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REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

By

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1. Introduction

Let $P_n(C)$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. Typical examples of real hypersurface in $P_n(C)$ are homogeneous ones. R. Takagi ([8]) showed that all homogeneous real hypersurfaces in $P_n(C)$ are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or 2. Namely, he proved the following

THEOREM 1.1. *Let M be a homogeneous real hypersurface of $P_n(C)$. Then M is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere* (*that is, a tube over a hyperplane* $P_{n-1}(C)$),

(A_2) *a tube over a totally geodesic* $P_k(C)$ $(1 \leq k \leq n-2)$,

(B) *a tube over a complex quadric* Q_{n-1},

(C) *a tube over* $P_1(C) \times P_{(n-1)/2}(C)$ *and* $n(\geq 5)$ *is odd*,

(D) *a tube over a complex Grassmann* $G_{2,5}(C)$ *and* $n = 9$,

(E) *a tube over a Hermitian symmetric space* $SO(10)/U(5)$ *and* $n = 15$.

On the other hand, many differential geometers have studied real hypersurfaces in $P_n(C)$ by making use of the almost contact metric structure induced from the complex structure J of $P_n(C)$ (see, §2). It is well-known that there does not exist a real hypersurface with parallel second fundamental tensor of $P_n(C)$. Moreover Hamada ([2]) showed that there does not exist a real hypersurface with recurrent second fundamental tensor A of $P_n(C)$, i.e., there exists a 1-form α such that $\nabla A = A \otimes \alpha$. In this paper we consider the weaker condition.

The second fundamental tensor A is called a birecurrent second fundamental tensor if there exists a covariant tensor field α of order 2 such that $\nabla^2 A = A \otimes \alpha$.

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We may regard the parallel condition and the recurrent condition as a special case. First, we show the nonexistence of real hypersurfaces with birecurrent second fundamental tensor of $P_n(C)$.

Next, we characterize a homogeneous real hypersurface of type (A_1) and (A_2) under the condition that the structure vector is principal (see, §2).

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2. Preliminaries

Let M be an orientable real hypersurface of $P_n(C)$ and let N be a unit normal vector field on M. The Riemannian connections $\tilde{\nabla}$ in $P_n(C)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{1}$$

$$\tilde{\nabla}_X N = -AX, \tag{2}$$

where g denote the Riemannian metric of M induced from the Fubini-Study metric G of $P_n(C)$ and A is the second fundamental tensor of M in $P_n(C)$. An eigenvector of A is called a *principal curvature vector*. Also an eigenvalue of A is called a *principal curvature*. It is known that M has an almost contact metric structure induced from the complex structure J on $P_n(C)$, that is, we define a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η on M by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, N)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \phi\xi = 0. \tag{3}$$

It follows from (1) that

$$(\nabla_X \phi) Y = \eta(X)AX - g(AX, Y)\xi, \tag{4}$$

$$\nabla_X \xi = \phi AX. \tag{5}$$

Let $\tilde{R}$ and R be the curvature tensors of $P_n(C)$ and M, respectively. Then we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{7}$$

In the following, we use the same terminology and notations as above unless otherwise stated. Now we prepare without proof the following results:

THEOREM 2.1 ([5]). *Let M be a real hypersurface of $P_n(C)$. Then the following are equivalent*:

(i) *M is locally congruent to one of homogeneous ones of type A_1 and A_2.*

(ii) $(\nabla_X A)Y = -\eta(Y)\phi X - g(\phi X, Y)\xi$ *for any* $X, Y \in TM$.

THEOREM 2.2 ([6]). *Let M be a real hypersurface of $P_n(C)$. Then the following are equivalent*:

(i) $\phi A = A\phi$.

(ii) *M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.*

THEOREM 2.3 ([4]). *There are no real hypersurfaces M with $(R(X, Y)A)Z = 0$ for $X, Y, Z \in TM$ in $P_n(C)$, $n \geq 2$.*

THEOREM 2.4 ([3]). *Let M be a real hypersurface of $P_n(C)$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

THEOREM 2.5 ([1]). *Let M be a real hypersurface of $P_n(C)$, $n \geq 3$. If the second fundamental tensor A satisfies*

$$(R(X, Y)A)Z = 0$$

for all tangent vectors X, Y, Z perpendicular to ξ, then M is locally congruent to a geodesic hypersphere.

PROPOSITION 2.6 ([5]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

PROPOSITION 2.7 ([5]). *Assume that ξ is a principal curvatur vector and corresponding principal curvature is α. If $AX = rX$ for $X \perp \xi$, then we have $A\phi X = ((\alpha r + 2)/(2r - \alpha))\phi X$.*

3. Main Theorem

First we consider a real hypersurface with birecurrent second fundamental tensor. Hamada ([2]) showed that there are no real hypersurfaces with recurrent second fundamental tensor of $P_n(C)$. We may regard the recurrent condition as special case of the birecurrent condition. We will prove the following:

THEOREM 1. *There exists no real hypersurfaces with birecurrent second fundamental tensor of $P_n(\boldsymbol{C})$.*

PROOF. The following equation holds for any $Y \in TM$.

$$\nabla_Y A^2 = (\nabla_Y A)A + A(\nabla_Y A).$$

Differentiating the above equation by $X \in TM$, we have

$$\nabla^2_{X,Y} A^2 = (\nabla^2_{X,Y} A)A + A(\nabla^2_{X,Y} A) + (\nabla_X A)(\nabla_Y A) + (\nabla_Y A)(\nabla_X A).$$

Here we suppose that the second fundamental tensor A is birecurrent, i.e., there exists a covariant tensor field α of order 2 satisfying $\nabla^2_{X,Y} A = \alpha(X, Y)A$. Hence we have from the above equation

$$\nabla^2_{X,Y} A^2 = 2\alpha(X, Y)A^2 + (\nabla_X A)(\nabla_Y A) + (\nabla_Y A)(\nabla_X A).$$

From this equation and commutativity of the trace and the derivation we obtain

$$\nabla^2_{X,Y}(tr\, A^2) = 2\alpha(X, Y)(tr\, A^2) + 2tr((\nabla_X A)(\nabla_Y A)).$$

Replacing X by Y, and subtracting from the above equation, we have

$$(\alpha(X, Y) - \alpha(Y, X))(tr\, A^2) = 0.$$

Since there exists no real hypersurfaces with $A = 0$, we have $\alpha(X, Y) = \alpha(Y, X)$. Then we obtain $(R(X, Y)A)Z = 0$ for $X, Y, Z \in TM$. This shows the assertion from Theorem 2.3. □

We denote by $\xi^\perp$ the subbundle of TM consisting of vectors perpendicular to ξ. In what follows $e_1, \ldots, e_{2n-2}$ stands for an orthonormal basis of $\xi^\perp$ at a point in M. Next, we will prove the following:

THEOREM 2. *There exist no real hypersurfaces in $P_n(\boldsymbol{C})$, $n \geq 3$, satisfying the following condition*:

$$g((\nabla^2_{X,Y} A)Z, W) = \alpha(X, Y)g(AZ, W) \quad \textit{for } X, Y, Z, W \in \xi^\perp, \tag{8}$$

where $\alpha(X, Y)$ is a covariant tensor field of order 2. And the structure vector ξ is principal.

PROOF.

$$
\begin{aligned}
\nabla^2_{X,Y}(tr\,A^2) &= \sum_{j=1}^{2n-2} g((\nabla^2_{X,Y}A^2)e_j, e_j) + g((\nabla^2_{X,Y}A^2)\xi, \xi) \\
&= \sum_{j=1}^{2n-2} g((\nabla^2_{X,Y}A)Ae_j, e_j) + \sum_{j=1}^{2n-2} g(A(\nabla^2_{X,Y}A)e_j, e_j) \\
&\quad + \sum_{j=1}^{2n-2} g((\nabla_X A)(\nabla_Y A)e_j, e_j) + \sum_{j=1}^{2n-2} g((\nabla_Y A)(\nabla_X A)e_j, e_j) \\
&\quad + g((\nabla^2_{X,Y}A)A\xi, \xi) + g(A(\nabla^2_{X,Y}A)\xi, \xi) \\
&\quad + g((\nabla_X A)(\nabla_Y A)\xi, \xi) + g((\nabla_Y A)(\nabla_X A)\xi, \xi).
\end{aligned}
$$

Since the structure vector ξ is principal, $Ae_j \in \xi^\perp$. By using the assumption (8), we have

$$
\begin{aligned}
\nabla^2_{X,Y}(tr\,A^2) &= 2\alpha(X, Y) \sum_{j=1}^{2n-2} g(A^2 e_j, e_j) + 2 \sum_{j=1}^{2n-2} g((\nabla_X A)e_j, (\nabla_Y A)e_j) \\
&\quad + g((\nabla^2_{X,Y}A)A\xi, \xi) + g(A(\nabla^2_{X,Y}A)\xi, \xi) + 2g((\nabla_X A)\xi, (\nabla_Y A)\xi).
\end{aligned}
$$

Replacing X by Y, and subtracting from the above equation, we have

$$
0 = (\alpha(X, Y) - \alpha(Y, X)) \sum_{j=1}^{2n-2} g(A^2 e_j, e_j), \tag{9}
$$

because the structure vector ξ is principal.

Here we assume that $\sum_{j=1}^{2n-2} g(A^2 e_j, e_j) = 0$. Let Z be a vector field orthogonal to ξ such that $AZ = \lambda Z$. Then it is known (see, Prop. 2.7) that

$$
A\phi Z = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi Z.
$$

Hence we have from the assumption

$$
\lambda = \frac{\alpha\lambda + 2}{2\lambda - \alpha} = 0.
$$

This is a contradiction. So (9) implies that $\alpha(X, Y)$ is symmetric tensor. Therefore we have from (8)

$$
g((R(X, Y)A)Z, W) = 0 \quad \text{for } X, Y, Z \text{ and } W \in \xi^\perp. \tag{10}
$$

Since the structure vector ξ is principal, from Gauss equation (6) we have the following

$$g((R(X, Y)A)Z, \xi) = 0 \quad \text{for } X, Y, Z \in \xi^{\perp}.$$

Hence we have

$$(R(X, Y)A)Z = 0 \quad \text{for } X, Y, Z \in \xi^{\perp}.$$

Therefore, in the case of $n \geq 3$, M is locally congruent to a real hypersurface of type (A_1) (cf. Theorem 2.5). So, we shall check the equation (8) for a real hypersurface of type (A_1). Then the second fundamental tensor A of type (A_1) is expressed as (cf. [8]):

$$AX = tX \quad \text{for } X \in \xi^{\perp},$$

where t is constant and $t > 0$. Making use of Theorem 2.1 and (5), and substituting the above equation into (8), we get

$$-g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) = \alpha(X, Y)g(Z, W).$$

Putting $Y = Z = \phi X$ and $W = X$, we have $\|X\|^2 = 0$ for any $X \in \xi^{\perp}$. This is a contradiction. $\square$

Next, our aim is to prove the following.

THEOREM 3. *Let M be a real hypersurface of $P_n(C)$, $n \geq 2$. If the second fundamental tensor A satisfies*

$$g((\nabla^2_{X,Y}A)Z, W) = -g(\phi AX, W)g(\phi Y, Z) - g(\phi AX, Z)g(\phi Y, W)$$

for any $X, Y, Z, W \in \xi^{\perp}$, and the structure vector ξ is principal. Then M is locally congruent to one of homogeneous real hypersurfaces of type (A_1) and (A_2).

PROOF. We assume that the second fundamental tensor A satisfies

$$\begin{aligned} &g((\nabla_X(\nabla_Y A))Z - (\nabla_{\nabla_X Y}A)Z, W) \\ &\quad = -g(\phi AX, W)g(\phi Y, Z) - g(\phi AX, Z)g(\phi Y, W). \end{aligned}$$

Exchanging X and Y in the above equation, we have the following

$$\begin{aligned} (11) \qquad &g((R(X, Y)A)Z, W) \\ &\quad = -g(\phi AX, W)g(\phi Y, Z) - g(\phi AX, Z)g(\phi Y, W) \\ &\quad\quad + g(\phi AY, W)g(\phi X, Z) + g(\phi AY, Z)g(\phi X, W). \end{aligned}$$

From Gauss equation (6) the left hand side of (11) is

$$\begin{aligned}
&g(R(X,Y)AZ - AR(X,Y)Z, W)\\
&\quad = g(Y,AZ)g(X,W) - g(X,AZ)g(Y,W)\\
&\qquad + g(\phi Y,AZ)g(\phi X,W) - g(\phi X,AZ)g(\phi Y,W)\\
&\qquad - 2g(\phi X,Y)g(\phi AZ,W) + g(AY,AZ)g(AX,W)\\
&\qquad - g(AX,AZ)g(AY,W) - g(Y,Z)g(X,AW)\\
&\qquad + g(X,Z)g(Y,AW) - g(\phi Y,Z)g(\phi X,AW)\\
&\qquad + g(\phi X,Z)g(\phi Y,AW) + 2g(\phi X,Y)g(\phi Z,AW)\\
&\qquad - g(AY,Z)g(AX,AW) + g(AX,Z)g(AY,AW).
\end{aligned}$$

And hence the equation (11) asserts that

$$\begin{aligned}
(12)\quad & g(HX,W)g(\phi Y,Z) + g(HX,Z)g(\phi Y,W)\\
& - g(HY,W)g(\phi X,Z) - g(HY,Z)g(\phi X,W) - 2g(HZ,W)g(\phi X,Y)\\
& - g(AY,Z)g(X,W) + g(Y,Z)g(AX,W)\\
& + g(AX,Z)g(Y,W) - g(X,Z)g(AY,W)\\
& - g(A^2Y,Z)g(AX,W) + g(A^2X,W)g(AY,Z)\\
& + g(A^2X,Z)g(AY,W) - g(A^2Y,W)g(AX,Z) = 0,
\end{aligned}$$

where H is tensor field of type (1,1) which is defined by

$$HX = (A\phi - \phi A)X.$$

Here let $e_1, \dots, e_{2n-2}$ be an orthonormal basis of $\xi^{\perp}$. And the index i runs from 1 to $2n-2$. Putting $Y = \phi e_i$, $Z = e_i$ in (12), and taking summation on i.

$$(2n+1)g(HX,W) + g(AHAX,W) = 0$$

for any $X, W \in \xi^{\perp}$.

Since the structure vector ξ is principal, the above equation holds for all tangent vectors X, W. Hence we have

$$(13)\qquad (2n+1)H + AHA = 0$$

at a point in M. From Proposition 2.7, we can take orthonormal vectors e_j, ϕe_j, ξ $(j = 1, \dots, n-1)$ which are principal vectors. Let λ_j and α be principal curvatures of e_j and ξ, respectively. Then principal curvatures of ϕe_j are $(\alpha\lambda_j + 2)/(2\lambda_j - \alpha)$, say λ_j'. By using the orthonormal basis we have

$$He_j = (A\phi - \phi A)e_j = (\lambda_j' - \lambda_j)\phi e_j,$$

$$H\phi e_j = (A\phi - \phi A)\phi e_j = (\lambda_j' - \lambda_j)e_j,$$

$$H\xi = 0.$$

Hence we have the following expression of H.

$$H = \left(\begin{array}{cccc|cccc|c}
 & & & & k_1 & & & 0 & 0 \\
 & & \mathbf{0} & & & \ddots & & & \vdots \\
 & & & & 0 & & & k_{n-1} & 0 \\
\hline
k_1 & & & 0 & & & & & 0 \\
 & \ddots & & & & & \mathbf{0} & & \vdots \\
0 & & & k_{n-1} & & & & & 0 \\
\hline
0 & \cdots & & 0 & 0 & \cdots & & 0 & 0
\end{array}\right),$$

where $k_j = \lambda_j' - \lambda_j$ $(j = 1, \dots, n-1)$.

By virture of the above expression of H, it follows from (13) that

$$(\lambda_j' - \lambda_j)(\lambda_j\lambda_j' + 2n + 1) = 0 \quad (j = 1, \dots, n-1).$$

Then by Proposition 2.6, we see that all principal curvatures are locally constant. Hence our real hypersurface M is homogeneous one by Theorem 2.4.

Due to Takagi's work ([8]), we find that a principal curvature of homogeneous real hypersurfaces in $P_n(C)$ is one of the following:

$$r_1 = t, \quad r_2 = -\frac{1}{t}, \quad r_3 = \frac{1+t}{1-t}, \quad r_4 = \frac{t-1}{t+1}, \quad \alpha = t - \frac{1}{t},$$

where $t = \cot\theta$ $(0 < \theta < \pi/4)$.

Here we assume that there exist k $(1 \le k \le n-1)$ such that $\lambda_k\lambda_k' = -2n-1$ (≤ -5). We note that a real hypersurface of type (A_1) has two distinct principal curvatures r_1 and α, type (A_2) has three distinct principal curvatures r_1, r_2 and α, type (B) has three distinct principal curvatures r_3, r_4, and α, a real hypersurface of type (C), (D) and (E) has five distinct principal curvatures. Now let $\lambda_k = r_i$ $(i = 1, 2)$. Then $\lambda_k' = r_i$ $(i = 1, 2)$ from Proposition 2.7. Hence we have $\lambda_k\lambda_k' = r_i^2$

$(i = 1, 2)$, which contradicts with the assumption. On the other hand, let $\lambda_k = r_3$. Then $\lambda_k = r_4$ from Proposition 2.7. Hence we have $\lambda_k \lambda'_k = -1$, which contradicts with the assumption. Hence we have $\lambda'_j - \lambda_j = 0$ $(j = 1, \ldots, n-1)$.

This means that $H = 0$ (i.e. $A\phi = \phi A$). By Theorem 2.2, M is one of homogeneous real hypersurfaces of type A_1 and A_2.

Conversely, a homogeneous real hypersurface of type (A) satisfies the assumption because of Theorem 2.1. This completes the proof of Theorem 3. □

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