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WEIERSTRASS POINTS AND RAMIFICATION LOCI ON SINGULAR PLANE CURVES

By

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Let X be a smooth compact Riemann surface (or a smooth projective curve) of genus g. A classical topic of study in Complex Analysis and Algebraic Geometry was the study of Weierstrass points of X. For a survey and the history of the subject up to 1986, see [EH]. For another survey containing the main definitions and results on Weierstrass points on singular Gorenstein curves, see [G]. For the case of a base field with positive characteristic, see [L]. Since Weierstrass points are "special" points on a curve, they have been very useful to study moduli problems. In particular, some subvarieties of the moduli space of smooth genus g curves are defined by the existence of suitable Weierstrass points. Several papers were devoted to the study of Weierstrass points on some interesting classes of projective curves (e.g. smooth plane curves and k-gonal curves). Our paper belong to this set of papers. We consider singular plane curves with ordinary cusps or nodes as only singularities. We believe that our paper gives a non-trivial contribution to the understanding of the existence of certain types of Weierstrass points and osculating points on these curves. In the first section we make easy extensions of [K2], Th. 1.1, to the case of singular curves. In the second section we use deformation theory to show the existence of several pairs (C, P) with C in integral nodal plane curve, $P \in C_{reg}$, such that the tangent line D of C at P has high order of contact with C at P (see Theorems 2.1 and 2.2 for precise statements). Calling X the normalization of C and seen P as a point of X, these pairs (C, P) satisfies the conditions of Proposition 1.1 below and hence P is a Weierstrass point of X. In the third section we consider the Weierstrass semigroup of the Weierstrass points obtained in this way. Here the main aim is to give a recipe to extract from the numerical calculations in [K2] as much informations as possible for the Weierstrass semigroup of the pair (X, P). The case of a total inflection point for nodal plane curves was considered in details in [CK]. Our recipe (see 3.1) gives

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some informations when the plane curve has ordinary nodes and ordinary cusps as only singularities. In section 4 we will construct nodal plane curves, say C, with a high order inflection point $P \in C_{reg}$ which is not a Weierstrass point of the normalization X of C. In particular this construction shows that the results in section 1 are reasonably sharp.

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§1. In this section we generalize [K2]. Th. 1.1, to the case of singular plane curves.

PROPOSITION 1.1. *Fix integers* d, u, m, x *with* $m > 0$, $d \geq u + 3 \geq 0$, $(u+2)(u+1)/2 > x \geq 0$, $m(d-u-3) \geq (d-2)(d-1)/2 - x \geq 0$. *Let* $t \leq d - u - 3$ *be the first integer with* $mt \geq (d-2)(d-1)/2 - x$. *Let* $C \subset \boldsymbol{P}^2$ *be an integral degree* d *curve with* x *double points (ordinary nodes or ordinary cusps) as only singularities. Let* X *be the normalization of* C *(hence* $g := g(X) = (d-2)(d-1)/2 - x$*). Assume the existence of a point* $P \in C_{reg}$ *such that the tangent line* D *to* C *at* P *as multiplicity at* P. *Then* P *(seen as a point of* C*) is a Weierstrass point of* X *with* $h^0(X, \mathcal{O}_X(gP)) \geq 1 + (d-t-1)(d-t-2)/2 - x$.

PROOF. By Serre duality and Riemann–Roch it is sufficient to check that $h^0(X, K_X(-gP)) \geq (d-t-1)(d-t-2)/2 - x$. The canonical bundle K_X is induced by the linear subsystem of $H^0(\boldsymbol{P}^2, \mathcal{O}(d-3))$ formed by the curves of degree $d-3$ passing through the x singular points of C. Thus it is sufficient to note the existence of several reducible degree $d-3$ curves $M = tD + Z$ with Z curve of degree $d-3-t$ containing $\mathrm{Sing}(X)$ and use the inequalities $g \leq mt \leq \mathrm{mult}_P(C \cdot M)$.

Now we consider the case in which the integral plane curve C has a smooth point P such that there is a degree $s > 1$ curve E which osculates to high order C at P. The proof of 1.1 gives with no change the following result.

PROPOSITION 1.2. *Fix integers* d, s, m, e, x *with* $e > 0$, $m > 0$, $s > 0$, $d - 3 \geq es \geq 0$, $(d-se-2)(d-se-1)/2 > x \geq 0$, $me \geq (d-2)(d-1)/2 - x \geq 0$. *Let* $C \subset \boldsymbol{P}^2$ *be an integral degree* d *curve with* x *double points (ordinary nodes or ordinary cusps) as only singularities. Let* X *be the normalization of* C *(hence* $g := g(X) = (d-2)(d-1)/2 - x$*). Assume the existence of a point* $P \in C_{reg}$ *such that there is a degree* s *curve* E *which intersects* C *at* P *with multiplicity* m. *Then*

P (seen as a point of X) is a Weierstrass point of X with $h^0(X, \mathcal{O}_X(gP)) \geq 1 + (d - se - 1)(d - se - 2)/2 - x$.

Now we consider the case in which the point P is a singular point of C and there is a line D (or a degree s curve E) which osculates with high order one of the branches of the curve C at P.

PROPOSITION 1.3. *Fix integers d, s, m, e, x with $e > 0$, $m > 0$, $s > 0$, $d - 3 \geq es \geq 0$, $(d - se - 2)(d - se - 1)/2 \geq x > 0$, $me \geq (d - 2)(d - 1)/2 - x \geq 0$. Let $C \subset \boldsymbol{P}^2$ be an integral degree d curve with x double points (ordinary nodes or ordinary cusps) as only singularities. Let X be the normalization of C (hence $g := g(X) = (d - 2)(d - 1)/2 - x$). Let P be an ordinary node of C. Assume the existence of a degree s curve E which intersects one of the two branches of C at P with multiplicity m. Then the point $P' \in X$ corresponding to this branch of C at P is a Weierstrass point of X with $h^0(X, \mathcal{O}_X(gP')) \geq 1 + (d + se - 1)(d - se - 2)/2 - x$.*

PROOF. To copy the proof of 1.1 it is sufficient to note that now we need to count only the curves, M, of the form $eE + Z$ with Z curve of degree $d - 3 - se$ containing $\mathrm{Sing}(C)\backslash\{P\}$.

The proof of 1.3 gives with no change the following result.

PROPOSITION 1.4. *Fix integers d, s, m, e, x with $e > 0$, $m > 0$, $s > 0$, $d - 3 \geq es \geq 0$, $(d - se - 2)(d - se - 1)/2 \geq x > 0$, $me \geq (d - 2)(d - 1)/2 - x \geq 0$. Let $C \subset \mathbf{P}^2$ be an integral degree d curve with x double points (ordinary nodes or ordinary cusps) as only singularities. Let X be the normalization of C (hence $g := g(X) = (d - 2)(d - 1)/2 - x$). Let P be an ordinary cusp of C. Assume the existence of a degree s curve E with $P \in E$ and such that E induces a degree ds divisor on X whose component supported by the point P' of X corresponding to P has degree m. Then P' is a Weierstrass point of X with $h^0(X, \mathcal{O}_X(gP)) \geq 1 + (d - se - 1)(d - se - 2)/2 - x$.*

§2. In this section we will give two related constructions of several pairs (C, P) satisfying all the conditions of Proposition 1.1.

We will use the following notations. Fix integers d, m, x with $d \geq m > 0$, $x \geq 0$. Fix $P \in \boldsymbol{P}^2$, a line D with $P \in D$ and the length m scheme B with $B \subset D$ and $B_{\mathrm{red}} = \{P\}$, i.e. the divisor mP on the line D. Let $V(B, d)$ be the set of degree d curves containing B. Since $m \leq d$, we have dim $(V(B, d)) = (d^2 + 3d)/$

$2-m$. If S is a finite subset of P^2, let $V(B,d,S)$ be the linear subsystem of $V(B,d)$ formed by the curves of $V(B,d)$ which are singular at each point of S. Since being singular at a fixed point imposes at most 3 conditions at each linear system of curves on a smooth surface, if $\dim(V(B,d)) \geq 3\mathrm{card}(S)$ we see that $V(B,d,S)$ is non empty and $\dim(V(B,d,S)) \geq (\dim(V(B,d)) - 3\mathrm{card}(S)$. Furthermore, by [AC], Lemma 4.2, we see easily that if S is general $\dim(V(B,d,S)) = \dim(V(B,d)) - 3\mathrm{card}(S)$ if $\dim(V(B,d)) \geq 3\mathrm{card}(S)$ and $V(B,d,S) = \varnothing$ if $\dim(V(B,d)) < 3\mathrm{card}(S)$. See Lemma 4.1 for more.

THEOREM 2.1. *Fix integers d,m,x with $d > m \geq 2, x \geq 0, 3x+m \leq (d^2+3d)/2$, $x \leq (d-2)(d-1)/2$. Let $S(x)$ be a general subset of P^2 with card$(S(x)) = x$. Then $V(B,d,S(x))$ is integral, non empty, of dimension $(d^2+3d)/2 - 3x - m$ and there is a pair (C,P) with C degree d integral plane curve with x ordinary nodes as only singularities and $P \in C_{reg}$ such that the tangent line D of C at P has multiplicity m at P unless $(d^2+3d)/2 = 3x+m$ and $d^2+6d = 8x+4m$; we may take as C a general element of $V(B,d,S(x))$. If $(d^2+3d)/2 = 3x+m$* and *$d^2+6d = 8x+4m$ and the unique $C \in V(B,d,S(x))$ is not integral with ordinary nodes at the points of $S(x)$ as only singularities, then $C = 2T$ with T integral curve of degree $d/2$ intersecting D at P with multiplicity $m/2$.*

PROOF. The proof is a modification of the proof of [AC], Prop. 4.1. The added difficulty is that now we are working on linear systems of curves containing B. However if we use proper notations, we will be able to "copy" the proof of [AC], Prop. 4.1. Look at that proof. Each time you see there a complete linear system $|C_i|$ in our situation this linear system would be better described as $|C_i|(-B_i)$ in the following sense. We have a linear system V contained in $|C_i|$ and a general $Z \in V$ intersects D in a divisor whose part supported by P is B_i; furthermore, if in [AC] there is a system $|C_i + C_j|$, the proper notation for us would be $|C_i + C_j|(-B_i - B_j)$ with $B_i + B_j$ sum of effective divisors on the line D. Since every two plane curves intersect, every part of the long proof of [AC], Prop. 4.1, which concerns curves C_i and C_j with $C_i \cdot C_j = 0$ may be ignored. In [AC] the letter δ is used instead of x. For the case $x = 0$ instead of the proof in [AC], p. 353, now it is sufficient to apply Bertini's theorem (characteristic 0). The use of Bertini's theorem at the beginning of step III of the proof of [AC], 4.1, is allowed because also in our situation the linear system $|2C_1|(-B_1)$ is not composed with a pencil. The exceptional case "$(d^2+3d)/2 = 3x+m$ and $d^2+6d = 8x+4m$" arises in the same way as the exceptional case in the statement of [AC], 4.1 (see [AC], end of p. 356). With these remarks, the theorem follows.

THEOREM 2.2. *Fix integers d, m, x with $d \geq m+2 \geq 4$, $0 \leq x \leq (d-2)(d-3)/2$. Then there is a pair (C,P) with C degree d integral plane curve with x ordinary nodes as only singularities and $P \in C_{reg}$ such that the tangent line D of C at P has multiplicity m at P.*

PROOF. Since $x \leq (d-2)(d-3)/2$ it is very well known the existence of an integral plane curve A with $\deg(A) = d-1$ and with x ordinary nodes as only singularities. We fix $P \in \boldsymbol{P}^2 \backslash A$ and a line D with $P \in D$ and D intersecting transversally A. Consider the length m scheme B with $B \subset D$ and $B_{\mathrm{red}} = \{P\}$, i.e. the divisor mP on the line D. Set $Y := A \cup D$. We claim the existence of a flat family of plane curves, all of them containing B and such that the general curve C of this family satisfies the thesis of the theorem. Let U be the normalization of A and $M = U \cup D$ (disjoint union) the normalization of Y. We consider the deformation theory of nodal plane curves containing the scheme B. This well-known theory is just the union of the classical deformation theory of plane curves (see e.g. [T]) and the deformation theory due to Kleppe (see e.g. [P], Th. 1.5) of projective varieties containing a fixed subscheme. By this theory (and see in particular [HH], § 2] to prove the claim it is sufficient to use the following two remarks. Note that the normal bundle of D in $\boldsymbol{P}^2$ has degree 1 and A intersects transversally D at exactly $d-1$ points,. Thus $1+(d-1)-m \geq -1$ and hence the line bundle of degree $1+(d-1)-m$ on D is not special, i.e., with the notations of [HH], § 2, $h^1(D,(N_Y|D)^-) = 0$. Calling $A \cap D$ the corresponding subset of $U \cap D$, note that $\deg((N_M|U^+)) = (d-1)^2 - x + d - 1 > 2g(M) - 2 + d - 1$; hence $h^1(D,(N_M|U)^+) = 0$ and the the restriction map $(N_M|U)^+) \to (N_M|U)^+|(A \cap D)$ is surjective.

§ 3. In this section we give a recipe to extract from the numerical calculations in [K2] as much informations as possible for the Weierstrass semigroup of the pair (X,P).

PROPOSITION 3.1. *Fix integers d, i, x, m with $d-3 \geq i$, $x \geq 0$, $d-3 \leq m \leq d-1$, and a finite set $S \subset \boldsymbol{P}^2$ with $card(S) = x$. Assume $h^1(P^2, \boldsymbol{I}_S(d-3-i)) = 0$. Let $C \subset \boldsymbol{P}^2$ be an integral degree d curve with $S = Sing(C)$ and with only ordinary nodes and ordinary cusps as singularities. Let*

X be the normalization of C (hence $g := g(X) = (d-2)(d-1)/2 - x$). Assume the existence of a point $P \in C_{reg}$ such that the tangent line D to C at P as multiplicity m at P. See P as a point of X, too. Then the part of the gap sequence of the pair (X, P) concerning the divisors tP with $t \leq im$ is the one described in [K2], Th. 2.4, for $m = d-1$, in [K2], Th. 2.5, for $m = d-2$, and one of the a priori possible ones described in [K2], Th. 3.1, for $m = d-2$.

PROOF. By Riemann–Roch and the adjunction formula for all integer $k \leq d-3$ we have $h^1(X, \mathcal{O}_X((d-2-k)mP)) - h^1(X, \mathcal{O}_X((d-3-k)mP)) = h^1(C, \mathcal{O}_C((d-2-k)mP)) - h^1(C, \mathcal{O}_C((d-3-k)mP)) + h^1(\boldsymbol{P}^2, \boldsymbol{I}_S((d-3-k)) - h^1(\boldsymbol{P}^2, \boldsymbol{I}_S((d-2-k))$. By assumption if $k \leq i$ the last two terms vanish. Hence we may repeat verbatim the corresponding parts of the numerical calculations in [K2], §2 and §3.

REMARK 3.2. If S is a general subset of $\boldsymbol{P}^2$ the condition "$h^1(\boldsymbol{P}^2, \boldsymbol{I}_S((d-3-i)) = 0$" is equivalent to the condition "$x \leq (d-i-2) \times (d-1-i)/2$." For the case $1 \leq x \leq 3$, see 3.4.

REMARK 3.3. The numerical calculations in [K2] gives easily many partial informations on the gap sequence of (X, P) also if $m < d-3$. Our recipe 3.1 applies also to these informations.

PROPOSITION 3.4. *Fix the notations and assumptions of* 3.1 *and* 3.2. *Assume also $1 \leq x \leq 3$, $m = d-1$ or $d-2$ and $d \geq 2x+2$. Then the $g-1$ gap values of the Weierstrass points P of X are the first $g-1$ of the $g-1+2x$ integers of the non gaps listed in [K2], Th. 2.4, for $m = d-1$, [K2]. Th. 2.5, for $m = d-2$.*

PROOF. Since $x \leq h^0(\boldsymbol{P}^2, \mathcal{O}(1))$ and if $x = 3$ S is assumed to be not collinear, by 3.1 it remains only to consider the gaps tP with $t > (d-4)m$ for $x = 1$ and $t > (d-3)m$ for $x = 2$, 3. Hence we take $i = d-3$ for $x = 1$ and $i = d-4$ for $x = 2$, 3 and copy the proofs in [K2] with no change. For instance if $m = d-1$ it remains only to prove that $(d-3)(d-1)+1$ is a nongap for P, i.e. we are in the case $i = j = d-2$ considered in the proof of [K2], Th. 2.4.

The proofs in [K2], §2 and §3, were based on a few lemmas on Weierstrass pairs proved in [K1]. We plan in a future paper to change the point of view and show that the work in this section may be applied to Weierstrass pairs (and Weierstrass triples, and so on) coming from ramification loci.

§4. Here we will show that Proposition 1.1 (and [K2], Th. 1.1) is reasonably sharp. Indeed we will construct for many integers d, m several pairs (C,P) with C degree d integral plane curve and $P \in C_{reg}$ such that the tangent line D to C at P has $m_P(C,D) = m$ and P is not a Weierstrass point of the normalization X of C. In 4.5 we will prove that in the boundary case $2m = d$ for smooth plane curves for a general such pair $(C,P)P$ is an ordinary Weierstrass point of C. First we will consider the case of smooth degree d plane curves and take as m any integer $\leq (d-1)/2$. We fix integers d, m $d \geq 4$, $m \leq (d-1)/2$, $P \in \boldsymbol{P}^2$, a line D with $P \in D$ and the length m subscheme B with $B \subset D$ and $B_{\mathrm{red}} = \{P\}$. As in section 2, if $S \subset \boldsymbol{P}^2$ set $V(B,d,S) :=$ {plane degree d curves C with $B \subset C$ and $S \subseteq \mathrm{Sing}(C)$} and $V(B,d) := V(B,d,\varnothing)$.

Lemma 4.1. *Assume* $x := card(S) \leq (m^2+m)/2$, $(d-m)^2 \geq 2x$ *and* $d \geq m+3$. *Then we have* $\dim(V(B,d,S)) = (d^2+3d)/2 - 3card(S) - m$ *and a general* $C \in V(B,d,S)$ *is integral and its only singularities are* x *ordinary nodes at the points of* S. *If* $x = 0$, *the same is true for every* $d \geq m$, *i.e. for every* $d \geq m$ *a general degree* d *plane curve containing* B *is smooth.*

Proof. To check the smoothness outside $\{P\} \cup S$ we will use Bertini's theorem. To check the assumptions of Bertini's theorem, just use the reducible curves $(A \cup T) \in V(B,d,S)$ with $A \in V(B,m)$, $S \subset A$, $\deg(T) = d-m$ and $S \subset T$. For general such A and T we see also that $A \cup T$ has ordinary nodes at S and it is smooth at P.

Proposition 4.2. *Fix integers* d, m *with* $d \geq 4$ *and* $m \leq [(d-1)/2]$. *Then a general* $C \in V(B,d)$ *is a smooth curve such that* P *is not a Weierstrass point of* C. *In particular there is a smooth degree* d *plane curve* X *with a inflectional tangent of order* m *at a point* P *which is not a Weierstrass point of* X.

Proof. Set $g := (d-1)(d-2)/2$. we assume by contradiction that for a general $C \in V(B,d)P$ is a Weierstrass point of C, i.e. there is a degree $d-3$ curve Y such that scheme $C \cap Y$ contains the divisor gP of C. We divide the proof into 4 steps.

Step 1. Since $V(B',d) \subseteq V(B,d)$ if $B \subset B'$ we may assume $m = [(d-1)/2]$.

Step 2. Here we will show that Y is singular at P. This step will work with no change for the case of plane nodal curves considered in 4.3. Note that the set of all curvilinear subschemes A of $\boldsymbol{P}$ with length$(A) = g$, $A = \{P\}$ and $B \subset A$ is an integral smooth variety of dimension $g-m$ (use for instance $g-1$ blowing ups to reduce the case $m=1$, i.e. to the case of the punctual Hilbert scheme of $C\{x,y\}$ considered in [Br]. Since $\dim V(B,d-3)=g-m-1$ and every curve T with $P\in T_{reg}$ has a unique divisor of degree g with P as support, we conclude.

Step 3. Here (and this is the key step) we will show that Y is not integral. We consider the surfaces $U(i)$, $0\le i\le g$, obtained from the plane with the following sequence of i blowing ups. Set $U(0):=P^2$ and let $U(1)$ be the blowing up of $U(0)$ at P. Call $E(1)$ the exceptional divisor of $U(1)$. For every $i>0$ $U(i+1)$ will be the blowing up of $U(i)$ at a suitable point $P(i)$ of the exceptional divisor $E(i)$ of $U(i)$ corresponding to the blowing up $U(i)\to U(i-1)$. If $i<m$, $P(i)$ is the point containing the strict transform of D. If $i\ge m$ we take as $P(i)$ a general point of $E(i)$. Let $C(i)$ (resp. $Y(i)$) be the strict transform of C (resp. $Y(i)$) into $U(i)$. By the generality of C and the assumption on Y we see that $A(i):=C(i)\cap E(i)\in Y(i)$. Call $V^{**}(i)$ the set of degree $d-3$ plane curves whose strict transform into $Y(i)$ contains $A(j)$ for all $j\le i$. We have $V(B,d)=V^{**}(m)$. Each $V^{**}(i)$ is a projective space, $\dim V^{**}(i+1)\le\dim V^{**}(i)\le\dim V^{**}(i+1)+1$. Since $Y\in V^{**}(g)$, we see the existence of a minimal integer $i<g$ with $V^{**}(i)=V^{**}(i+1)$. By the generality of the point $A(i)$ this implies that every $W\in V^{**}(i)$ has a strict transform which is singular at $A(i)$. We claim that this implies that the difference between the arithmetic genus of the partial normalization of Y which resolve only the singularity of Y at P is at least i. Indeed this difference is given by the sum over all infinitely near points P_j of Y at P of $m_j(m_j-1)/2$ with m_j multiplicity of Y at P_j. Since for $j\le i$ the curve Y has multiplicity at $A(j)$ at least equal to the multiplicity at $A(i)$ which is ≥ 2, we obtain the claim. Our second claim is that $i\ge -(2d-3)$. Indeed, since the Proposition is trivially true for $d=4,5$, we may work by induction from the case $d'=d-2$ to the case $d'=d$. The case $d'=d-2$ shows exactly that $V^{**}(i)\ne V^{**}(i)$ for $i<(d-3)(d-4)/2$. Hence the second claim. Since $g-(2d-3)>p_a(Y)$, by the second claim Y is not integral.

Step 4. Let $W_1,\dots,W_t, t\ge 2$, be the irreducible components of Y (taking c times in this list a component which occur with multiplicity t). Set $e_i=m_P(C,W_i)$ and $d_i:=\deg(W_i)$. We have $m_P(C,Y)=\sum_i e_i\ge g$. Since $m(d-3)<g$, there is an integer i with $e_i\ge m$. Thus $B\subset W_i$. This implies that either $W_i=D$ or $d_i\ge m$. Since C and D are smooth at P, if either $1<d_i<m$ or $d_i=1$ but $W_i\ne D$ we have $e_i:=m_P(C,D)\le d_i$ (see [Fu], Ex. 1.1, part 6). By Step 1 we may assume $m=[(d-1)/2]$. Hence there are at most one component of

Y, say W_1, with $\deg(W_i) \geq m$; by induction on d we may assume $e_i \leq (d_1+1)(d_1+2)/2$. Hence, since $d_1 < d-3$ we find $\sum_i e_i < g$, contradiction.

THEOREM 4.3. *Fix integers d, m, x with $2 \leq m \leq [(d-1)/2]$, $m(d-3) < g := (d-1)(d-2)/2 - x$, $3x+m < (d^2+d)/2$ and $d \geq x+7$. Fix a general $S \subset \boldsymbol{P}^2$ with card$(S) = x$. Then a general $C \in V(B,d,S)$ is an integral curve with x nodes at S as unique singularities and P is not a Weierstrass point of the normalization X of C.*

PROOF. The first assertions is a particular case of Lemma 4.1. Look at the proof of 4.2 in our new set-up. Now set $g := p_a(X) = (d-1)(d-2)/2 - x$. Step 1 and Step 2 of the proof of 4.2 work verbatim. Step 3 works with no change, but only because we have introduced the very restrictive condition "$d \geq x+7$" which is equivalent to the condition "$g - (2d-3) > p_a(Y)$". Then Step 4 works with no change.

Here is the analogous of Proposition 1.2. Its proof follows with no change from the proofs of 4.2 and 4.3.

PROPOSITION 4.4. *Fix integers d, s, m, e, x with $e > 0$, $s > 0$, $d-3 \leq es$, $(d-se-2)(d-se-2-1)/2 > x \geq 0$, $me < (d-2)(d-1)/2 - x$ and $d \geq x+7$. Fix a general $S \subset \boldsymbol{P}^2$ with card$(S) = x$. Let E be a degree s integral curve and $P \in E_{reg}$. Let B be the length m subscheme of E with P as support. Set $W(B,d,S) :=$ {plane degree d curves C with $B \subset C$ and $S \subseteq$ Sing(C)}. Then a general curve $C \in W(B,d,S)$ is integral, with ordinary nodes at the points of S as only singularities and P is not a Weierstrass point of the normalization X of C.*

The proof of Proposition 4.2 gives with no change the following result.

PROPOSITION 4.5. *Assume $2m = d$. Then a general $C \in V(B,d)$ is a smooth plane curve such that P is an ordinary Weierstrass point of C, i.e. $h^0(C, \mathcal{O}_C((g-1)P)) = 1$ and $h^0(C, \mathcal{O}_C(gP)) = h^0(C, \mathcal{O}_C((g+1)P)) = 2$.*

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