

Let $(X, d)$ be a metric space. The *Hausdorff topology* $\tau_{H(d)}$ on $\mathscr{F}(X)$ is generated by the *Hausdorff distance* $H(d)$ associated to $d$, hence it depends essentially on the metric $d$. Let us recall that $H(d)$ defines as

$$H(d)(S, T) = \sup\{d(S, x) + d(x, T) : x \in S \cup T\}, \quad S, T \in \mathscr{F}(X).$$

It is well-known that $\tau_V$ coincides with $\tau_{H(d)}$ if and only if $X$ is compact [14], while, in general, these two topologies are not comparable.

There are many interesting investigations about properties of the *Vietoris extension* $(\mathscr{F}(X), \tau_V)$ of a topological space $X$, most of them related to the following general question: *Do there exist properties* $\mathscr{P}$ *and* $\mathscr{F}_V(\mathscr{P})$ *such that* $X \in \mathscr{P}$ *if and only if* $(\mathscr{F}(X), \tau_V) \in \mathscr{F}_V(\mathscr{P})$?

Here are two particular results in this direction which will be important for the proper understanding of this paper. The first one is related to disconnectedness-like properties of $\tau_V$ and states that a space $X$ is strongly zero-dimensional if and only if $(\mathscr{F}(X), \tau_V)$ is zero-dimensional, see [14]. Here, a space $Z$ is *strongly zero-dimensional* if $\dim(Z) = 0$, and $Z$ is *zero-dimensional* if it has a base of clopen sets (i.e., if $\operatorname{ind}(Z) = 0$).

The second result gives that a strongly zero-dimensional metrizable space $X$ is Čech complete if and only if $\mathscr{F}(X)$ has a $\tau_V$-continuous selection [8, 10, 15]. Here, a map $f : \mathscr{F}(X) \to X$ is a *selection* for $\mathscr{F}(X)$ if $f(S) \in S$ for every $S \in \mathscr{F}(X)$. In case $\tau$ is a topology on $\mathscr{F}(X)$, a selection $f$ for $\mathscr{F}(X)$ is $\tau$-*continuous* if it is continuous with respect to $\tau$.

In the present paper we deal with similar problems, but this time about relations between *topological properties* of hyperspaces and *compatible metrics* on the base space. Briefly, let $\mathscr{D}(X)$ be the set of all *compatible* metrics on a metrizable space $X$, and let, for every $d \in \mathscr{D}(X)$, a topology $\tau_{\mathscr{R}(d)}$ on $\mathscr{F}(X)$ be defined (i.e., "$\tau_{\mathscr{R}}$" stands for a generic class of metric-generated hyperspace topologies on $\mathscr{F}(X)$). *Do there exist* $\mathscr{M} \subset \mathscr{D}(X)$ *and a topological property* $\mathscr{F}_{\mathscr{R}}(\mathscr{M})$ *such that* $d \in \mathscr{M}$ *if and only if* $(\mathscr{F}(X), \tau_{\mathscr{R}(d)}) \in \mathscr{F}_{\mathscr{R}}(\mathscr{M})$?

The collection of all Hausdorff topologies $\tau_{H(d)}$, $d \in \mathscr{D}(X)$, provides an example of a generic class "$\tau_H$" of metric-generated hyperspace topologies on $\mathscr{F}(X)$. Note that $(\mathscr{F}(X), \tau_{H(d)})$ is metrizable for every $d \in \mathscr{D}(X)$.

A central place in the present paper is occupied by another collection of admissible metric-generated hyperspace topologies $\tau_{\delta(d)}$, $d \in \mathscr{D}(X)$, on $\mathscr{F}(X)$ which is, in fact, intermediate between the Vietoris hyperspace and the corresponding Hausdorff hyperspaces (actually, it is obtained by "mixing" these hypertopologies in a suitable way). For a given metric $d \in \mathscr{D}(X)$, the topology $\tau_{\delta(d)}$ is known as the *d-proximal topology* on $\mathscr{F}(X)$ [5], and is generated by all *d-modifications* of the basic $\tau_V$-neighbourhoods, i.e. by all collections of the form

$$\langle\!\langle \mathscr{V} \rangle\!\rangle_d = \{S \in \langle \mathscr{V} \rangle : D_d(S, X \setminus \bigcup \mathscr{V}) > 0\},$$

where $\mathscr{V}$ is a finite family of open subsets of $X$ and

$$D_d(S,T) = \inf\{d(x,y) : x \in S,\ y \in T\}, \quad \text{whenever } S, T \subset X.$$

In what follows, for technical reasons only, let us agree that $D_d(S, \emptyset) = +\infty$ for every non-empty $S \subset X$.

Let us mention that a $d$-proximal topology $\tau_{\delta(d)}$ is metrizable if and only if $(X,d)$ is totally bounded [5], which is in turn equivalent to the normality of $(\mathscr{F}(X), \tau_{\delta(d)})$ [12]. Also, for metrics $d, \rho \in \mathscr{D}(X)$, we have that $\tau_{\delta(d)} = \tau_{\delta(\rho)}$ if and only if $d$ and $\rho$ are *uniformly equivalent* [5]. Finally, we always have the following (usually strong) inclusion:

$$\tau_{\delta(d)} \subset \tau_V \cap \tau_{H(d)}.$$

We are now ready to state more precisely the main purpose of this paper. In the first place, we characterize those compatible metrics $d \in \mathscr{D}(X)$ on a metrizable space $X$ which give rise to a connected $d$-proximal hyperspace topology (Theorem 2.1). Further, we demonstrate that the space of the irrational numbers $\boldsymbol{P}$ has a complete compatible metric $p \in \mathscr{D}(\boldsymbol{P})$ such that $(\mathscr{F}(\boldsymbol{P}), \tau_{\delta(p)})$ is connected (Example 2.6). In particular, this implies that $\mathscr{F}(\boldsymbol{P})$ does not admit any $\tau_{\delta(p)}$-continuous selection, which provides a negative answer to a question of [11].

In the second place, we show that a $d$-proximal hyperspace is zero-dimensional if and only if $d$ is uniformly equivalent to an ultrametric (Theorem 3.3). We apply this fact to show that a zero-dimensional metrizable space $X$ is compact if and only if any $d \in \mathscr{D}(X)$ is uniformly equivalent to an ultrametric (Theorem 4.3). Other results in classifying metrizable spaces are provided (see Theorems 4.1, 4.5 and 5.9).

Finally, the paper contains also results about the selection problem on "metric"-proximal hyperspaces (see Section 5). Related to the result of [8, 10, 15] mentioned before, we extend [11, Theorem 1.2] showing that, for a completely metrizable space $X$ and a $d \in \mathscr{D}(X)$, if $(X,d)$ has a base of *d-clopen sets*, then

$\mathscr{F}(X)$ has a $\tau_{\delta(d)}$-continuous selection (Theorem 5.1). We also show that the assumption above can be weakened to requiring the subspace $X \backslash \{z\}$, obtained by removing some single point $z$, to have a base of $d$-clopen sets (Corollary 5.7).

## 2. Which Metrics Do Give Rise to a Connected Proximal Topology?

Let $(X, d)$ be a metric space. We shall say that a subset $A$ of a metric space $(X, d)$ is *d-clopen* if $D_d(A, X \backslash A) > 0$. Note that every $d$-clopen set is clopen but the converse is not true (see, for instance, Examples 2.5 and 2.6). On the other hand, by the definition of $D_d$, the subsets $X$ and $\varnothing$ are always $d$-clopen. Now, we shall say that a metric space $(X, d)$ is *d-connected* if $X$ and $\varnothing$ are the only $d$-clopen subsets of $(X, d)$.

The following theorem will be proven in this section.

THEOREM 2.1. *A metric space $(X, d)$ is d-connected if and only if $(\mathscr{F}(X), \tau_{\delta(d)})$ is connected.*

To prepare for the proof of Theorem 2.1, we provide some relations between clopen subsets of $(\mathscr{F}(X), \tau_{\delta(d)})$ and $d$-clopen subsets of $(X, d)$. To this end, we need the following property of the Vietoris hyperspace; such a property was also stated, in a slightly weaker form, in [6].

LEMMA 2.2. *Let $X$ be a topological space, $\mathscr{C} \subset \mathscr{F}(X)$ be a $\tau_V$-closed set, and let $\mathscr{M}$ be a non-empty subset of $\mathscr{C}$ which is a chain with respect to the usual set-theoretical inclusion. Then, there exists $M \in \mathscr{C}$ such that $\bigcup \mathscr{M} \subset M$.*

PROOF. Let $M = \overline{\bigcup \mathscr{M}}$, and let us show that $M \in \mathscr{C}$. Take a basic $\tau_V$-neighbourhood $\langle \mathscr{U} \rangle$ of $M$. For every $U \in \mathscr{U}$ there exists $M_U \in \mathscr{M}$ such that $M_U \cap U \neq \varnothing$ because $U$ is open and $\overline{\bigcup \mathscr{M}} \cap U \neq \varnothing$. Hence, $M_{\mathscr{U}} = \bigcup \{M_U : U \in \mathscr{U}\} \in \langle \mathscr{U} \rangle$ because $M_{\mathscr{U}} \subset M$. On the other hand, $M_{\mathscr{U}} \in \mathscr{M} \subset \mathscr{C}$ because $\mathscr{M}$ is a chain in $\mathscr{C}$. Therefore, $\langle \mathscr{U} \rangle \cap \mathscr{C} \neq \varnothing$. This finally implies that $M \in \mathscr{C}$ because $\mathscr{C}$ is $\tau_V$-closed in $\mathscr{F}(X)$. □

The following consequence of Lemma 2.2 regards the $d$-clopen subsets of $(X, d)$ as an indication about the possible clopen subsets of $(\mathscr{F}(X), \tau_{\delta(d)})$.

COROLLARY 2.3. *Let $(X, d)$ be a metric space, $\mathscr{U} \subset \mathscr{F}(X)$ be a $\tau_{\delta(d)}$-clopen set, with $\mathscr{U} \neq \varnothing$, and let $\mathscr{A} \subset \mathscr{U}$ be a maximal chain with respect to the usual set-*

*theoretical inclusion. Then, $\mathscr{A}$ has a maximal element which is a d-clopen subset of $(X, d)$.*

Proof. It follows from Lemma 2.2 that $\mathscr{A}$ has a maximal element $A$ because $\mathscr{A} \subset \mathscr{U}$ is a maximal chain and $\mathscr{U}$ is, in particular, a $\tau_V$-closed set. Since $\mathscr{U}$ is also $\tau_{\delta(d)}$-open, there now exists a basic $\tau_{\delta(d)}$-open set $\langle\!\langle \mathscr{V} \rangle\!\rangle_d$ such that $A \in \langle\!\langle \mathscr{V} \rangle\!\rangle_d \subset \mathscr{U}$. The only possibility is $A = \bigcup \mathscr{V}$. Indeed, $x \in \bigcup \mathscr{V}$ implies $A \cup \{x\} \in \langle\!\langle \mathscr{V} \rangle\!\rangle_d \subset \mathscr{U}$ because $d(x, X \backslash \bigcup \mathscr{V}) > 0$. Since $\mathscr{A}$ is maximal, we finally get that $A \cup \{x\} \in \mathscr{A}$. That is, $A = \bigcup \mathscr{V}$ holds and, by definition, $D_d(A, X \backslash A) = D_d(A, X \backslash \bigcup \mathscr{V}) > 0$. □

We conclude the preparation for the proof of Theorem 2.1 with the following proposition which may read as a partial converse of Corollary 2.3. Below, and in the sequel, for a subset $A \subset X$ we set $\langle\!\langle A \rangle\!\rangle_d = \langle\!\langle \{A\} \rangle\!\rangle_d$ and, respectively, $\langle A \rangle = \langle \{A\} \rangle$.

Proposition 2.4. *For a clopen subset $A$ of a metric space $(X, d)$, the following conditions are equivalent*:

(a) *$A$ is d-clopen.*
(b) *$\langle\!\langle A \rangle\!\rangle_d$ is $\tau_{\delta(d)}$-clopen.*
(c) *$\langle A \rangle$ is $\tau_{\delta(d)}$-open.*

Proof. In case $A = \varnothing$, this is trivial. Suppose $A \neq \varnothing$. Then, (a) $\to$ (b) follows from the definition of a $d$-clopen set. For (b) $\to$ (c), take a maximal chain $\mathscr{A}$ in $\langle\!\langle A \rangle\!\rangle_d$. Then, by Corollary 2.3, $\mathscr{A}$ has a maximal element which is a $d$-clopen subset of $(X, d)$, and it is clear that such an element must be $A$ itself; thus, $\langle A \rangle = \langle\!\langle A \rangle\!\rangle_d$. Finally, (c) $\to$ (a) is a consequence of $A \in \langle A \rangle$. □

Proof of Theorem 2.1. In case $(\mathscr{F}(X), \tau_{\delta(d)})$ is connected, by Proposition 2.4, the space $(X, d)$ must be $d$-connected.

As for the inverse implication, suppose that $(X, d)$ is $d$-connected but there exists a $\tau_{\delta(d)}$-clopen $\mathscr{A} \subset \mathscr{F}(X)$, with $\varnothing \neq \mathscr{A} \neq \mathscr{F}(X)$. Then $\mathscr{F}(X) \backslash \mathscr{A}$ has the same properties, and either $\mathscr{A}$ or $\mathscr{F}(X) \backslash \mathscr{A}$ does not contain $X$. So, Corollary 2.3 gives a $d$-clopen set $A$ with $\varnothing \neq A \neq X$ which is impossible. □

Note that every connected metric space $(X, d)$ is certainly $d$-connected which, together with Theorem 2.1, gives a list of examples of connected $d$-proximal topologies. However, the converse is not true and this is what we will establish in the rest of the section. To this end, let us observe that $(X,d)$ is $d$-connected if and only if $D_d(A, X\setminus A) = 0$ for every non-empty proper (closed) subset $A$ of $X$. In particular, this implies the following immediate example of strongly zero-dimensional $d$-connected metric spaces $(X,d)$.

Example 2.5. *Let $X$ be a dense subset of the real line $\boldsymbol{R}$, and let $d$ be the standard Euclidean metric on $X$. Then, $(X,d)$ is $d$-connected.*

The rational numbers $\boldsymbol{Q}$ and the irrational numbers $\boldsymbol{P}$ are among the most important zero-dimensional dense subsets of the real line. Unfortunately, both the metric spaces $(\boldsymbol{Q},d)$ and $(\boldsymbol{P},d)$ are not complete. From a topological point of view, however, the space of the irrational numbers $\boldsymbol{P}$ is Čech complete. As we will see, this is only a part of our motivation for the next key example.

Example 2.6. *There exists a complete metric $p \in \mathscr{D}(\boldsymbol{P})$ on the irrational line $\boldsymbol{P}$ such that $(\boldsymbol{P},p)$ is $p$-connected.*

Proof. Let $d$ be the standard Euclidean metric on $\boldsymbol{P}$. We will describe the metric $p$ in an explicit way. In fact, $p$ is the metric on $\boldsymbol{P}$ obtained by modifying $d$ to a complete metric on $\boldsymbol{P}$ by the help of the countable complement $\boldsymbol{Q}$ of $\boldsymbol{P}$ in $\boldsymbol{R}$. Namely, let $\{q_i : i \in \boldsymbol{N}\}$ be a one-to-one indexing of the rational numbers $\boldsymbol{Q}$. Then, the formula

$$p(x,y) = d(x,y) + \sum_{i=1}^{\infty} \frac{1}{2^i} \min\left\{1, \left|\frac{1}{d(x,q_i)} - \frac{1}{d(y,q_i)}\right|\right\}, \quad x, y \in \boldsymbol{P},$$

certainly defines a complete compatible metric $p$ on $\boldsymbol{P}$. Turning to the verification that $(\boldsymbol{P},p)$ is $p$-connected, let $B$ be a proper non-empty closed subset of $\boldsymbol{P}$, and let $\varepsilon > 0$. What we have to show is that $D_p(B, \boldsymbol{P}\setminus B) < \varepsilon$. For the purpose, let $k \in \boldsymbol{N}$ be such that $1/2^{k-1} < \varepsilon/3$. It will be now sufficient to find a point $b \in B$ and a point $c \in \boldsymbol{P}\setminus B$ such that

(i) $d(b,c) < \varepsilon/3$, and
(ii) $|1/d(b,q_i) - 1/d(c,q_i)| < \varepsilon/3$ for every $i \in \boldsymbol{N}$, with $1 \le i \le k-1$.

Indeed, let $b \in B$ and $c \in \boldsymbol{P}\setminus B$ be as in (i) and (ii). Then,

$$p(b,c) = d(b,c) + \sum_{i=1}^{\infty} \frac{1}{2^i} \min\left\{1, \left|\frac{1}{d(b,q_i)} - \frac{1}{d(c,q_i)}\right|\right\}$$

$$< \frac{\varepsilon}{3} + \sum_{i=1}^{k-1} \frac{1}{2^i} \min\left\{1, \left|\frac{1}{d(b,q_i)} - \frac{1}{d(c,q_i)}\right|\right\}$$

$$+ \sum_{i=k}^{\infty} \frac{1}{2^i} \min\left\{1, \left|\frac{1}{d(b,q_i)} - \frac{1}{d(c,q_i)}\right|\right\}$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\sum_{i=1}^{k-1} \frac{1}{2^i} + \sum_{i=k}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{1}{2^{k-1}} < \varepsilon,$$

and therefore $D_p(B, \boldsymbol{P} \setminus B) \le p(b,c) < \varepsilon$.

Thus, to finish the proof, it only remains to define such points $b \in B$ and $c \in \boldsymbol{P} \setminus B$. Since there exists $x \in \boldsymbol{P} \setminus B$, we may suppose that there exists $y \in B$ with $y > x$ (the case $y < x$ is symmetric). Let $a = \sup\{z \in [x, y) : [x, z] \cap B = \varnothing\}$. We distinguish the following two situations.

In case $a \in \boldsymbol{P}$, we have that $a \in B$ because $B$ is closed in $\boldsymbol{P}$. Then, set $b = a$. As for $c$, take any point $c \in [x, b) \cap \boldsymbol{P}$ such that $d(b,c) < \varepsilon/3$ and $|1/d(b,q_i) - 1/d(c,q_i)| < \varepsilon/3$ for every $i \in \boldsymbol{N}$, with $1 \le i \le k-1$. Clearly, $b$ and $c$ are as required in (i) and (ii).

In case $a \in \boldsymbol{Q}$, it follows that $a = q_j$ for some $j \in \boldsymbol{N}$. Hence, there exists a strictly decreasing sequence $\{y_n : n \in \boldsymbol{N}\} \subset B$ such that $\lim_{n\to\infty} y_n = a$ and $(2a - y_1) \in [x, a]$. For every $n \in \boldsymbol{N}$, let $x_n = 2a - y_n$ be the element symmetric to $y_n$ with respect to $a$. Then, $\{x_n : n \in \boldsymbol{N}\} \subset \boldsymbol{P} \setminus B$ is a strictly increasing sequence which is convergent to $a$. Note that $\lim_{n\to\infty} |1/d(x_n, q_i) - 1/d(y_n, q_i)| = 0$ for every $i \in \boldsymbol{N}$. Indeed, if $i \ne j$, then this follows from the fact that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Otherwise, merely note that $|1/d(x_n, q_j) - 1/d(y_n, q_j)| = 0$ for every $n \in \boldsymbol{N}$. In this way, there is now an $m \in \boldsymbol{N}$ such that $d(x_m, y_m) < \varepsilon/3$ and $|1/d(x_m, q_i) - 1/d(y_m, q_i)| < \varepsilon/3$ for every $i \in \boldsymbol{N}$ with $1 \le i \le k-1$. Then, in this case, $b = y_m$ and $c = x_m$ are as required in (i) and (ii). □

By Example 2.6, we have the following interesting consequence which provides, in particular, a negative answer to a question of [11].

Corollary 2.7. *Let $X$ be a completely metrizable space which contains a closed copy of the irrational line $\boldsymbol{P}$. Then, there exists a complete compatible metric $d$ on $X$ such that $\mathscr{F}(X)$ does not admit any $\tau_{\delta(d)}$-continuous selection.*

Proof. Let $p \in \mathscr{D}(\boldsymbol{P})$ be as in Example 2.6. By a result of [3], $p$ extends to a complete compatible metric $d$ on $X$. Suppose, by contradiction, that $f : \mathscr{F}(X) \to X$ is a $\tau_{\delta(d)}$-continuous selection, and let $S \in \mathscr{F}(X)$ be a proper clopen subset of $\boldsymbol{P}$. By [5, Lemma 4.1], $(\mathscr{F}(\boldsymbol{P}, \tau_{\delta(p)})$ coincides with $\mathscr{F}(\boldsymbol{P})$ equipped with the relative topology of $(\mathscr{F}(X), \tau_{\delta(d)})$. Hence, by Theorem 2.1 and Example 2.6, $\mathscr{F}(\boldsymbol{P})$ is a connected subset of $(\mathscr{F}(X), \tau_{\delta(d)})$. On the other hand, the set $f^{-1}(S) \cap \mathscr{F}(\boldsymbol{P})$ is clopen in $(\mathscr{F}(\boldsymbol{P}), \tau_{\delta(p)})$, so $\mathscr{F}(\boldsymbol{P}) \subset f^{-1}(S)$. However this is impossible because $f$ is a selection and therefore $f^{-1}(S)$ will contain all the singletons of points of $S$ but will not contain any singleton of points of $\boldsymbol{P} \backslash S$. □

## 3. Ultrametrics and Disconnectedness-Like Properties of Proximal Hyperspaces

In this section, we first establish an equivalence between a suitable remetrization property on a metrizable space $X$ and the topological property of zero-dimensionality of the corresponding proximal hyperspaces. Let $d \in \mathscr{D}(X)$, $x, y \in X$ and let $\delta > 0$. We shall say that the points $x$ and $y$ are *$\delta$-chainable* in $(X, d)$, and shall write that $\mathrm{ch}_d(x, y) < \delta$, if there exists an $n \in \boldsymbol{N}$ and points $z_0, \dots, z_n \in X$ such that $z_0 = x$, $z_n = y$ and $d(z_{i-1}, z_i) < \delta$ for $i = 1, \dots, n$ (cf. [2]).

For a non-empty subset $A$ of $X$ and $\varepsilon > 0$, we define an *$\varepsilon$-chain neighbourhood* of $A$ in $(X, d)$ by

$$\mathscr{CN}_\varepsilon^d(A) = \{y \in X : \mathrm{ch}_d(x, y) < \varepsilon \text{ for some } x \in A\}.$$

Also, we will use $\mathscr{N}_\varepsilon^d(A)$ to denote the *open $\varepsilon$-neighbourhood* of $A$ in $(X, d)$, i.e. $\mathscr{N}_\varepsilon^d(A) = \{y \in X : d(y, A) < \varepsilon\}$. In the special case of a singleton $A = \{x\}$, we set $\mathscr{CN}_\varepsilon^d(x) = \mathscr{CN}_\varepsilon^d(\{x\})$ and, respectively, $\mathscr{N}_\varepsilon^d(x) = \mathscr{N}_\varepsilon^d(\{x\})$.

Note that $\mathscr{N}_\delta^d(A) \subset \mathscr{CN}_\delta^d(A)$ is always valid but the converse is related to special properties of the metric $d$. Let us recall that a metric $d \in \mathscr{D}(X)$ on $X$ is said to be an *ultrametric*, or a *non-Archimedean* one, if $d(x, y) \le \max\{d(x, z), d(z, y)\}$ for every $x, y, z \in X$.

Proposition 3.1. *Let $X$ be a metrizable space. For a metric $d \in \mathscr{D}(X)$, the following two conditions are equivalent*:

(a) *$d$ is an ultrametric.*
(b) *$\mathscr{CN}_\varepsilon^d(x) = \mathscr{N}_\varepsilon^d(x)$ for every $x \in X$ and $\varepsilon > 0$.*

Proof. In case $d$ is an ultrametric, we have that $\mathscr{N}_\varepsilon^d(x) =$

$\bigcup\{\mathscr{N}_\varepsilon^d(y) : y \in \mathscr{N}_\varepsilon^d(x)\}$ for every $x \in X$ and $\varepsilon > 0$. This easily entails, by an inductive argument, the implication (a) $\to$ (b). Suppose now that $d$ is as in (b). Also, take points $x, y, z \in X$, and let $\delta = \max\{d(x,z), d(z,y)\}$. Then, for every $\varepsilon > \delta$, we get that $y \in \mathscr{C}\mathscr{N}_\varepsilon^d(x) = \mathscr{N}_\varepsilon^d(x)$ which finally implies that $d(x,y) \leq \delta$. □

In what follows, to every subset $A$ of a metric space $(X,d)$ we associate a real number $\Delta_d(A)$, or the infinite number $\Delta_d(A) = +\infty$, defined as $\Delta_d(A) = D_d(A, X\backslash A)$. By definition, $A$ is a $d$-clopen subset of $(X,d)$ if and only if $\Delta_d(A) > 0$. On the other hand, $\Delta_d(A) = +\infty$ if and only if either $A = X$ or $A = \varnothing$. The following simple observation, whose verification is left to the reader, presents some important relations between $d$-clopen sets and $\delta$-chain neighbourhoods.

Proposition 3.2. *For a non-empty subset $A$ of a metric space $(X,d)$ and $\delta > 0$, the following holds*:

(1) $\mathscr{C}\mathscr{N}_\delta^d(A) = A$ *if and only if* $\Delta_d(A) \geq \delta$.
(2) $\mathscr{C}\mathscr{N}_\delta^d(\mathscr{C}\mathscr{N}_\delta^d(A)) = \mathscr{C}\mathscr{N}_\delta^d(A)$.

We are now ready to prove the following theorem.

Theorem 3.3. *For a metric space $(X,d)$, the following conditions are equivalent*:

(a) *There exists an ultrametric on $X$ which is uniformly equivalent to $d$.*
(b) $(\mathscr{F}(X), \tau_{\delta(d)})$ *is zero-dimensional.*
(c) *For every $A \in \mathscr{F}(X)$ and $\varepsilon > 0$ there exists a $d$-clopen subset $B$ of $(X,d)$ with $A \subset B \subset \mathscr{N}_\varepsilon^d(A)$.*
(d) *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathscr{C}\mathscr{N}_\delta^d(x) \subset \mathscr{N}_\varepsilon^d(x)$ for all $x \in X$.*

Proof. (a) $\to$ (b). Let $\rho$ be an ultrametric on $X$ which is uniformly equivalent to $d$. By a result of [5], we have that $\tau_{\delta(\rho)} = \tau_{\delta(d)}$. Hence, it suffices to show that $(\mathscr{F}(X), \tau_{\delta(\rho)})$ is zero-dimensional. Towards this end, let $A \in \mathscr{F}(X)$ and let $\langle\!\langle \mathscr{U} \rangle\!\rangle_\rho$ be a basic $\tau_{\delta(\rho)}$-neighbourhood of $A$. Then, there exists $\delta > 0$ such that $\mathscr{N}_\delta^\rho(A) \subset \bigcup \mathscr{U}$. Next, for every $U \in \mathscr{U}$ pick a fixed point $x(U) \in A \cap U$ and $\delta(U) > 0$ with $\mathscr{N}_{\delta(U)}^\rho(x(U)) \subset U \cap \mathscr{N}_\delta^\rho(A)$. Finally, set $\mathscr{V} = \{\mathscr{N}_\delta^\rho(A)\} \cup \{\mathscr{N}_{\delta(U)}^\rho(x(U)) : U \in \mathscr{U}\}$. In this way, by Propositions 3.1 and 3.2, we get a family

$\mathscr{V}$ of $\rho$-clopen subsets of $(X, \rho)$. Then, by Proposition 2.4, $\langle\!\langle \mathscr{V} \rangle\!\rangle_\rho$ defines a $\tau_{\delta(\rho)}$-clopen neighbourhood of $A$ because

$$\begin{aligned}\langle\!\langle \mathscr{V} \rangle\!\rangle_\rho &= \bigcap\{\langle\!\langle \{\mathscr{N}_\delta^\rho(A), \mathscr{N}_{\delta(U)}^\rho(x(U))\}\rangle\!\rangle_\rho : U \in \mathscr{U}\} \\ &= \langle\!\langle \mathscr{N}_\delta^\rho(A)\rangle\!\rangle_\rho \setminus \bigcup\{\langle\!\langle X \setminus \mathscr{N}_{\delta(U)}^\rho(x(U))\rangle\!\rangle_\rho : U \in \mathscr{U}\}.\end{aligned}$$

Since $\langle\!\langle \mathscr{V} \rangle\!\rangle_\rho \subset \langle\!\langle \mathscr{U} \rangle\!\rangle_\rho$, (b) holds.

(b) $\to$ (c). Let $A \in \mathscr{F}(X)$, and let $\varepsilon > 0$. Note that $\mathscr{V} = \langle\!\langle \mathscr{N}_\varepsilon^d(A)\rangle\!\rangle_d$ defines a $\tau_{\delta(d)}$-neighbourhood of $A$. Then, by (b), there exists a $\tau_{\delta(d)}$-clopen neighbourhood $\mathscr{U}$ of $A$ with $\mathscr{U} \subset \mathscr{V}$. Let $\mathscr{B}$ be a maximal chain in $\mathscr{U}$ such that $A \in \mathscr{B}$. Then, by Corollary 2.3, there exists $B = \max \mathscr{B}$, and it is a $d$-clopen subset of $(X, d)$. In particular, $A \subset B \subset \mathscr{N}_\varepsilon^d(A)$ which is the statement of (c).

(c) $\to$ (d). Suppose that (d) fails. Hence, there exists a $\gamma > 0$ such that for every $n \in \boldsymbol{N}$ one can find points $x_n, y_n \in X$ with $\mathrm{ch}_d(x_n, y_n) < 1/n$ and $d(x_n, y_n) \geq 4\gamma$. According to the Efremovic Lemma (see [4]), there now exists a strictly increasing sequence $\{n_i : i \in \boldsymbol{N}\} \subset \boldsymbol{N}$ such that $d(x_{n_i}, y_{n_j}) \geq \gamma$ for every $i, j \in \boldsymbol{N}$. Setting then $A = \overline{\{x_{n_i} : i \in \boldsymbol{N}\}}$, we get that $\mathscr{C}\mathscr{N}_\delta^d(A) \setminus \mathscr{N}_\gamma^d(A) \neq \varnothing$ for every $\delta > 0$ because $y_{n_i} \in \mathscr{C}\mathscr{N}_{1/n_i}^d(A) \setminus \mathscr{N}_\gamma^d(A)$ for every $i \in \boldsymbol{N}$. On the other hand, by condition, there is $B \subset X$ such that $\Delta_d(B) > 0$ and $A \subset B \subset \mathscr{N}_\gamma^d(A)$. Hence, by Proposition 3.2,

$$\mathscr{C}\mathscr{N}_{\Delta_d(B)}^d(A) \subset \mathscr{C}\mathscr{N}_{\Delta_d(B)}^d(B) = B \subset \mathscr{N}_\gamma^d(A).$$

A contradiction.

(d) $\to$ (a). Let $\{\delta_n : n \in \boldsymbol{N}\}$ be a decreasing sequence of positive real numbers such that, for every $n \in \boldsymbol{N}$ and $x, y \in X$, we have $d(x, y) < 1/n$ provided $\mathrm{ch}_d(x, y) < \delta_n$. Set $\mathscr{U}_0 = \{X\}$ and $\mathscr{U}_n = \{\mathscr{C}\mathscr{N}_{\delta_n}^d(x) : x \in X\}$. By Proposition 3.2, each $\mathscr{U}_n$, $n > 0$, is a disjoint open cover of $X$ which refines both $\{\mathscr{N}_{1/n}^d(x) : x \in X\}$ and $\mathscr{U}_{n-1}$. Also, $\bigcup\{\mathscr{U}_n : n \in \boldsymbol{N}\}$ is a base for the topology $X$. Therefore, $\{\mathscr{U}_n : n \in \boldsymbol{N}\}$ is a discrete development in the sense of [9]. Then, according to [9, Proposition 1.5], we may consider the compatible ultrametric $\rho$ on $X$ defined by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y \\ \dfrac{1}{r(x, y)} & \text{if } x \neq y, \end{cases}$$

where $r(x, y) = \min\{n \in \boldsymbol{N} : y \notin \mathscr{C}\mathscr{N}_{\delta_n}^d(x)\}$. If $\rho(x, y) < 1/n$ for some $x, y \in X$ and $n \in \boldsymbol{N}$, then $\mathrm{ch}_d(x, y) < \delta_n$ and, therefore, $d(x, y) < 1/n$. Thus, to prove that $\rho$ and $d$ are uniformly equivalent, it only remains to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(x, y) < \varepsilon$ provided $d(x, y) < \delta$. For a given $\varepsilon > 0$, let $m \in \boldsymbol{N}$ be such that $1/m \leq \varepsilon$. Then, $d(x, y) < \delta_m$ for some $x, y \in X$ certainly implies that $y \in \mathscr{C}\mathscr{N}_{\delta_m}^d(x)$ and, hence, $\rho(x, y) < 1/m \leq \varepsilon$. □

The rest of the section is devoted to a more precise reading of the remetrization condition stated in (a) of Theorem 3.3. Suppose that $X$ is a metrizable space. We consider a relation $\preceq$ of partial order on $\mathscr{D}(X)$ by letting, for $\rho, d \in \mathscr{D}(X)$, that $\rho \preceq d$ if and only if the uniformity generated by $\rho$ is coarser than the one generated by $d$ or, equivalently, if for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that, whenever $x, y \in X$, $d(x, y) < \eta(\varepsilon)$ implies $\rho(x, y) < \varepsilon$. Note that two metrics $\rho, d \in \mathscr{D}(X)$ are uniformly equivalent if and only if $\rho \preceq d$ and $d \preceq \rho$. On the other hand, it could be easy observed that $\rho \preceq d$ implies $\tau_{\delta(\rho)} \subset \tau_{\delta(d)}$.

The following observation shows that the existence of an ultrametric $\rho$ with $d \preceq \rho$ is a topological property and, hence, it cannot be applied to recognize special $d$-proximal hyperspaces.

Proposition 3.4. *For a metrizable space $X$, the following two conditions are equivalent*:

(a) *For every $d \in \mathscr{D}(X)$ there exists an ultrametric $\rho \in \mathscr{D}(X)$ with $d \preceq \rho$.*
(b) *$X$ is strongly zero-dimensional.*

Proof. The implication (a) → (b) is obvious. Suppose that $X$ is strongly zero-dimensional. We follow the construction in the last part of the previous proof. Namely, we set $\mathscr{U}_0 = \{X\}$. Since $X$ is strongly zero-dimensional, for every $n > 0$ there exists a disjoint open cover $\mathscr{U}_n$ of $X$ which refines both $\{\mathscr{N}_{1/n}^d(x) : x \in X\}$ and $\mathscr{U}_{n-1}$. In this way, we get a discrete development $\{\mathscr{U}_n : n \in \boldsymbol{N}\}$ of $X$. Also, for every point $x \in X$ and every $n \in \boldsymbol{N}$ there exists exactly one $U_n(x) \in \mathscr{U}_n$ with $x \in U_n(x)$. Then, as before, we may consider the compatible ultrametric $\rho$ on $X$ defined by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y \\ \dfrac{1}{r(x, y)} & \text{if } x \neq y \end{cases}$$

where $r(x, y) = \min\{n \in \boldsymbol{N} : y \notin U_n(x)\}$. If $\rho(x, y) < 1/n$ for some $x, y \in X$ and $n \in \boldsymbol{N}$, then $x, y \in U$ for some $U \in \mathscr{U}_n$. Therefore, $d(x, y) < 2/n$ which finally implies that $d \preceq \rho$. □

The reverse inequality certainly implies a special metric property which is the statement of our next result. To state this, for a metrizable space $X$ and $d, \rho \in \mathscr{D}(X)$, let us agree to write that $\rho \leq d$ if and only if $\rho(x, y) \leq d(x, y)$ for every $x, y \in X$. Note that $\rho \leq d$ implies $\rho \preceq d$.

Theorem 3.5. *For a metric space* $(X, d)$, *the following conditions are equivalent*:

(a) *There exists an ultrametric* $\rho \in \mathscr{D}(X)$ *with* $\rho \leq d$.
(b) *There exists an ultrametric* $\rho \in \mathscr{D}(X)$ *with* $\rho \preceq d$.
(c) *For every* $x \in X$ *there exists a neighbourhood* $L$ *of* $x$ *and an ultrametric* $\rho \in \mathscr{D}(L)$ *with* $\rho \preceq d \mid L \times L$.
(d) $(X, d)$ *has a base of d-clopen sets.*
(e) *For every* $x \in X$ *and* $\varepsilon > 0$ *there is* $\delta = \delta(x, \varepsilon) > 0$ *with* $\mathscr{C}\mathscr{N}_\delta^d(x) \subset \mathscr{N}_\varepsilon^d(x)$.

To prepare for the proof of Theorem 3.5, we need the following statements about $d$-clopen sets and $\delta$-chain neighbourhoods.

Proposition 3.6. *Let* $(X, d)$ *be a metric space,* $x \in X$, $L$ *be a neighbourhood of* $x$, *and let* $\rho \in \mathscr{D}(L)$ *be an ultrametric on* $L$ *such that* $\rho \preceq d \mid L \times L$. *Then,* $L$ *contains a d-clopen neighbourhood* $G$ *of* $x$.

Proof. Let $\gamma > 0$ and $\varepsilon > 0$ be such that $\mathscr{N}_\varepsilon^\rho(x) \subset \mathscr{N}_\gamma^d(x) \subset \mathscr{N}_{2\gamma}^d(x) \subset L$. Then, $G = \mathscr{N}_\varepsilon^\rho(x)$ is a $d$-clopen subset of $(X, d)$. Indeed, let $\eta(\varepsilon) > 0$ be as in the definition of the relation $\rho \preceq d \mid L \times L$, and let $\delta = \min\{\eta(\varepsilon), \gamma\}$. Take a point $y \in G$ and a point $z \in X$ such that $d(y, z) < \delta$. Note that $\mathscr{N}_\delta^d(y) \subset \mathscr{N}_\gamma^d(y) \subset \mathscr{N}_{2\gamma}^d(x) \subset L$ because $y \in G \subset \mathscr{N}_\gamma^d(x)$. Therefore, $\delta \leq \eta(\varepsilon)$ implies

$$z \in \mathscr{N}_\delta^d(y) \subset \mathscr{N}_{\eta(\varepsilon)}^d(y) \cap L \subset \mathscr{N}_\varepsilon^\rho(y) = \mathscr{N}_\varepsilon^\rho(x) = G$$

because $\rho$ is an ultrametric. That is, $\Delta_d(G) \geq \delta > 0$. □

Proposition 3.7. *Let* $(X, d)$ *be a metric space,* $x \in X$ *and let* $\delta > 0$. *Then,* $\mathscr{C}\mathscr{N}_\delta^d(y) = \mathscr{C}\mathscr{N}_\delta^d(x)$ *for every* $y \in \mathscr{C}\mathscr{N}_\delta^d(x)$.

Proof. Follows from the definition of $\delta$-chainable points. □

Proposition 3.8. *Let* $(X, d)$ *be a metric space,* $x \in X$, *and let* $\delta > 0$. *Also, let* $B$ *be a d-clopen subset of* $(X, d)$, *and let* $\gamma = \min\{\delta, \Delta_d(B)\}$. *Then,* $\mathscr{C}\mathscr{N}_\gamma^d(y) \subset \mathscr{C}\mathscr{N}_\delta^d(x) \setminus B$ *for every point* $y \in \mathscr{C}\mathscr{N}_\delta^d(x) \setminus B$.

PROOF. Easy. □

Now, to every family $\mathscr{V}$ of subsets of a metric space $(X,d)$ we associate the number $\Delta_d(\mathscr{V}) = \inf\{\Delta_d(V) : V \in \mathscr{V}\}$. The following trivial property of $\Delta_d(\mathscr{V})$ will be found useful in our next considerations.

PROPOSITION 3.9. *Let $(X,d)$ be a metric space, and let $\mathscr{V}$ be a family of subsets of $X$. Then, $\Delta_d(\mathscr{V}) \le \min\{\Delta_d(\bigcup \mathscr{V}), \Delta_d(\bigcap \mathscr{V})\}$.*

For a metric space $(X,d)$ and $\gamma > 0$, we consider the following families of subsets of $X$:

$$\mathscr{CN}[X,d] = \{\mathscr{CN}_\delta{}^d(x) : x \in X \text{ and } \delta > 0\},$$

and

$$\mathscr{CN}_\gamma[X,d] = \{\mathscr{CN}_\delta{}^d(x) : x \in X \text{ and } \delta \ge \gamma\}.$$

Note that $\mathscr{CN}[X,d] = \bigcup\{\mathscr{CN}_\gamma[X,d] : \gamma > 0\}$.

PROPOSITION 3.10. *Let $(X,d)$ be a metric space, $\gamma > 0$, and let $\mathscr{V} \subset \mathscr{CN}_\gamma[X,d]$. Then, there is a disjoint refinement $\mathscr{U}$ of $\mathscr{V}$ such that $\bigcup \mathscr{U} = \bigcup \mathscr{V}$ and $\mathscr{U} \subset \mathscr{CN}_\gamma[X,d]$.*

PROOF. By Proposition 3.7, $\mathscr{U} = \{\mathscr{CN}_\gamma{}^d(x) : x \in \bigcup \mathscr{V}\}$ is as required because for every $V \in \mathscr{V}$ and $x \in V$ there exists $\delta \ge \gamma$ with $\mathscr{CN}_\gamma{}^d(x) \subset \mathscr{CN}_\delta{}^d(x) = V$. □

LEMMA 3.11. *Let $(X,d)$ be a metric space, and let $\mathscr{V} \subset \mathscr{CN}[X,d]$. Then, there exists a disjoint family $\mathscr{U} \subset \mathscr{CN}[X,d]$ such that $\mathscr{U}$ refines $\mathscr{V}$ and $\bigcup \mathscr{U} = \bigcup \mathscr{V}$.*

PROOF. Whenever $n \ge 1$, set $\mathscr{V}_n = \{V \in \mathscr{V} : V \in \mathscr{CN}_{1/n}[X,d]\}$. By Proposition 3.10, the family $\mathscr{V}_1$ is refined by a disjoint family $\mathscr{U}_1 \subset \mathscr{CN}_1[X,d]$ such that $\bigcup \mathscr{U}_1 = \bigcup \mathscr{V}_1$. By Propositions 3.2 and 3.9, $\bigcup \mathscr{U}_1$ is a $d$-clopen set with $\Delta_d(\bigcup \mathscr{U}_1) \ge 1$. Hence, by Proposition 3.8, there exists a family $\mathscr{W}_2 \subset \mathscr{CN}_{1/2}[X,d]$ which refines $\mathscr{V}_2$ and $\bigcup \mathscr{W}_2 = (\bigcup \mathscr{V}_2)\backslash(\bigcup \mathscr{U}_1)$. Then, by Proposition 3.10, we find a disjoint family $\mathscr{U}_2 \subset \mathscr{CN}_{1/2}[X,d]$ which refines $\mathscr{W}_2$ (and, hence, $\mathscr{V}_2$) and $\bigcup \mathscr{U}_2 = \bigcup \mathscr{W}_2$. In this way, by induction, for every $n > 1$ there exists a disjoint family $\mathscr{U}_n \subset \mathscr{CN}_{1/n}[X,d]$ which refines $\mathscr{V}_n$ and $\bigcup \mathscr{U}_n = (\bigcup \mathscr{V}_n)\backslash(\bigcup \mathscr{V}_{n-1})$. The family $\mathscr{U} = \bigcup\{\mathscr{U}_n : n \ge 1\}$ satisfies all our requirements because $\mathscr{V} = \bigcup\{\mathscr{V}_n : n \ge 1\}$. □

We accomplish the preparation for the proof of Theorem 3.5 by the following consequence of Lemma 3.11.

Proposition 3.12. *Let $(X, d)$ be a metric space for which $\mathscr{CN}[X, d]$ constitutes a base for the topology of X. Then, every open cover $\mathscr{V}$ of X admits a disjoint refinement $\mathscr{U} \subset \mathscr{CN}[X, d]$.*

Proof of Theorem 3.5. The implications (a) $\to$ (b) $\to$ (c) are obvious, while the implication (c) $\to$ (d) follows by Proposition 3.6.

(d) $\to$ (e). Suppose that $X$ is not a singleton. Also, let $x \in X$ and let $\varepsilon > 0$. By (d), there exists a proper $d$-clopen subset $W$ of $(X, d)$ such that $x \in W \subset \mathscr{N}_\varepsilon^d(x)$. We now merely set $\delta = \Delta_d(W) > 0$. Then, by Proposition 3.2, $\mathscr{CN}_\delta^d(x) \subset \mathscr{CN}_\delta^d(W) = W \subset \mathscr{N}_\varepsilon^d(x)$.

(e) $\to$ (a). Again, we suppose that $X$ is not a singleton. Note that, by (e), the family $\mathscr{CN}[X, d]$ constitutes a base for the topology of $X$. Hence, by Proposition 3.2, $X$ contains a non-empty proper $d$-clopen subset $A$. Then, by Proposition 3.12, there exists a disjoint cover $\mathscr{U}_0 \subset \mathscr{CN}[X, d]$ of $X$ which refines $\{A, X \backslash A\}$. Relying once again on Proposition 3.12, for every $n > 0$ we also construct a disjoint cover $\mathscr{U}_n \subset \mathscr{CN}[X, d]$ of $X$ which refines both $\{\mathscr{N}_{1/n}^d(x) : x \in X\}$ and $\mathscr{U}_{n-1}$. Thus, we get a family $\bigcup\{\mathscr{U}_n : n \in \boldsymbol{N}\} \subset \mathscr{CN}[X, d]$ which is a base for the topology of $X$. Hence, for every point $x \in X$ there exists exactly one decreasing sequence $\{U_n(x) \in \mathscr{U}_n : n \in \boldsymbol{N}\}$ such that $\{x\} = \bigcap\{U_n(x) : n \in \boldsymbol{N}\}$. Take a point $x \in X$. For every $n \in \boldsymbol{N}$ we now have that $\Delta_d(U_n(x)) \in (0, +\infty)$ because, by construction, $U_n(x)$ is a non-empty proper $d$-clopen subset of $(X, d)$. Let us also note that

$$\mathscr{CN}_{\Delta_d(U_n(x))}^d(x) = U_n(x) \quad \text{for every } n \in \boldsymbol{N}. \tag{1}$$

Indeed, by Proposition 3.7, $U_n(x) \in \mathscr{CN}[X, d]$ implies the existence of a $\delta > 0$ with $U_n(x) = \mathscr{CN}_\delta^d(x)$. Hence, by Proposition 3.2, we get that $\Delta_d(U_n(x)) \geq \delta$. Finally, by the same proposition, we have that $\mathscr{CN}_{\Delta_d(U_n(x))}^d(x) \subset \mathscr{CN}_{\Delta_d(U_n(x))}^d(U_n(x)) = U_n(x)$.

According to (1), we get that

$$\{\Delta_d(U_n(x)) : n \in \boldsymbol{N}\} \quad \text{is a decreasing sequence,} \tag{2}$$

and, more precisely, that

$$\Delta_d(U_{n+1}(x)) \geq \Delta_d(U_n(x)) \quad \text{implies} \quad U_{n+1}(x) = U_n(x). \tag{3}$$

Merely, if $\Delta_d(U_{n+1}(x)) \geq \Delta_d(U_n(x))$ for some $n \in \boldsymbol{N}$, then

$$U_n(x) = \mathscr{CN}_{\Delta_d(U_n(x))}^d(x) \subset \mathscr{CN}_{\Delta_d(U_{n+1}(x))}^d(x) = U_{n+1}(x).$$

Whenever $x, y \in X$ are different, we now set $n(x, y) = \min\{n \in \boldsymbol{N} : y \notin U_n(x)\}$. Next, we define a function $\rho : X \times X \to \boldsymbol{R}$ by letting for every $x, y \in X$ that $\rho(x, y) = \max\{\Delta_d(U_{n(x,y)}(x)), \Delta_d(U_{n(x,y)}(y))\}$ if $x \neq y$ and $\rho(x, y) = 0$ otherwise. Clearly, $\rho(x, y) = \rho(y, x) \geq 0$ is always valid, and $\rho(x, y) = 0$ if and only if $x = y$. Let us check the following important property of $\rho$. Take points $x, y \in X$ and an $n \in \boldsymbol{N}$. Then,

$$y \notin U_n(x) \quad \text{implies} \quad \rho(x, y) \geq \max\{\Delta_d(U_n(x)), \Delta_d(U_n(y))\}. \tag{4}$$

Indeed, $y \notin U_n(x)$ implies $n(x, y) \leq n$. Therefore, by (2), $\Delta_d(U_{n(x,y)}(x)) \geq \Delta_d(U_n(x))$. According to the definition of $\rho$, this implies that $\rho(x, y) \geq \Delta_d(U_n(x))$. Hence, $\rho(x, y) \geq \Delta_d(U_n(y))$ holds too because $y \notin U_n(x)$ is equivalent to $x \notin U_n(y)$.

We now complete the proof showing that $\rho$ is as required in (a). First, we show that $\rho$ is an ultrametric on $X$. Take two different points $x, y \in X$. Then, for a point $z \in X$, we distinguish the following two cases. If $z \notin U_{n(x,y)}(x) \cup U_{n(x,y)}(y)$, then (4) implies that

$$\rho(x, y) = \max\{\Delta_d(U_{n(x,y)}(x)), \Delta_d(U_{n(x,y)}(y))\} \leq \max\{\rho(x, z), \rho(z, y)\}.$$

If $z \in U_{n(x,y)}(x) \cup U_{n(x,y)}(y)$, then either $z \notin U_{n(x,y)}(x)$ or $z \notin U_{n(x,y)}(y)$. Hence, there exists a point $t \in \{x, y\}$ such that $z \notin U_{n(x,y)}(t)$ and $\{U_{n(x,y)}(z), U_{n(x,y)}(t)\} = \{U_{n(x,y)}(x), U_{n(x,y)}(y)\}$. Therefore, by (4), we get that

$$\begin{aligned}
\max\{\rho(x, z), \rho(z, y)\} \geq \rho(z, t) &\geq \max\{\Delta_d(U_{n(x,y)}(z)), \Delta_d(U_{n(x,y)}(t))\} \\
&= \max\{\Delta_d(U_{n(x,y)}(x)), \Delta_d(U_{n(x,y)}(y))\} \\
&= \rho(x, y).
\end{aligned}$$

Next, we show that $\rho \leq d$. Take two different points $x, y \in X$. Then, merely note that $d(x, y) \geq \max\{\Delta_d(U_{n(x,y)}(x)), \Delta_d(U_{n(x,y)}(y))\} = \rho(x, y)$.

We finally show that $\rho$ is a compatible metric on $X$. Towards this end, let $x \in X$ and let $k \in \boldsymbol{N}$. Since $\bigcup\{\mathscr{U}_n : n \in \boldsymbol{N}\}$ is a base for the topology of $X$, it suffices to show that $\mathscr{N}^{\rho}_{\Delta_d(U_k(x))}(x) = U_k(x)$. By (4), we get that $\mathscr{N}^{\rho}_{\Delta_d(U_k(x))}(x) \subset U_k(x)$. Take a point $y \in U_k(x) \setminus \{x\}$. Then, $n(x, y) > n(x, y) - 1 \geq k$ and $U_{n(x,y)-1}(x) = U_{n(x,y)-1}(y)$. Therefore, by (2) and (3), this implies that

$$\Delta_d(U_k(x)) \geq \Delta_d(U_{n(x,y)-1}(x)) > \max\{\Delta_d(U_{n(x,y)}(x)), \Delta_d(U_{n(x,y)}(y))\} = \rho(x, y).$$

So, $y \in \mathscr{N}^{\rho}_{\Delta_d(U_k(x))}(x)$. □

## 4. Characterizing Certain Zero-Dimensional Metrizable Spaces

The present section contains some possible applications of our previous results. By a *Polish space* we mean a completely metrizable separable space. As Example 2.6 demonstrates, there exists a zero-dimensional Polish space and a complete compatible metric on it such that the corresponding "metric"-proximal hyperspace is connected. In effect, this is the only non-trivial example, as our first result here states.

Theorem 4.1. *For a non-singleton zero-dimensional Polish space $X$, the following two conditions are equivalent*:

(i) *$X$ is homeomorphic to the space of irrational numbers $\boldsymbol{P}$.*
(ii) *$(\mathscr{F}(X), \tau_{\delta(d)})$ is connected for some $d \in \mathscr{D}(X)$.*

Proof. The implication (i) $\to$ (ii) follows from Example 2.6 and Theorem 2.1. Suppose that $d$ is as in (ii). Then, $(X, d)$ has only two $d$-clopen subsets. This implies that $X$ has no points of local compactness. Hence, by a result of [1], $X$ is homeomorphic to $\boldsymbol{P}$. □

To prepare for our next application, we need the following example.

Example 4.2. *There exists a complete metric $d \in \mathscr{D}(\boldsymbol{N})$ on the set of natural numbers $\boldsymbol{N}$ which is not uniformly equivalent to any ultrametric.*

Proof. For every natural $n \geq 1$, let $Y_n = \{0, 1/n, \dots, (n-1)/n, 1\}$ and $X_n = Y_n \times \{n\}$. Also, let $d_n$ be the metric on $X_n$ defined by $d_n((y', n), (y'', n)) = |y' - y''|$. Finally, let $X = \bigcup\{X_n : n \geq 1\}$, and let $d$ be the metric on $X$ defined by $d(x, y) = d_n(x, y)$ if $x, y \in X_n$ for some $n \geq 1$, and $d(x, y) = 1$ otherwise. Obviously, $(X, d)$ is a countable discrete metric space. Therefore, it is homeomorphic to $\boldsymbol{N}$. Also, $d$ is clearly complete. We will show that $d$ is as required. Suppose that $\rho$ is an ultrametric on $X$ which is uniformly equivalent to $d$. Then, in particular, there exists $\delta > 0$ such that $d(x, y) \leq 1/2$ provided $x, y \in X$ and $\rho(x, y) < \delta$. On the other hand, there also exists a natural $m > 0$ such that $\rho(x, y) < \delta$ provided $x, y \in X$ and $d(x, y) \leq 1/m$. Since $d((i/m, m), ((i+1)/m, m)) = |i/m - (i+1)/m| = 1/m$ for every $i \in \{0, 1, \dots, m-1\}$, we have that $\rho((i/m, m), ((i+1)/m, m)) < \delta$. Hence,

$$\rho((0, m), (1, m)) \leq \max\{\rho((0, m), (1/m, m)), \dots, \rho((m-1)/m, m), (1, m))\} < \delta$$

and therefore $d((0,m),(1,m)) \leq 1/2$. However, by definition, $d((0,m),(1,m)) = |0-1| = 1$. A contradiction. □

Theorem 4.3. *For a zero-dimensional metrizable space $X$, the following conditions are equivalent*:

(i) *$X$ is compact.*
(ii) *Any $d \in \mathscr{D}(X)$ is uniformly equivalent to an ultrametric.*
(iii) *$(\mathscr{F}(X), \tau_{\delta(d)})$ is zero-dimensional for any $d \in \mathscr{D}(X)$.*

Proof. If $X$ is compact, then every two compatible metrics on $X$ are uniformly equivalent which is the implication (i) $\to$ (ii). The implication (ii) $\to$ (iii) follows by Theorem 3.3. Suppose finally that (iii) holds but $X$ is not compact. Then, $X$ must contain a closed copy of the natural numbers $\boldsymbol{N}$. Let $p \in \mathscr{D}(\boldsymbol{N})$ be as in Example 4.2. By the Hausdorff extension theorem (see, for example, [7, Theorem 3.2, ch. II]), $p$ extends to a compatible metric $d$ on $X$. Then, by Example 4.2, the metric $d$ is not uniformly equivalent to any ultrametric on $X$, so, by Theorem 3.3, $(\mathscr{F}(X), \tau_{\delta(d)})$ is not zero-dimensional. A contradiction. □

To prepare for our next result, we need an example of a special metric on another "standard" space. Namely, we consider the hedgehog $J(\omega)$ of weight $\omega$. We recall here its definition: as a set, $J(\omega) = Y/\sim$, where $Y = \omega \times [0,1]$ and "$\sim$" is the equivalence relation on $Y$ defined by $(\alpha, x) \sim (\beta, y)$ iff ($x = y = 0$ or $(\alpha, x) = (\beta, y)$). The topology of $J(\omega)$ is that induced by the metric $d$, defined as

$$d(\langle \alpha, x\rangle, \langle \beta, y\rangle) = \begin{cases} |x-y| & \text{if } \alpha = \beta, \\ x+y & \text{if } \alpha \neq \beta, \end{cases}$$

where $\langle \alpha, x\rangle$ and $\langle \beta, y\rangle$ are the equivalence classes associated to $(\alpha, x)$, and respectively, $(\beta, y)$. Also, we consider the subset $J_0(\omega)$ of $J(\omega)$ defined by $\langle \alpha, x\rangle \in J_0(\omega)$ if and only if either $x = 0$ or $x = 1/n$ for some natural $n > 0$, and we put

$$d' = d \mid J_0(\omega) \times J_0(\omega).$$

Example 4.4. *There exists a metric $\rho \in \mathscr{D}(J_0(\omega))$ such that $(J_0(\omega), \rho)$ has no base of $\rho$-clopen sets.*

Proof. For every $\alpha \in \omega$, pick a fixed sequence $\{x_n^\alpha : n \in \boldsymbol{N}\} \subset [0,1]$ such that

$$x_1^\alpha = 1, \quad |x_n^\alpha - x_{n+1}^\alpha| < 1/\alpha, n \geq 1, \quad \text{and} \quad \lim_{n\to\infty} x_n^\alpha = 0 = x_0^\alpha.$$

Next, define $X = \{\langle\alpha, x_n^\alpha\rangle : \alpha \in \omega \text{ and } n \in \boldsymbol{N}\}$, and then set $d'' = d \mid X \times X$. Topologically, $X$ is a copy of $J_0(\omega)$: actually, a homeomorphism may be constructed using, for example, the fact that for every $n \in \omega$, the two sets $\mathcal{N}_{1/n}^{d'}(\langle 0,0\rangle)\backslash\mathcal{N}_{1/(n+1)}^{d'}(\langle 0,0\rangle)$ and $\mathcal{N}_{1/n}^{d''}(\langle 0,0\rangle)\backslash\mathcal{N}_{1/(n+1)}^{d''}(\langle 0,0\rangle)$ are infinite.

Let us show that $d''$ has the properties of the metric $\rho$ in the statement. Suppose $V$ is any neighbourhood of $\langle 0,0\rangle$ such that $V \subset \mathcal{N}_1^{d''}(\langle 0,0\rangle)$. Note that $\langle\alpha, x_1^\alpha\rangle \notin V$ for every $\alpha \in \omega$. Then, whenever $\alpha \in \omega$, set $\alpha(V) = \min\{n \geq 2 : \langle\alpha, x_n^\alpha\rangle \in V\}$. Hence,

$$D_{d''}(V, X\backslash V) \leq |x_{\alpha(V)}^\alpha - x_{\alpha(V)-1}^\alpha| < 1/\alpha.$$

So, $D_{d''}(V, X\backslash V) \leq \inf_{\alpha\in\omega} 1/\alpha = 0$ which completes the proof. □

Theorem 4.5. *For a zero-dimensional metrizable space X, the following conditions are equivalent*:

(i) *For every $d \in \mathscr{D}(X)$ there exists an ultrametric $\rho \in \mathscr{D}(X)$ with $\rho \preceq d$.*
(ii) *$(X, d)$ has a base of d-clopen sets for any $d \in \mathscr{D}(X)$.*
(iii) *$X$ is locally compact.*

Proof. The implication (i) → (ii) follows by Theorem 3.5. If $X$ is not locally compact, then it contains a closed copy of $J_0(\omega)$. Let $p$ be a metric on $J_0(\omega)$ as that in Example 4.4. By the Hausdorff extension theorem, $p$ extends to a compatible metric $d$ on $X$. Then, by virtue of Example 4.4, the space $X$ doesn't admit a base of $d$-clopen sets. That is, (ii) → (iii) holds. Since every two metrics on a compact space are uniformly equivalent, Theorem 3.5 completes the proof. □

## 5. On the Selection Problem for the Proximal Hyperspaces

This last section of the paper is devoted to some further results concerning the selection problem for the proximal hyperspaces. The first one states the following generalization of [11, Theorem 1.2].

Theorem 5.1. *Let $X$ be a completely metrizable space, and let $d \in \mathscr{D}(X)$ be such that $(X, d)$ has a base of d-clopen sets. Then $\mathscr{F}(X)$ has a $\tau_{\delta(d)}$-continuous selection.*

To prepare for the proof of Theorem 5.1, we need a result about special approximate selections on subsets of proximal hyperspaces. For a topological space $X$ and a subset $A \subset X$, we let $\mathscr{F}_X(A) = \{S \in \mathscr{F}(X) : S \cap A \neq \varnothing\}$. Suppose now that $X$ is metrizable, $d, \rho \in \mathscr{D}(X)$, and $\varepsilon > 0$. We shall say that a map $f : \mathscr{F}_X(A) \to X$ is a $\tau_{\delta(d)}$-*continuous* $(\varepsilon, \rho)$-*selection* for $\mathscr{F}_X(A)$ provided

(1) $f$ is continuous with respect to the topology on $\mathscr{F}_X(A)$ induced by $\tau_{\delta(d)}$,
(2) $\rho(f(F), F) < \varepsilon$ for every $F \in \mathscr{F}_X(A)$.

Lemma 5.2. *Let $(X, d)$ be a metric space which has a base of d-clopen sets, and let $A$ be an open subset of $X$. Then, for every $\rho \in \mathscr{D}(X)$ and $\varepsilon > 0$ there exists a $\tau_{\delta(d)}$-continuous $(\varepsilon, \rho)$-selection $f : \mathscr{F}_X(A) \to A$ for $\mathscr{F}_X(A)$.*

Proof. Let $\rho \in \mathscr{D}(X)$, and let $\varepsilon > 0$. Since $A$ is open, there is a cover $\mathscr{V}$ of $A$ which consists of $d$-clopen subsets of $(X, d)$ and $\mathrm{diam}_\rho(V) < \varepsilon$, $V \in \mathscr{V}$. To every $V \in \mathscr{V}$ we associate a number $n(V) \in \boldsymbol{N}$ by letting $n(V) = \min\{k \in \boldsymbol{N} : \Delta_d(V) \geq 1/k\}$. Take now a well-ordering $\ll$ on the set $\mathscr{V}$. Next, define another well-ordering $\prec$ on $\mathscr{V}$ by $W \prec V$ provided either $n(W) < n(V)$ or $n(W) = n(V)$ and $W \ll V$. Finally, for every $V \in \mathscr{V}$ we set

$$\mathscr{T}_V = \{F \in \mathscr{F}(X) : F \cap V \neq \varnothing \text{ and } F \cap W = \varnothing \text{ for every } W \prec V\}.$$

Obviously, this defines a disjoint cover $\{\mathscr{T}_V : V \in \mathscr{V}\}$ of $\mathscr{F}_X(A)$. Let us show that each $\mathscr{T}_V$ is $\tau_{\delta(d)}$-open. Suppose that $C \in \mathscr{T}_V$. Then, $\mathscr{C}_1 = \{F \in \mathscr{F}(X) : F \cap V \neq \varnothing\}$ is a $\tau_{\delta(d)}$-neighbourhood of $C$. On the other hand, $\Delta_d(\{W \in \mathscr{V} : W \prec V\}) \geq 1/(n(V))$ because $W \prec V$ implies $n(W) \leq n(V)$. Therefore, by virtue of Proposition 3.9, $\mathscr{C}_2 = \{F \in \mathscr{F}(X) : D_d(\bigcup\{W : W \prec V\}, F) > 0\}$ is also a $\tau_{\delta(d)}$-neighbourhood of $C$. This completes the verification because $C \in \mathscr{C}_1 \cap \mathscr{C}_2 \subset \mathscr{T}_V$. Define now a $\tau_{d(d)}$-continuous map $f : \mathscr{F}_X(A) \to A$ by setting $f|\mathscr{T}_V : \mathscr{T}_V \to V$ to be a constant map whenever $\mathscr{T}_V$ is nonempty. This $f$ is as required. Indeed, for every $F \in \mathscr{F}_X(A)$ there exists exactly one $V(F) \in \mathscr{V}$ with $F \in \mathscr{T}_{V(F)}$. Then, $f(F) \in V(F)$ implies that $\rho(f(F), F) \leq \mathrm{diam}_\rho(V(F)) < \varepsilon$. $\square$

Proof of Theorem 5.1. Let $(X, d)$ be as in the statement. Note that, by Theorem 3.5, $X$ is strongly zero-dimensional. Then, take a complete ultrametric $\rho \in \mathscr{D}(X)$. It will be now sufficient to construct a sequence $\{f_n\}$ of $\tau_{\delta(d)}$-continuous $(2^{-n}, \rho)$-selections $f_n$ for $\mathscr{F}(X)$ such that $\rho(f_n(F), f_{n+1}(F)) < 2^{-n}$ for every $F \in \mathscr{F}(X)$ and $n \in \boldsymbol{N}$. This is what we shall do. Since the existence of $f_0$ follows from Lemma 5.2, we may suppose that $f_n$ has already been constructed and we have to define $f_{n+1}$. Since $\rho$ is an ultrametric, $\mathscr{U} = \{\mathscr{N}^\rho_{2^{-n}}(x) : x \in X\}$ defines a disjoint open cover of $X$. Then, $f_n^{-1}(\mathscr{U})$ defines a disjoint $\tau_{\delta(d)}$-open cover of $\mathscr{F}(X)$. On the other hand, $f_n^{-1}(U) \subset \mathscr{F}_X(U)$ for every $U \in \mathscr{U}$, and to every $F \in \mathscr{F}(X)$ it corresponds exactly one $U(F) \in \mathscr{U}$ such that $\mathscr{N}^{\rho}_{2^{-n}}(f_n(F)) = U(F)$ and $F \cap U(F) \neq \emptyset$ (because $\rho$ is an ultrametric). Now, by Lemma 5.2, for every $U \in \mathscr{U}$ there exists a $\tau_{\delta(d)}$-continuous $(2^{-(n+1)}, \rho)$-selection $f_U : \mathscr{F}_X(U) \to U$ for $\mathscr{F}_X(U)$. Then, we may define a $\tau_{\delta(d)}$-continuous $(2^{-(n+1)}, \rho)$-selection $f_{n+1}$ for $\mathscr{F}(X)$ by letting $f_{n+1}|f_n^{-1}(U) = f_U|f_n^{-1}(U)$ for every $U \in \mathscr{U}_n$. This $f_{n+1}$ is as required. Indeed, $F \in \mathscr{F}(X)$ implies $f_{n+1}(F) = f_{U(F)}(F) \in U(F) = \mathscr{N}^{\rho}_{2^{-n}}(f_n(F))$. □

By Theorems 5.1 and 4.5, we get the following consequence.

COROLLARY 5.3. *Let $X$ be a zero-dimensional locally compact metrizable space. Then, for every compatible metric $d$ on $X$ there exists a $\tau_{\delta(d)}$-continuous selection for $\mathscr{F}(X)$.*

Concerning the right place of Theorem 5.1, a word should be said. As the proof of this theorem shows, our approach is based on the metric generation of proximal hyperspaces, for a natural generalization of Theorem 5.1 in terms of "hit-and-miss" topologies on $\mathscr{F}(X)$ we refer the interested reader to [13]. Let $\mathscr{S}_{\tau_\delta}(X)$ be the set of those metrics $d \in \mathscr{D}(X)$ for which $\mathscr{F}(X)$ has a $\tau_{\delta(d)}$-continuous selection. Then, by Theorem 3.5, we get the following equivalent reading of Theorem 5.1 in terms of special relations with the compatible ultrametrics on a metrizable space.

COROLLARY 5.4. *Let $X$ be a completely metrizable space. Then, for every ultrametric $\rho \in \mathscr{D}(X)$ we have that $\{d \in \mathscr{D}(X) : \rho \preceq d\} \subset \mathscr{S}_{\tau_\delta}(X)$.*

Relying once again on Theorem 3.5 and the fact that, for $\rho, d \in \mathscr{D}(X)$, the relation $\rho \preceq d$ implies $\tau_{\delta(\rho)} \subset \tau_{\delta(d)}$, we might read Corollary 5.4 (hence, Theorem 5.1 as well) as the fact that $\mathscr{S}_{\tau_\delta}(X)$ contains all compatible ultrametrics on a completely metrizable space $X$. Concerning the selection problem for the proximal hyperspaces on strongly zero-dimensional metrizable spaces, this presents a bit more information, but related especially to the role of the metric property of *completeness*. From this point of view, our next result presents an improvement in the direction of an *ultrametric* condition.

THEOREM 5.5. *Let $X$ be a completely metrizable space, and let $d \in \mathscr{D}(X)$ be such that, for some point $z \in X$, the subspace $X \backslash \{z\}$ has a base of $d$-clopen sets. Then $\mathscr{F}(X)$ has a $\tau_{\delta(d)}$-continuous selection.*

To prepare for the proof of Theorem 5.5, we need the following proposition.

Proposition 5.6. *Let $(X,d)$ be a metric space, and let $Z$ be a non-empty $d$-clopen subset of $(X,d)$. Define a map $\varphi : \mathscr{F}_X(Z) \to \mathscr{F}(Z)$ by letting $\varphi(S) = S \cap Z$ for every $S \in \mathscr{F}_X(Z)$. Then, $\varphi$ is continuous with respect to the topologies induced by $\tau_{\delta(d)}$ on $\mathscr{F}_X(Z)$ and $\tau_{\delta(d \mid Z\times Z)}$ on $\mathscr{F}(Z)$, respectively.*

Proof. Let $p = d \mid Z \times Z$. Note that, by [5, Lemma 4.1], the space $(\mathscr{F}(Z), \tau_{\delta(p)})$ coincides with $\mathscr{F}(Z)$ equipped with the relative topology of $(\mathscr{F}(X), \tau_{\delta(d)})$. Then, take an $S \in \mathscr{F}_X(Z)$, and let $\langle\!\langle \mathscr{U} \rangle\!\rangle_p$ be a basic $\tau_{\delta(p)}$-neighbourhood of $\varphi(S)$ in $\mathscr{F}(Z)$. Since $Z$ is a $d$-clopen subset of $(X,d)$, we now have that $\langle\!\langle \mathscr{U} \rangle\!\rangle_d = \langle\!\langle \mathscr{U} \rangle\!\rangle_p$. Then, let $\mathscr{V} = \{X \backslash Z\} \cup \mathscr{U}$ if $S \neq \varphi(S)$ and $\mathscr{V} = \mathscr{U}$ otherwise. In this way, we get a $\tau_{\delta(d)}$-neighbourhood $\langle\!\langle \mathscr{V} \rangle\!\rangle_d$ of $S$ with $\varphi(\langle\!\langle \mathscr{V} \rangle\!\rangle_d) \subset \langle\!\langle \mathscr{U} \rangle\!\rangle_p$. □

Proof of Theorem 5.5. Let $X$ and $d \in \mathscr{D}(X)$ be as in the statement of this theorem. By condition, there exists a point $z \in X$ such that every $x \in Z = X \backslash \{z\}$ has a local base of $d$-clopen subset of $(X,d)$. Suppose that $Z \neq \varnothing$. Next, for every $x \in Z$ pick a fixed $d$-clopen subset $Z_x$ of $(X,d)$ such that $x \in Z_x$ and $z \notin Z_x$, and then set $\ell = \min\{1, \sup\{\Delta_d(Z_x) : x \in Z\}\}$. For every $n \in \boldsymbol{N}$ we now define a non-empty set $Z_n = \bigcup\{Z_x : x \in Z \text{ and } \Delta_d(Z_x) \geq \ell/2^n\}$ which, by Proposition 3.9, is a $d$-clopen subset of $(X,d)$ with $\Delta_d(Z_n) \geq \ell/2^n$. It is clear that $Z = \bigcup\{Z_n : n \in \boldsymbol{N}\}$. Then, define a map $g : \mathscr{F}_X(Z) \to \boldsymbol{N}$ by $g(S) = \min\{n \in \boldsymbol{N} : S \cap Z_n \neq \varnothing\}$, $S \in \mathscr{F}_X(Z)$. For every $n \in \boldsymbol{N}$, we also define a map $\varphi_n : \mathscr{F}_X(Z_n) \to \mathscr{F}(Z_n)$ by letting $\varphi_n(S) = S \cap Z_n$ for $S \in \mathscr{F}_X(Z_n)$. Finally, for every $n \in \boldsymbol{N}$ we set $d_n = d \mid Z_n \times Z_n$. By virtue of Proposition 5.6, each $\varphi_n$ is continuous with respect to the topologies induced by $\tau_{\delta(d)}$ on $\mathscr{F}_X(Z_n)$ and $\tau_{\delta(d_n)}$ on $\mathscr{F}(Z_n)$, respectively. By Theorem 5.1, for every $n \in \boldsymbol{N}$ there exists a $\tau_{\delta(d_n)}$-continuous selection $f_n$ for $\mathscr{F}(Z_n)$ because each $Z_n$ has a base of $d_n$-clopen subsets. We now define a map $f : \mathscr{F}(X) \to X$ by $f(S) = f_{g(S)}(\varphi_{g(S)}(S))$ if $S \in \mathscr{F}_X(Z)$ and $f(S) = z$ otherwise. In this way, we get a selection $f$ for $\mathscr{F}(X)$ which is $\tau_{\delta(d)}$-continuous at $\{z\}$ (let $W$ be any neighbourhood of $z$: then $S \in \langle\!\langle W \rangle\!\rangle_d$ implies $f(S) \in S \subset W$). So, to finish the proof, it only remains to show that $f$ is $\tau_{\delta(d)}$-continuous at the points of $\mathscr{F}_X(Z)$. Take any $S \in \mathscr{F}_X(Z)$ and, for reasons of convenience, set $Z_{-1} = \varnothing$. Since $Z_{g(S)}$ and $Z_{g(S)-1}$ are $d$-clopen subsets of $(X,d)$, the set

$$\mathscr{T}_S = \{F \in \mathscr{F}(X) : F \cap Z_{g(S)} \neq \varnothing \text{ and } D_d(F, Z_{g(S)-1}) > 0\}$$

defines a $\tau_{\delta(d)}$-neighbourhood of $S$ in $\mathscr{F}_X(Z)$. On the other hand, $F \in \mathscr{T}_S$ implies

$g(F) = g(S)$. Hence, $\mathcal{T}_S \subset \mathcal{F}_X(Z_{g(S)})$ and, therefore, $f|\mathcal{T}_S$ is $\tau_{\delta(d)}$-continuous because $f|\mathcal{T}_S = f_{g(S)} \circ \varphi_{g(S)} | \mathcal{T}_S$. □

By Theorem 5.5 (see, also, Theorem 4.5 and Example 4.4), we get the following interesting consequence. Here, a space $X$ is *locally compact modulo one point* if $X \backslash \{x\}$ is locally compact for some $x \in X$; observe that every metrizable space, which is locally compact modulo one point, is completely metrizable.

Corollary 5.7. *Let $X$ be a zero-dimensional space which is locally compact modulo one point. Then, for every compatible metric $d$ on $X$ there exists a $\tau_{\delta(d)}$-continuous selection for $\mathcal{F}(X)$, i.e. $\mathcal{S}_{\tau_\delta}(X) = \mathcal{D}(X)$. In particular, $\mathcal{S}_{\tau_\delta}(J_0(\omega)) = \mathcal{D}(J_0(\omega))$.*

We conclude the paper by suggesting some possible lines of development for the subjects we have dealt with, and pointing out related open questions. The hypothesis on the metric $d$ in Theorem 5.5 defines the following natural class of "metric"-disconnected spaces. Namely, one can say that a metric space $(X, d)$ is *totally disconnected* with respect to $d$, or *totally d-disconnected*, if every singleton of $X$ is an intersection of $d$-clopen subsets of $(X, d)$. Here is an example of the most natural (strongly) 0-dimensional metrizable space for which this property fails.

Example 5.8. *There exists a compatible metric $\sigma$ on the disjoint sum $J_0(\omega) \oplus J_0(\omega)$ such that $(J_0(\omega) \oplus J_0(\omega), \sigma)$ is not totally $\sigma$-disconnected.*

Proof. Let $Z$ be the set $\omega \times [0,1] \times \{1,2\}$, and introduce on $Z$ the equivalence relation "$\approx$" defined by:

$$(\alpha, x, i) \approx (\beta, y, j) \Leftrightarrow ((x = y = 0 \text{ and } i = j) \quad \text{or} \quad (\alpha = \beta \text{ and } x = y = 1) \quad \text{or} \quad (\alpha, x, i) = (\beta, y, j)).$$

Consider the metric $p$ on $Z/\approx$, defined by

$$p(\langle \alpha, x, i \rangle, \langle \beta, y, j \rangle) = (1 - |i - j|) \cdot d(\langle \alpha, x \rangle, \langle \beta, y \rangle) + |i - j| \cdot \min\{d(\langle \alpha, x \rangle, \langle \alpha, 1 \rangle) + d(\langle \alpha, 1 \rangle, \langle \beta, y \rangle), d(\langle \alpha, x \rangle, \langle \beta, 1 \rangle) + d(\langle \beta, 1 \rangle, \langle \beta, y \rangle)\},$$

where $d$ is the standard metric on $J(\omega)$.

For $i = 1, 2$, put $Z_i = \{\langle \alpha, x, i \rangle : \langle \alpha, x \rangle \in X\}$, where $X$ is the subset of $J(\omega)$ defined in Example 4.4. Since, for $i = 1, 2$, the function $\varphi : X \to Z_i$ defined by $\varphi(\langle \alpha, x \rangle) = \langle \alpha, x, i \rangle$ is an isometry with respect to $d'' = d \mid X \times X$ and $p_i = p \mid Z_i \times Z_i$, the space $Z_i$ is a copy of $J_0(\omega)$, and it is easy to check that $Z_1 \cup Z_2$ is homeomorphic to $J_0(\omega) \oplus J_0(\omega)$ (we may consider the points of kind $\langle \alpha, 1, i \rangle$ as belonging, as well, to the first or second $J_0(\omega)$). On the other side, by virtue of Example 4.4, $Z_1 \cup Z_2$ is not totally disconnected with respect to the metric induced by $p$. □

The characterization below makes more transparent the interest to the class of totally $d$-disconnected metric spaces.

THEOREM 5.9. *For a zero-dimensional metrizable space $X$, the following two conditions are equivalent*:

(i) *$(X, d)$ is totally $d$-disconnected for any $d \in \mathscr{D}(X)$.*
(ii) *$X$ is locally compact modulo one point.*

PROOF. If $X$ is not locally compact modulo one point, then it has a nonempty proper clopen subset $Z$ such that both $Z$ and $X \backslash Z$ are not locally compact. Then, each of the spaces $Z$ and $X \backslash Z$ contains a closed copy of $J_0(\omega)$. Therefore, in this case, $X$ contains a closed copy of $J_0(\omega) \oplus J_0(\omega)$. Let $\sigma$ be a metric on $J_0(\omega) \oplus J_0(\omega)$ as that in Example 5.8. By the Hausdorff extension theorem, $\sigma$ extends to a compatible metric $d$ on $X$. Then, by Example 5.8, the metric space $(X, d)$ fails to be totally $d$-disconnected. This shows (i) → (ii). Since the inverse implication is obvious, the proof completes. □

According to Theorem 5.9 and Corollary 5.7, the following question is of certain interest.

QUESTION 1. Let $X$ be a (strongly zero-dimensional) completely metrizable space, and let $d \in \mathscr{D}(X)$ be such that $(X, d)$ is totally $d$-disconnected. Does there exist a $\tau_{\delta(d)}$-continuous selection for $\mathscr{F}(X)$?

In view Example 5.8, the following more particular question is also open.

QUESTION 2. Let $X$ be a metrizable scattered space, and let $d \in \mathscr{D}(X)$ be such that $(X, d)$ is totally $d$-disconnected. Does there exist a $\tau_{\delta(d)}$-continuous selection for $\mathscr{F}(X)$?

Example 5.8 also suggests a problem in another direction, i.e., weighing how important is this metric property of disconnectedness.

Question 3. Let $X$ be a metrizable scattered space. Does $\mathscr{S}_{\tau_\delta}(X)$ coincide with $\mathscr{D}(X)$?

The above question is open even in the special case of $X = J_0(\omega) \oplus J_0(\omega)$. Finally, a last question which naturally arises from Corollary 2.7.

Question 4. Let $X$ be a metrizable space which is scattered with respect to compact subsets, i.e. every non-empty closed subset of $X$ contains a non-empty compact and relatively open subset. Does $\mathscr{S}_{\tau_\delta}(X)$ coincide with $\mathscr{D}(X)$?

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