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MODULES OF INFINITE PROJECTIVE DIMENSION OVER ALGEBRAS WHOSE IDEMPOTENT IDEALS ARE PROJECTIVE

By

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Abstract. Let A be a finite dimension algebra over an algebraically closed field such that all its idempotent ideals are projective. We show that if A is representation-infinite and not hereditary, then there exist infinitely many nonisomorphic indecomposable A-modules of infinite projective dimension.

In [2], M. Auslander, M. I. Platzeck and G. Todorov have studied homological properties of the idempotent ideals of an artin algebra A. They gave there a characterization of the idempotent two sided ideals which are projective left A-modules. Their main motivation for this study came from the work of Cline Parshall-Scott [6], Dlab-Ringel [7, 8] and Burgess-Fuller [5]. Also, in [9], Platzeck has studied artin rings with the property that all their idempotent ideals are projective. In particular, she has shown that the finitistic projective dimension of such a ring is at most one.

Let A be a finite dimensional k-algebra, where k is an algebraically closed field, and assume in addition that each idempotent ideal of A is a projective A-module. By Platzeck's result, the projective dimension of any indecomposable nonprojective A-module is either one or infinite. Therefore, if A is not hereditary, then there always exist nonprojective indecomposable A-modules of infinite projective dimension. In case A is representation-infinite (that is, such that there exist infinitely many nonisomorphic indecomposable A-modules) and not hereditary, one can ask if the number of nonisomorphic indecomposable A-modules of infinite projective dimension can be finite. The main aim of this paper is to show the following result.

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THEOREM. *Let A be a finite dimension indecomposable algebra over an algebraically closed field. Suppose that all idempotent (bilateral) ideals are projective A-modules. If A is representation-infinite and not hereditary, then there exist infinitely many nonisomorphic indecomposable A-modules of infinite projective dimension.*

The proof of this theorem will be given in section 3. Observe that if A is such that all idempotent ideals are projective and there exist only finitely many nonisomorphic indecomposable A-modules of infinite projective dimension, then the projective dimension of all but finitely many nonisomorphic indecomposable A-modules is at most one, because the finitistic projective dimension is at most one. We will use the fact that the latter condition is equivalent to saying that A is a right glueing of tilted algebras in the sense studied by I. Assem and F. U. Coelho in [1]. In section 1 we shall recall some basic facts in representation theory of algebras, and also recall the notion and basic properties on right glued algebras. Section 2 will be devoted to some preliminary results on algebras whose idempotent ideals are projective modules.

1. Preliminaries

1.1. Unless otherwise stated, all algebras in this paper are basic, connected finite dimensional algebras over a fixed algebraically closed field k. Therefore, any algebra A can be viewed as a quotient $kQ(A)/I$ of a path algebra $kQ(A)$, where $Q(A)$ is a finite quiver and I is an admissible ideal of $kQ(A)$. Recall that an ideal I of $kQ(A)$ is said to be *admissible* if there exists an n such that $J^2 \supset I \supset J^n$, where J is the ideal of $kQ(A)$ generated by the arrows from $Q(A)$. The elements of an admissible ideal are called *admissible relations*. The uniquely determined quiver $Q(A)$ will be referred to as the *ordinary quiver* of A. For a given quiver Q, we shall denote by Q_0 and by Q_1, the set of vertices and arrows of Q, respectively. If α is an arrow in Q_1 then $s(\alpha)$ and $e(\alpha)$ denote, respectively, the *start* and the *end* vertices of α. A *loop* is an arrow α such that $s(\alpha) = e(\alpha)$. Following [4], we shall sometimes equivalently consider an algebra as a k-linear category. An ideal is always a two sided ideal.

Let I be an admissible ideal in a path algebra kQ and let $a, b \in Q_0$. We denote by $I(a,b)$ the set of the elements $\sum \lambda_i, \gamma_i \in I$, where, for each $i, \lambda_i \in k$, and the path γ_i starts at a and ends at b.

1.2. For a given algebra A, let A-mod denote the category of finitely generated left A-modules. All modules and maps are in A-mod. Denote by A-ind the category with one representative of each isoclasse of indecomposable A-modules.

Given $M \in A$-mod, denote by add M the full subcategory of A-mod consisting of all finite direct sums of summands of M.

Let $A = kQ(A)/I$ be an algebra and let $a \in (Q(A))_0$. Denote by $S(a)$ the simple A-module associated to a and by $P(a)$ the projective cover of $S(a)$. It is well-known that there exists an equivalence between the category A-mod and the category of the $(Q(A), I)$-representations. Recall that a $(Q(A), I)$-representation X is given by $X = ((X_i)_{i \in (Q(A))_0}, (f_\alpha)_{\alpha \in (Q(A))_1})$, where for each $i \in (Q(A))_0$, X_i is a finite-dimensional k-vector space, for each $\alpha \in (Q(A))_1$, f_α is a linear transformation from $X_{s(\alpha)}$ to $X_{e(\alpha)}$, and such that these linear transformations are subjected to the relations of I. We shall now agree to identify a $kQ(A)/I$-module with the corresponding $(Q(A), I)$-representation.

We denote by pd$X$ the projective dimension of the module X. Also, the global and the finitistic projective dimensions of A are defined, respectively, by

$$\text{gl.dim}\, A = \max\{\text{pd}X : X \in A\text{-ind}\} \text{ and}$$

$$\text{fpd}\, A = \max\{\text{pd}\, X : X \in A\text{-ind and pd}\, X < \infty\}$$

1.3. We shall now recall the notion of right glued algebras introduced in [1] that will be needed in the proof of our main theorem. Let $B_1, \ldots, B_t$ be representation-infinite tilted algebras having complete slices $\Sigma_1, \ldots, \Sigma_t$ respectively, in the preinjective components and no projectives in these components, $B = B_1 \times \cdots \times B_t$ and C be a representation-finite algebra. An algebra A is called a *right glueing of* $B_1, \ldots, B_t$ *by* C *along the slices* $\Sigma_1, \ldots, \Sigma_t$ or, more briefly, to be a *right glued algebra* if $A = C$ or:

(RG1) each of $B_1, \ldots, B_t$ and C is a full convex subcategory of A, and any object in A belongs to one of these subcategories;
(RG2) no injective A-module is a proper predecessor of the union $\Sigma_1 \cup \cdots \cup \Sigma_t$, considered as embedded in A-ind; and
(RG3) B-ind is cofinite in A-ind.

The algebra C being an arbitrary representation-finite algebra, the component of the Auslander-Reiten quiver Γ_A of A containing $\Sigma_1 \cup \cdots \cup \Sigma_t$ may contain periodic modules and oriented cycles: it is actually an $\imath$-component containing all the injective A-modules (see [1] for details). On the other hand, the projective A-modules are either projective B-modules or belong to the $\imath$-component containing the Σ_i's. Consequently, the ordinary quiver of A is the union of the ordinary quivers of $B_1, \ldots, B_t$ and C together with some additional arrows of the form $x \longrightarrow y$, with x in the quiver of C, and y in the quiver of some B_i. In particular, a right glued algebra A may be written as a lower triangular matrix algebra

$$A \cong \begin{pmatrix} C & 0 \\ N & B \end{pmatrix}$$

where N is a B-C-bimodule.

The next result, proven in [1], will be very useful. We say that a property holds for *almost all* A-modules if it holds for all but finitely many of the indecomposable A-modules.

THEOREM. *Let A be a finite dimensional k-algebra, where k is an algebraically closed field. Then A is a right glued algebra if and only if $pdX \leq 1$ for almost all indecomposable A-modules X.*

We refer the reader to [1] for details on right (and its dual left) glued algebras. For unexplained notations and notions in representation theory, we refer the reader to [3, 10].

2. Algebras whose idempotent ideals are projective

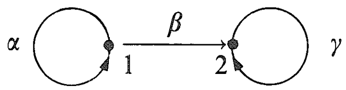
2.1. Let A be an algebra. If M, N are A-modules, denote by $\tau_M(N)$ the *trace of M in N*, that is, the submodule of N generated by all homomorphic images of M in N. If P is a projective A-module, then $\tau_P(A)$ is an idempotent ideal of A, and any such ideal is obtained in this way. Observe also that if P and P' are projective A-modules, then $\tau_P(A) = \tau_{P'}(A)$ if and only if $\operatorname{add} P = \operatorname{add} P'$ (see [2]).

We are particularly interested in the situation when the algebra A satisfies the following property:

(IIP) All idempotent (bilateral) ideals of A are projective A-modules.

The class of algebras satisfying (IIP) clearly includes the hereditary and the local algebras, but it also contains other algebras as shown by the following examples.

EXAMPLES. Let Q be the quiver



(a) Let A_1 be the k-algebra given by Q with relations $\alpha^3 = 0$ and $\gamma^2 = 0$. Clearly, $\tau_{P(1)}(A_1) = P(1)$ and $\tau_{P(2)}(A_1) = P(2) \oplus P(2) \oplus P(2) \oplus P(2)$, and hence A_1 satisfies (IIP) (see [9](1.2)).

(b) Let A_2 be the k-algebra given by Q with relations $\alpha^2 = 0$, $\gamma^2 = 0$ and $\beta\alpha = 0$. In this case, $\tau_{P(1)}(A_2) = P(1)$ and $\tau_{P(2)}(A_2) = P(2) \oplus P(2)$, and hence A_2 also satisfies (IIP) (see [9](1.2)).

2.2. The next result has been proven in [9](2.5).

THEOREM. *If A is an algebra satisfying (IIP), then fpd $A \leq 1$.*

COROLLARY. *Let A be an algebra satisfying (IIP). Then there exists an indecomposable A-module M of infinite projective dimension if and only if A is not hereditary.*

Our main result states that, if A is a representation-infinite algebra with (IIP) which is not hereditary, then, in fact, there are infinitely many non-isomorphic indecomposable A-modules with infinite projective dimension. Therefore, from now on, we shall concentrate our attention on the study of algebras satisfying (IIP) which are not hereditary. For such an algebra $A = kQ(A)/I$, with $I \neq 0$, we shall see that $Q(A)$ has always a loop and I is generated by relations which contain always summands starting at loops. We shall also discuss the notion of *suitable* arrows for A. The rest of this section will be devoted to these questions.

2.3. We recall the following result from [9](2.1), which holds for artin algebras.

PROPOSITION. *Let A be an (artin) algebra with (IIP), and let P and P' be indecomposable projective A-modules such that $Hom_A(P, P') \neq 0$. Then $\tau_P(P') \cong P^r$, for some $r > 0$. Consequently, if P is not isomorphic to P', then $Hom_A(P', P) = 0$.*

This proposition has the following nice consequence. Let A be an algebra with (IIP). Then the indecomposable projective A-modules $P_1, \ldots, P_n$ can be indexed in such a way that $\mathrm{Hom}_A(P_i, P_j) = 0$ whenever $i < j$. In particular, the ordinary quiver $Q(A)$ has no oriented cycles involving arrows which are not loops. Example (2.1) shows that loops can occur in $Q(A)$, and, in fact, we shall show now that they do occur if A is not hereditary. This fact will be a consequence of the next proposition.

2.4. Proposition. *Let A be an (artin) algebra satisfying (IIP). If P is an indecomposable projective A-module whose endomorphism ring is a division ring, then $radP$ is projective. As a consequence, $pd(P/radP) \leq 1$.*

Proof. Since P is an indecomposable projective module, then $P = P_i$ for some i, in the indexing given in (2.3). Therefore, the projective cover of $\operatorname{rad} P_i$, P', belongs to $\operatorname{add}(P_i \oplus \cdots \oplus P_n)$. However, by hypothesis, $\operatorname{Hom}_A(P_i, \operatorname{rad} P_i) = 0$ and then $P' \in \operatorname{add}(P_{i+1} \oplus \cdots \oplus P_n)$. Then $\tau_{P'}(P_i) = \operatorname{rad} P_i$, and hence, $\operatorname{rad} P_i$ is projective, as required. □

2.5. Corollary. *Let A be a basic (artin) algebra satisfying (IIP). Then A is hereditary if and only if $End_A(P)$ is a division ring for every indecomposable projective module P.*

Proof. By (2.4), $\operatorname{rad} P$ is a projective A-module for every indecomposable projective module P, and therefore the algebra A is hereditary. The converse is direct. □

2.6. Corollary. *Let $A = kQ(A)/I$ be a finite dimension k-algebra with (IIP). If $Q(A)$ has no loops, then A is hereditary, that is, $I = 0$.*

Proof. Let $P(i)$ be the indecomposable projective associated with the vertex i. Then, $P(i) = Ae_i$, where e_i is an idempotent of A. Since $Q(A)$ has no loops and A satisfies (IIP), we infer that there are no oriented cycles in $Q(A)$ (2.3). Therefore

$$\operatorname{End}_A(P(i)) = \frac{e_i(kQ(A))e_i}{I(i,i)}$$

is a division ring. This being true for each vertex i, we conclude, by (2.5), that A is hereditary. □

2.7. For the rest of this section let $A = kQ(A)/I$ be a nonhereditary algebra over the algebraically closed field k, and satisfying (IIP). We shall look now at the relations which generate I. Fix $u \in (Q(A))_0$ and let $\beta_i : u \longrightarrow v_i$, for $i = 1, \ldots, n$, be all the arrows starting at u which are not loops. Let $P = \bigoplus_{i=1}^n P(v_i)$, where $P(x)$ denotes the indecomposable projective module corresponding to the vertex x. Let

$$(c_1, \dots, c_m) : \bigoplus_{i=1}^{n} P(v_i)^{d_i} \longrightarrow \tau_P(P(u))$$

be the projective cover of $\tau_P(P(u))$. Since A satisfies (IIP), then $(c_1, \dots, c_m)$ is indeed an isomorphism. Denote $P' = \bigoplus_{i=1}^{n} P(v_i)^{d_i}$.

PROPOSITION. *Under the above hypothesis, there exists an isomorphism*

$$(\beta_1, \dots, \beta_n, b_{n+1}, \dots, b_m) : P' \longrightarrow \tau_P(P(u))$$

(*reordering the summands of* P' *if necessary*), *with* $b_{n+1}, \dots, b_m \in rad^2 A$.

PROOF. We first show that there exists an isomorphism

$$(c_1, \dots, c_{l-1}, \beta_1, c_{l+1}, \dots c_m) : \bigoplus_{i=1}^{n} P(v_i)^{d_i} \longrightarrow \tau_P(P(u))$$

(using the above notations). Indeed, since $\beta_1 \in \tau_P(P(u))$, and $(c_1, \dots, c_m)$ is an epimorphism, there exists

$$(\delta_1, \dots, \delta_m) \in \bigoplus_{i=1}^{n} P(v_i)^{d_i}, \text{such that } \beta_1 = \sum_{i=1}^{m} \delta_i c_i.$$

Observe that $c_i \in \operatorname{rad} A$ because $v_i \neq u$ for each $i = 1, \dots, n$, and the relations are all admissible. Then, there exists an l such that $0 \neq \delta_l \notin \operatorname{rad} A$. Therefore $\delta_l = \lambda_l v_l$, $\lambda_l \in k$. We claim that $f' = (c_1, \dots, c_{l-1}, \beta_1, c_{l+1}, \dots c_m)$ is also an isomorphism. It suffices to show that it is an epimorphism. This is indeed the case, because if $x \in \tau_P(P(u))$, then

$$x = \sum_{j=1}^{m} \mu_j c_j = \mu_l \left(\lambda_l^{-1} \beta_1 - \sum_{j \neq l} \lambda_l^{-1} \delta_j c_j \right) + \sum_{j \neq l} \mu_j c_j$$

and then $x \in \operatorname{Im} f'$, which proves the claim. Reordering the summands of P', we have an isomorphism

$$(\beta_1, c_1, \dots, c_{l-1}, c_{l+1}, \dots c_m) : P' \longrightarrow \tau_P(P(u)).$$

Observe that the same procedure can be repeated for $\beta_2, \dots, \beta_n$. In the i-th step one can choose the element c_{l_i} we removed to be different from $\beta_1, \dots \beta_{i-1}$ because the relations are admissible. Then, reordering the summands of P' we will end up with an isomorphism $(\beta_1, \dots, \beta_n, b'_{n+1}, \dots, b'_m)$ from P' to $\tau_P(P(u))$.

By subtracting, for each $i=1,\dots,m-n$, from the b'_{n+i} appropriate linear combinations of $\beta_1,\dots,\beta_n$, we obtain $b_{n+1},\dots,b_m$ in $\operatorname{rad}^2 A$ such that $(\beta_1,\dots,\beta_n,b_{n+1},\dots,b_m)$ is the required isomorphism. $\square$

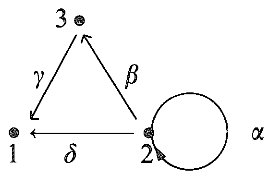
2.8. We have seen in (2.6) that, since $A=kQ(A)/I$ is not hereditary, $Q(A)$ has loops. The next result, which follows easily from the above proposition, shows that there is a set of relations, each of them with a summand starting at a loop, which generates I.

COROLLARY. *Let $A=kQ(A)/I$ be an algebra satisfying (IIP), and $\beta_1,\dots,\beta_n$ be arrows in $Q(A)$ which are not loops and starting at the same vertex. If $r=\sum_{i=1}^n \gamma_i\beta_i\in I$, for some linear combinations of paths $\gamma_1,\dots,\gamma_n$, then $\gamma_i\in I$, for each $i=1,\dots,n$.*

PROOF. By (2.7), there exists an isomorphism $(\beta_1,\dots,\beta_n,b_{n+1},\dots,b_m)$ from P' to $\tau_P(P(u))$. The hypothesis implies that the element $(\gamma_1,\dots,\gamma_n,0,\dots,0)$ goes to zero under this isomorphism. Therefore, $\gamma_i=0$, for each $i=1,\dots,n$. $\square$

2.9. In the proof of our main theorem in section 3, we will consider the following construction. We shall start with a subquiver Q' of $Q(A)$ and extend a representation of Q' to one of $Q(A)$ through an arrow $\alpha\notin(Q')_1$. Clearly, this can not be done always because of the relations involving α. However, we shall show that, under the hypothesis of the theorem, we can always find a suitable arrow for this extension. We shall prove now some preliminary results in this direction. We start with an example.

EXAMPLE. Let $A=kQ(A)/I$, where $Q(A)$ is the quiver



and I is generated by the relations $\delta\alpha-\gamma\beta$, $\beta\alpha$ and α^2. We leave to the reader to show that A satisfies (IIP). Consider the full subquiver Q' of Q containing the vertices 1 and 3 and let V be an indecomposable Q'-representation. Observe that we can extend V to a $Q(A)$-representation $\bar{V}$ given by $\bar{V}_1=V_1$, $\bar{V}_2=V_1$, $\bar{V}_3=V_3$, $\bar{f}_\gamma=f_\gamma$, $\bar{f}_\delta=Id$, and $\bar{f}_\beta=\bar{f}_\alpha=0$. However, if one tries to extend V

to $\bar{V}$ through the arrow β in the same fashion, that is, by putting, $\bar{V}_1 = V_1$, $\bar{V}_3 = V_3$, $\bar{V}_2 = V_3, \bar{f}_\gamma = f_\gamma, \bar{f}_\beta = Id$, and $\bar{f}_\delta = \bar{f}_\alpha = 0$, this does not define a $Q(A)$-representation because of the relation $\delta\alpha - \gamma\beta$. In the sense given by the definition below, the arrow δ is suitable and the arrow β is not suitable.

DEFINITION. An arrow $\beta_1 : u \to v_1$ which is not a loop, is called *suitable* if there are no relations of the type

$$\sum_{i=1}^{r} \gamma_i\beta_i + \sum_{i,j} \gamma_{ij}\beta_i\delta_{ij}$$

where arrow γ_i and γ_{ij} are linear combinations of paths from v_i to a (fixed) vertex w, $\gamma_1 \neq 0$ and $\delta_{ij} \in \operatorname{rad} A$, that is, it is a nonzero linear combination of α_j^l, which $l > 0$ and α_j is a loop around u.

2.10. Fix $u \in (Q(A))_0$ and let $\beta_i : u \to v_i$, for $i = 1, \dots, n$, be all the arrows starting at u which are not loops. We will show that if there are no loops at the vertices $v_1, \dots, v_n$ and u has a unique loop around it, then there exists a suitable arrow starting at u. We need the following result.

PROPOSITION. *Let $u \in (Q(A))_0$ and let $\beta_i : u \to v_i$, for $i = 1, \dots, n$ be all the arrows starting at u which are not loops. β_1 is not suitable, then there exists a path of length greater than zero from v_1 to v_i, for some i.*

PROOF. Assume that β_1 satisfies a relation

$$\sum_{i=1}^{r} \gamma_i\beta_i + \sum_{i,j} \gamma_{ij}\beta_i\delta_{ij} \tag{$*$}$$

where γ_i and γ_{ij} are linear combinations of paths from v_i to a (fixed) vertex w, $\gamma_1 \neq 0$ and $\delta_{ij} (\in \operatorname{rad} A)$ are nonzero linear combinations of α_j^l, with $l > 0$ and α_j is a loop around u_i. Let $P = \bigoplus_{i=1}^n P(v_i)$, and $P' = \bigoplus_{i=1}^n P(v_i)^{d_i} \cong \tau_P(P(u))$. We know by (2.7) that there exists an isomorphism

$$(\beta_1, \dots, \beta_n, b_{n+1}, \dots, b_m) : P' \to \tau_P(P(u))$$

with $b_1 = \beta_1, \dots, b_n = \beta_n$ and $b_{n+1}, \dots, b_m \in \operatorname{rad}^2 A$. Since $\beta_i\delta_{ij} \in \tau_{P(v_i)}(P(u)) \subset \tau_{P'}(P(u))$, we can write

$$\beta_i\delta_{ij} = \sum_l \mu_{ijl} b_l \tag{$**$}$$

with μ_{ijl} linear combinations from the end of b_l to the end of β_i. In particular, since $b_1 = \beta_1$, we have that μ_{ij1} goes from v_1 to v_i. Observe that $\mu_{ij1} \in \operatorname{rad} A$ because $\beta_i \delta_{ij} \in \operatorname{rad}^2 A$, all relations are admissible and $(b_1, \ldots, b_m)$ is a monomorphism. Assume that there are no paths of length greater than zero from v_1 to v_i, for any i. Hence $\mu_{ij1} = 0$, for each i. Replacing now $(**)$ in $(*)$, we get a relation, which is a linear combination of $b_1, \ldots, b_m$ with the coefficients of $b_1 = \beta_1$ equal to γ_1. Using again that $(b_1, \ldots, b_m)$ is a monomorphism, we conclude that $\gamma_1 = 0$, a contradiction. Therefore, there exists a path from v_1 to some v_i, as required. □

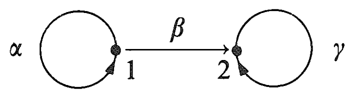
2.11. The next result will be essential in the proof of our main theorem.

Corollary. *Let $u \in (Q(A))_0$ and let $\beta_i : u \to v_i$, for $i = 1, \ldots, n$, be all the arrows starting at u which are not loops. Suppose, furthermore, that there are no loops at the vertices v_i's. Then one of the β_i's is suitable.*

Proof. Since there are no oriented cycles involving $v_1, \ldots, v_n$, there exists a partial order for these vertices given by: $v_i \leq v_j$ if and only if there exists a path from v_i to v_j. Let v_l be a maximum element under this order. The corresponding arrow β_l is, clearly, by (2.10), a suitable arrow. □

2.12. We end this section with the following example which shows that the hypothesis of the nonexistence of loops around the vertices v_i's is essential for the validity of (2.11).

Example. Let A be the algebra given by the quiver



with relations $\alpha^2 = 0$, $\gamma^2 = 0$ and $\beta\alpha = \gamma\beta$. Observe that A satisfies (IIP) but there are no suitable arrows.

3. The main theorem

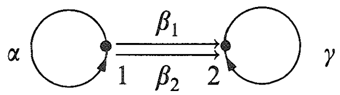
3.1. In this section we shall prove our main result, that is, that any representation-infinite artin algebra satisfying (IIP) and not hereditary has an infinite number of nonisomorphic indecomposable modules of infinite projective dimension. First we start observing that there are many such algebras, showing examples of them. Then we will prove some preliminary results needed in the proof of the theorem.

EXAMPLES. The following are examples of representation-infinite artin algebras satisfying (IIP) and not hereditary.

(a) Representation-infinite local algebras. Our main theorem is trivial in this case, since the only modules of finite projective dimension are projective.

(b) Let Q be a quiver with at least one loop but not oriented cycles containing an arrow which is not a loop. Consider the algebra $A = kQ/I$, where I is an admissible ideal generated by linear combinations of products of loops. Then A satisfies (IIP) and it is not hereditary. Many of these algebras are representation-infinite.

(c) Let A be the algebra given by the quiver



with relations $\alpha^3 = 0$, $\gamma^2 = 0$ and $\beta_1\alpha = 0$. Clearly, A satisfies the required conditions.

3.2. We shall need the following lemma.

LEMMA. *Let R and B be algebras, M be a B-R-bimodule and*

$$A = \begin{pmatrix} R & 0 \\ M & B \end{pmatrix}$$

If A satisfies (IIP), then B also does.

PROOF. Observe that we have an embedding of categories $B-\mathrm{mod} \hookrightarrow A$-mod, which preserves projective modules and resolutions. Let I be an idempotent ideal of B. Therefore

$$J = \begin{pmatrix} 0 & 0 \\ IM & I \end{pmatrix}$$

is an idempotent ideal of A. In fact,

$$\begin{pmatrix} R & 0 \\ M & B \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ IM & I \end{pmatrix} \cdot \begin{pmatrix} R & 0 \\ M & B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ IM & I \end{pmatrix} \cdot \begin{pmatrix} R & 0 \\ M & B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ IM & I \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} 0 & 0 \\ IM & I \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ IM & I \end{pmatrix}$$

Since A satisfies (IIP), we have that J is a projective A-module. On the other hand, the natural epimorphism

$$\pi : A \to B \qquad \text{given by} \qquad \pi\begin{pmatrix} r & 0 \\ m & b \end{pmatrix} = b$$

takes projective modules to projective modules. Since $\pi(J) = I$, we conclude that I is projective, as required. □

3.3 Corollary. *Let A be a right glueing algebra of $B_1, \ldots, B_t$ by C. If A satisfies (IIP), then B_i is hereditary for each i.*

Proof. By (1.3), we know that

$$A \cong \begin{pmatrix} C & 0 \\ M & B \end{pmatrix}$$

where $B = B_1 \times \cdots \times B_t$ and M is a B-C-bimodule. By the lemma above, B also satisfies (IIP) and, in particular, $\operatorname{fpd} B$ is at most one (2.2). On the other hand, since B is a product of tilted algebras, we have that $\operatorname{gl.dim} B \leq 2$. Hence, $\operatorname{gl.dim} B \leq 1$, and B is a product of hereditary algebras. □

3.4. We shall now prove our main theorem.

Theorem. *Let A be a finite dimension indecomposable algebra over an algebraically closed field k, satisfying (IIP). If A is representation-infinite and not hereditary, then there exist infinitely many nonisomorphic indecomposable A-modules of infinite projective dimension.*

Proof. By (2.2), we know that $\operatorname{fpd} A$ is at most one. Assume that there are only finitely many nonisomorphic indecomposable A-modules of infinite projective dimension. Therefore, $\operatorname{pd} M \leq 1$ for almost all indecomposable A-modules M.

By (1.3), A is then a right glueing of $B_1, \ldots, B_t$ by C, where C is representation-finite and, for each i, B_i is a representation-infinite tilted algebra. By (3.3), each B_i is hereditary.

If now $C = 0$, then A is hereditary, a contradiction. Therefore $C \neq 0$. By the description of right glued algebras, the ordinary quivers of $B_1, \ldots, B_t$, and C are full convex subquivers of $Q(A)$, and there are neither arrows from a vertex of

$Q(B_i)$ to a vertex of $Q(B_j)$, if $i \neq j$, nor arrows from a vertex of $Q(B_i)$ to $Q(C)$, for $i = 1, \ldots, t$. Since $Q(A)$ is connected, there are arrows from $Q(C)$ to each $Q(B_i)$. For a given i, B_i is a connected representation-infinite hereditary algebra, and then each vertex of $Q(B_i)$ belongs to the support of infinitely many indecomposable B_i-modules. The strategy now is the following: we start with an infinite family of indecomposable B_i-modules with support containing a vertex which is the end of a convenient arrow β which starts at $Q(C)$. We shall then extend each module of this family, through the arrow β, to an indecomposable A-module which is not a B-module, leading to a contradiction to the fact that B-ind is cofinite in A-ind. The key point of this proof is the choice of the arrow β.

Claim. There exists a suitable arrow from a vertex of $Q(C)$ to a vertex of $Q(B)$.

Let $F = \{u \in (Q(C))_0 : \text{there exists an arrow } u \to v, \text{with } v \in (Q(B))_0\}$. Since the only oriented cycles are sequences of loops, there exists a vertex $u_0 \in F$ such that there are no paths in $Q(C)$ from u_0 to any other vertex of F. Let i be such that there is an arrow from u_0 to a vertex of $Q(B_i)$. Without loss of generality, suppose $i = 1$. Let $\beta_1 : u_0 \to v_1, \ldots, \beta_n : u_0 \to v_n$ be all the arrows from u_0 to a vertex of $Q(B_1)$. Observe that, by the choice of u_0, any path from u_0 to a vertex of $Q(B_1)$ has to pass through one of the β_i's. On the other hand, observe that there are no loops around the vertices v_i's because they belong to $(Q(B_1))_0$ and B_1 is hereditary. By (2.11), we infer that one of the β_i's is a suitable arrow. This proves the claim.

Denote by $\beta : u \to v$ a suitable arrow in $(Q(A))_1$, with $u \in (Q(C))_0$ and $v \in (Q(B_1))_0$. Let now $\mathscr{X}_v$ be the (infinite) set of all nonisomorphic indecomposable B_1-modules whose support contains v. This means that if $((M_i)_{i \in (Q(B_1))_0}, (f_\gamma)_{\gamma \in (Q(B_1))_1}) \in \mathscr{X}_v$, then $M_v \neq 0$. We shall construct an infinite set of non-isomorphic indecomposable A-modules which are not B_1-modules using the (suitable) arrow β. For an $X = ((X_i)_{i \in (Q(B_1))_0}, (f_\gamma)_{\gamma \in (Q(B_1))_1}) \in \mathscr{X}_v$ define $\bar{X} = ((\bar{X}_i)_{i \in (Q(A))_0}, (\bar{f}_\gamma)_{\gamma \in (Q(A))_1})$, by

$$\bar{X}_i = \begin{cases} X & \text{if } i \in (Q(B_1))_0 \\ X_v & \text{if } i = u \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{f}_\gamma = \begin{cases} f_\gamma & \text{if } \gamma \text{ is an arrow in } (Q(B_1))_1 \\ Id & \text{if } \gamma = \beta \\ 0 & \text{otherwise} \end{cases}$$

Since β is a suitable arrow, the representation $\bar{X}$ as defined above satisfies all the relations required to be an A-module. We shall show now that $\bar{X}$ is indecomposable. Suppose $\bar{X} = Y_1 \oplus Y_2$, and define, for each i, Y_i' by $(Y_i')_j = (Y_i)_j$ if $j \in (Q(B_1)_0)$ and $(Y_i')_j = 0$, otherwise. Then $X = Y_1' \oplus Y_2'$ in B_1-ind. Since X is indecomposable, it follows that either $Y_1' = 0$ or $Y_2' = 0$. Therefore, either Y_1 or Y_2 is a sum of copies of the simple $S(u)$ associated to u, contradicting the hypothesis that $\bar{f}_\beta = id$. Moreover, if X and X' are two nonisomorphic indecomposable B_1-modules in $\mathscr{X}_v$, then $\bar{X}$ and $\bar{X}'$ are also nonisomorphic. Therefore, there exist infinitely many indecomposable A-modules which are not B_1-modules, a contradiction to the fact that B_1-ind is cofinite in A-ind, and the result is proven. □

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