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HELICES AND ISOMETRIC IMMERSIONS

By

Shigeo FUEKI

Abstract. Let $f: M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\tilde{M}$. We study the geometry of submanifolds under various assumptions with respect to the first curvature $\tilde{\lambda}_1$ and the second curvature $\tilde{\lambda}_2$ of $\tilde{\sigma} = f \circ \sigma$ in $\tilde{M}$ for a helix σ in M.

Introduction

Let $f: M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\tilde{M}$. K. Nomizu and K. Yano [4] proved the following fact:

If, for some $r > 0$, every circle of radius r in M is a circle in $\tilde{M}$, then M is an extrinsic sphere in $\tilde{M}$. Conversely if M is an extrinsic sphere in $\tilde{M}$, then every circle in M is a circle in $\tilde{M}$.

In this paper, we study relations between isometric immersions and helices. We set $\tilde{\sigma} = f \circ \sigma$ for a curve σ in M. Let p be a point of M and $d \geq 2$. Let $\lambda_1, \ldots, \lambda_{d-1}$ be positive constants. We consider the following conditions (C_1), (C_2) and (C_3):

$$(C_1) \quad \begin{cases} \text{The first curvature } \tilde{\lambda}_1 \text{ of } \tilde{\sigma} \text{ is constant along } \tilde{\sigma} \text{ for every helix } \sigma \\ \text{of order } d \text{ through } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \ (1 \leq i \leq d-1), \end{cases}$$

$$(C_2) \quad \begin{cases} (C_1) \text{ holds and the second curvature } \tilde{\lambda}_2 \text{ of } \tilde{\sigma} \text{ is constant along } \tilde{\sigma} \\ \text{for every helix } \sigma \text{ of order } d \text{ though } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \\ (1 \leq i \leq d-1), \end{cases}$$

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$$(\mathrm{C}_3)\quad \begin{cases} (\mathrm{C}_2) \text{ holds and the second curvature } \tilde{\lambda}_2 \text{ of } \tilde{\sigma} \text{ is independent of the} \\ \text{choice of helix } \sigma \text{ of order } d \text{ through } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \\ (1 \leq i \leq d-1). \end{cases}$$

The result of Nomizu and Yano is given under the condition (C_1) in the case where $d = 2$ and $\tilde{\sigma}$ is a circle for every circle σ. In Section 1, we give notations and equations which are used in this paper. In section 2, we obtain some results under the condition (C_1). In Section 3, we treat the conditions (C_2) and (C_3). In Section 4, we study some curves under the condition (C_2) where $\tilde{M}$ is of constant curvature.

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§1. Preliminaries

In this paper, the differentiability of all geometric objects will be C^∞. Let $f : M \to \tilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold M into an m-dimensional Riemannian manifold $\tilde{M}$. For all local formulas and computations, we may assume f as an imbedding and thus we shall often identify $p \in M$ with $f(p) \in \tilde{M}$. The tangent space T_pM is identified with a subspace $f_*(T_pM)$ of $T_p\tilde{M}$ where f_* is the differential map of f. Letters X, Y and Z (resp. ξ, η and ζ) vector fields tangent (resp. normal) to M. We denote the tangent bundle of M (resp. $\tilde{M}$) by TM (resp. $T\tilde{M}$), unit tangent bundle of M by UM and the normal bundle by $T^\perp M$. Let $\tilde{\nabla}$ and ∇ be the Levi-Civita connections of $\tilde{M}$ and M, respectively. Then the Gauss formula is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where h denotes the second fundamental form. The Weingarten formula is given by

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where A denotes the shape operator and $\nabla^\perp$ the normal connection. Clearly A is related to h as $\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle$, where $\langle\ ,\ \rangle$ denotes the Riemannian metrics of M and $\tilde{M}$. We put $\|\tilde{x}\| = \sqrt{\langle \tilde{x}, \tilde{x}\rangle}$ for $\tilde{x} \in T\tilde{M}$. For the second fundamental form and the shape operator, we define their covariant derivatives by

$$(Dh)(Z, X, Y) = \nabla^\perp_Z(h(X, Y)) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y),$$

$$(DA)_\xi(Y, X) = \nabla_Y(A_\xi X) - A_{\nabla^\perp_Y \xi} X - A_\xi(\nabla_Y X).$$

Furthermore we define the k-th covariant derivative of h as follows:

$$(D^k h)(X_1, X_2, \dots, X_{k+2}) = \nabla_{X_1}((D^{k-1}h)(X_2, \dots, X_{k+2})) - \sum_{i=2}^{k+2} (D^{k-1}h)(X_2, \dots, \nabla_{X_1} X_i, \dots, X_{k+2})$$

where $k \geq 1$ and $D^0 h = h$. If, for the non-negative integers $i_1, i_2, \dots, i_j$ $(j \geq 1)$ satisfying that $i_1 + i_2 + \cdots + i_j = k + 2$ $(k \geq 0)$, $X_1 = X_2 = \cdots = X_{i_1} = X$, $X_{i_1+1} = \cdots = X_{i_2} = Y, \dots,$ $X_{i_{j-1}+1} = \cdots = X_{i_j} = Z$, then a normal vector $(D^k h)(X_1, X_2, \dots, X_{k+2})$ is written as $(D^k h)(X^{i_1}, Y^{i_2}, \dots, Z^{i_j})$. Moreover a tangent vector $(DA)_\xi(X, X)$ will be written as $(DA)_\xi(X^2)$. The submanifold M in $\tilde{M}$ is said to be *isotropic at* $p \in M$ of a constant normal curvature μ if the normal vector $h(x^2)$ satisfies

$$\langle h(x^2), h(x^2) \rangle = \mu^2 \langle x, x \rangle^2$$

for every $x \in T_p M$. The above isotropic condition is equivalent with

$$\mathfrak{S}\langle h(x,y), h(z,w) \rangle = \mathfrak{S}\mu^2 \langle x, y \rangle \langle z, w \rangle \tag{1.1}$$

for x, y, z, $w \in T_p M$, where $\mathfrak{S}$ denote the cyclic sum with respect to x, y and z. (cf. B. O'Neill [5]). If there exists a non-negative function μ on M such that M is isotropic at p of the constant normal curvature $\mu(p)$ for every point of M, then M is said to be an *isotropic submanifold*. In particular, when μ is constant on M, M is said to be *constant isotropic*. The mean curvature vector field H of M is defined by

$$H := \frac{1}{n} \sum_{i=1}^{n} h(e_i^2),$$

where $e_1, \dots, e_n$ is an orthonormal frame at each point of M. If the second fundamental form h satisfies $h(X, Y) = \langle X, Y \rangle H$, then M is called a totally umbilical submanifold. The mean curvature vector field H is said to be parallel if $\nabla^\perp H = 0$. A totally umbilical submanifold with the parallel mean curvature vector field is called an *extrinsic sphere*. If the second fundamental form h vanishes identically, then we call M a totally geodesic submanifold of $\tilde{M}$.

Next we shall define a helix of order d in a Riemannian manifold N. Let $\sigma : I \to N (s \mapsto \sigma(s))$ be a smooth curve in N, where I is an open interval of the real line $\boldsymbol{R}$. We denote the tangent vector field $d\sigma/ds$ of σ by v_1. We call s a *d-regular point of* σ if dim Span$\{\nabla_{v_1}^k v_1(s) \mid k = 0, \dots, d-1\} = d$ where $\nabla_{v_1}^0 v_1 = v_1$ and $\nabla_{v_1}^k v_1 = \nabla_{v_1}(\nabla_{v_1}^{k-1} v_1)$ for $k \geq 1$. If every $s \in I$ is a d-regular point of σ, then σ

is said to be a *d-regular curve*. Note that 1-regular curve is a usual regular curve. Hereafter, in this paper, we assume that all curves are regular and parametrized by arc length. If σ is a d-regular curve, then we put

$$(1.2)\qquad \begin{cases} v_0 := 0, \quad w_0 := v_1, \quad \lambda_0 := 1, \\ v_i := \dfrac{w_{i-1}}{\lambda_{i-1}}, \quad w_i := \nabla_{v_1} v_i + \lambda_{i-1} v_{i-1} \quad \text{and} \quad \lambda_i := \|w_i\| \quad \text{for} \quad 1 \le i \le d. \end{cases}$$

We call λ_i $(1 \le i \le d)$ (resp. w_i) the *i-th curvature* (resp. the *i-th curvature vector field*) and v_i $(2 \le i \le d)$ the $(i-1)$*-th normal vector field.* If σ is a d-regular curve and the d-th curvature λ_d of σ vanishes on I, then we call such a curve a *curve of order d* and $v_1, \dots, v_d$ the *Frenet frame field.* Note that a curve of order one is a geodesic. In the case where σ is a curve of order d, we put

$$(1.3)\qquad v_i := 0, \quad w_i := 0 \quad \text{and} \quad \lambda_i := 0 \quad \text{for } i > d.$$

From (1.2) and (1.3), we have the following Frenet formula of σ

$$(1.4)\qquad \nabla_{v_1} v_j + \lambda_{j-1} v_{j-1} = \lambda_j v_{j+1}$$

for $j \ge 1$. If σ is a curve of order d and λ_i are constant along σ, then we call this a *helix of order d.* Note that a helix of order two is a circle.

§2. Helices in a Riemannian submanifold

Let $f : M \to \tilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold into an m-dimensional Riemannian manifold $\tilde{M}$. Let σ be a helix of order d in M with the i-th curvature $\lambda_i (1 \le i \le d-1)$ and the Frenet frame field $v_1, \dots, v_d$. We set $\tilde{\sigma} := f \circ \sigma$. We have $\tilde{v}_1 = d\tilde{\sigma}/ds = v_1$. From the Gauss formula and the Frenet formula of σ, we get $\tilde{\nabla}_{v_1} v_1 = \lambda_1 v_2 + h(v_1^2)$. Since $\tilde{\sigma}$ is a regular curve, we have

$$(2.1)\qquad \tilde{w}_1 = \lambda_1 v_2 + h(v_1, v_1), \quad \tilde{\lambda}_1^2 = \lambda_1^2 + \langle h(v_1^2), h(v_1^2) \rangle,$$

where $\tilde{w}_1$ is the first curvature vector field of $\tilde{\sigma}$. First we prove the following lemma.

LEMMA 2.1. *Let $d \ge 1$ and $\lambda_1, \dots, \lambda_{d-1}$ be positive constants. Let μ be non-negative constant and $p \in M$. Then the following conditions are equivalent:*

(a) *The first curvature $\tilde{\lambda}_1$ of $\tilde{\sigma}$ at p is equal to μ for every helix σ of order d through p in M with the i-th curvature $\lambda_i (1 \le i \le d-1)$,*

(b) *M is isotropic at p in $\tilde{M}$ of the normal curvature $\sqrt{\mu^2 - \lambda_1^2}$.*

PROOF. Suppose that (a) holds. Let x_0 be any unit tangent vector at p in M. We take a helix σ of order d in M with the i-th curvature $\lambda_i (1 \leq i \leq d-1)$ satisfying that $\sigma(0) = p$ and $v_1(0) = x_0$ where v_1 is the tangent vector field of σ. From (2.1), we have $\mu^2 = \lambda_1^2 + \langle h(x_0^2), h(x_0^2)\rangle$. Hence we get $\langle h(x^2), h(x^2)\rangle = \mu^2 - \lambda_1^2$ for every $x \in U_pM$. Therefore we see that M is isotropic at p. Hence we get (b).

Suppose that (b) holds. Let x_0 be any unit tangent vector at p in M. We take a helix σ of order d in M with the i-th curvature λ_i satisfying that $\sigma(0) = p$ and $v_1(0) = x_0$ where v_1 is the tangent vector field of σ. Set $\tilde{\lambda}_1$ the first curvature of $\tilde{\sigma}$. From (2.1), we have

$$\tilde{\lambda}_1^2(0) = \lambda_1^2 + \langle h(x_0^2) h(x_0^2)\rangle = \lambda_1^2 + (\mu^2 - \lambda_1^2) = \mu^2.$$

Hence we get (a). □

REMARK. If M is isotropic at p of a normal curvature μ, then it is clear from (1.1) that

$$A_{h(x^2)}x = \mu^2 x \quad \text{for } x \in U_pM.$$

Let p be a point of M, $d \geq 2$ and $\lambda_1, \dots, \lambda_{d-1}$ positive constants. We consider the following the condition (C_1):

$$(\mathrm{C}_1) \qquad \left\{ \begin{array}{l} \text{The first curvature } \tilde{\lambda}_1 \text{ of } \tilde{\sigma} \text{ is constant along } \tilde{\sigma} \text{ for every helix } \sigma \\ \text{of order } d \text{ through } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i (1 \leq i \leq d-1). \end{array} \right.$$

From Lemma 2.1, we obtain the following Lemma.

LEMMA 2.2. *Let $d \geq 2$ and $\lambda_1, \dots, \lambda_{d-1}$ be positive constants. Let p be a point of M satisfying* (C_1). *Then M is isotropic at p of the normal curvature $\sqrt{\tilde{\lambda}_1^2 - \lambda_1^2}$ (i.e., $\tilde{\lambda}_1$ is independent of the choice of σ). Moreover we get*

$$\begin{aligned}(2.2) \qquad & \langle h(v,z), (Dh)(y,x,w)\rangle + \langle h(w,z), (Dh)(y,x,v)\rangle \\ & + \langle h(x,z), (Dh)(y,w,v)\rangle + \langle h(w,v), (Dh)(y,x,z)\rangle \\ & + \langle h(x,v), (Dh)(y,w,z)\rangle + \langle h(x,w), (Dh)(y,v,z)\rangle = 0\end{aligned}$$

for every $x, y, z, v, w \in T_pM$.

PROOF. Let x and y be any orthonormal tangent vectors at p in M. We take a helix σ of order d in M with the i-th curvature λ_i satisfying that $\sigma(0) = p$,

$v_1(0) = x$ and $v_2(0) = y$ where v_1 (resp. v_2) is the tangent vector field of σ (resp. the first normal vector field of σ). From (2.1), we get $\tilde{\lambda}_1^2 = \lambda_1^2 + \langle h(v_1^2), h(v_1^2)\rangle$. Applying $\tilde{\nabla}_{v_1}$ to this equation and using the Frenet formula of σ, we get

$$\langle (Dh)(v_1^3), h(v_1^2)\rangle + 2\lambda_1\langle h(v_1, v_2), h(v_1^2)\rangle = 0. \tag{2.3}$$

Moreover, applying $\tilde{\nabla}_{v_1}$ to (2.3) and using the Frenet formula of σ, we get

$$\begin{aligned} (2.4)\quad & \langle (D^2h)(v_1^4), h(v_1^2)\rangle + \langle (Dh)(v_1^3), (Dh)(v_1^3)\rangle + \lambda_1\langle (Dh)(v_2, v_1^2), h(v_1^2)\rangle \\ & + 4\lambda_1\langle (Dh)(v_1^2, v_2), h(v_1^2)\rangle + 4\lambda_1\langle (Dh)(v_1^3), h(v_1, v_2)\rangle \\ & + 4\lambda_1^2\langle h(v_1, v_2), h(v_1, v_2)\rangle + 2\lambda_1^2\langle h(v_1^2), h(v_2^2)\rangle \\ & - 2\lambda_1^2\langle h(v_1^2), h(v_1^2)\rangle + 2\lambda_1\lambda_2\langle h(v_1^2), h(v_1, v_3)\rangle = 0. \end{aligned}$$

From (2.3), we get

$$\langle (Dh)(x^3), h(x^2)\rangle + 2\lambda_1\langle h(x, y), h(x^2)\rangle = 0.$$

Since x and $-y$ are orthonormal tangent vectors and $\lambda_1 > 0$, we obtain that

$$\langle (Dh)(x^3), h(x^2)\rangle = \langle h(x, y), h(x^2)\rangle = 0.$$

Hence we have $\langle h(x^2), h(x, y)\rangle = 0$ for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$ and

$$\langle (Dh)(x^3), h(x^2)\rangle = 0 \tag{2.5}$$

for every $x \in T_pM$. Therefore we get M is isotropic at p of the normal curvature $\sqrt{\tilde{\lambda}_1^2 - \lambda_1^2}$. From Lemma 2.1, we see that $\tilde{\lambda}_1$ is independent of the choice of σ. Also, from (1.1) and (2.4), we get

$$\begin{aligned} & \langle (D^2h)(x^4), h(x^2)\rangle + \langle (Dh)(x^3), (Dh)(x^3)\rangle \\ & \quad + \lambda_1\langle (Dh)(y, x^2), h(x^2)\rangle + 4\lambda_1\langle (Dh)(x^2, y), h(x^2)\rangle + 4\lambda_1\langle (Dh)(x^3), h(x, y)\rangle = 0 \end{aligned}$$

for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$. Since x and $-y$ are orthonormal and $\lambda_1 > 0$, we get

$$\langle (Dh)(y, x^2), h(x^2)\rangle + 4\langle (Dh)(x^2, y), h(x^2)\rangle + 4\langle (Dh)(x^3), h(x, y)\rangle = 0 \tag{2.6}$$

for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$. From (2.5), we have

$$\langle (Dh)(y, x^2), h(x^2)\rangle + 2\langle (Dh)(x^2, y), h(x^2)\rangle + 2\langle (Dh)(x^3), h(x, y)\rangle = 0 \tag{2.7}$$

for every $x, y \in T_pM$. From (2.6) and (2.7), it follows that

$$\langle (Dh)(y,x^2),h(x^2)\rangle = \langle (Dh)(x^2,y),h(x^2)\rangle + \langle (Dh)(x^3),h(x,y)\rangle = 0$$

for every $x,y \in U_pM$ such that $\langle x,y\rangle = 0$. Hence, from (2.5), we see that

$$\langle h(x^2),(Dh)(y,x^2)\rangle = 0 \text{ for every } x,y \in T_pM.$$

Since h is symmetric, we have (2.2) for any tangent vectors x,y,z,v and w at p. □

From Lemma 2.2, we get

Proposition 2.3. *Let M be an n-dimensional connected Riemannian submanifold in an m-dimensional Riemannian manifold $\tilde{M}$ isometrically immersed by f and $n \geq 2$. Let $d \geq 2$ and $\lambda_1,\dots,\lambda_{d-1}$ be positive constants. If the condition* (C_1) *holds at every point of M, then M is a constant isotropic submanifold of $\tilde{M}$.*

Proof. By Lemma 2.2, we see that M is an isotropic submanifold. Then there exists a non-negative function μ on M such that M is isotropic at p of the constant normal curvature $\mu(p)$ for every point p of M. We shall show that the derivative of μ^2 vanishes on M. Let $p \in M$ and $x \in U_pM$ be arbitrarily fixed. For a unit vector field Y on a neighborhood of p, we have

$$x\mu^2 = x\langle h(Y^2),h(Y^2)\rangle = 2\langle (Dh)(x,Y^2),h(Y^2)\rangle|_{\mathrm{at}\,p} + 4\langle h(\nabla_xY,Y),h(Y^2)\rangle|_{\mathrm{at}\,p}.$$

Since the equation (2.2) holds and $\langle \nabla_xY,Y\rangle = 0$, we get $x\mu^2 = 0$. Hence we see that M is constant isotropic. □

§3. The discriminant of the second fundamental form

Let M, $\tilde{M}$ and f be as in §2. Let σ be a helix of order d in M with the i-th curvature $\lambda_i (1 \leq i \leq d-1)$ and the Frenet frame field $v_1,\dots,v_d$. Let $\tilde{\lambda}_i (1 \leq i)$ be the i-th curvature of $\tilde{\sigma}$. By a routine calculation, we have the following lemma.

Lemma 3.1. *The tangent vector field $\tilde{v}_1$ and the first curvature vector field $\tilde{w}_1$ of $\tilde{\sigma}$ are given by*

$$\tilde{v}_1 = v_1, \quad \tilde{w}_1 = \lambda_1 v_2 + h(v_1^2).$$

If $\tilde{\lambda}_1$ is constant along $\tilde{\sigma}$ then the second curvature vector field $\tilde{w}_2$ of $\tilde{\sigma}$ is given by

$$\tilde{\lambda}_1\tilde{w}_2 = (\tilde{\lambda}_1^2 - \lambda_1^2)v_1 + \lambda_1\lambda_2 v_3 - A_{h(v_1^2)}v_1 + 3\lambda_1 h(v_1,v_2) + (Dh)(v_1^3). \tag{3.1}$$

Moreover, If $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are constant along $\tilde{\sigma}$, then the third curvature vector field $\tilde{w}_3$ of $\tilde{\sigma}$ is given by

$$(3.2)\quad \tilde{\lambda}_1\tilde{\lambda}_2\tilde{w}_3 = \lambda_1(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2-\lambda_1^2-\lambda_2^2)v_2+\lambda_1\lambda_2\lambda_3 v_4-(DA)_{h(v_1^2)}(v_1^2)$$
$$-5\lambda_1 A_{h(v_1,v_2)}v_1-\lambda_1 A_{h(v_1^2)}v_2-2A_{(Dh)(v_1^3)}v_1-h(v_1,A_{h(v_1^2)}v_1)$$
$$+(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2-4\lambda_1^2)h(v_1^2)+3\lambda_1^2 h(v_2^2)+4\lambda_1\lambda_2 h(v_1,v_3)$$
$$+5\lambda_1(Dh)(v_1^2,v_2)+\lambda_1(Dh)(v_2,v_1^2)+(D^2h)(v_1^4).$$

We prove the following lemma.

LEMMA 3.2. *Let p be a point of $M, d\geq 2$ and $\lambda_1,\cdots,\lambda_{d-1}$ positive constants. If, for every helix σ of order d through p in M with the i-th curvature λ_i $(1\leq i\leq d-1)$,*

$$(3.3)\quad v_1\langle h(v_1,v_2),(Dh)(v_1^3)\rangle=0 \text{ at } p$$

where v_1 (resp. v_2) is the tangent vector field of σ (resp. the first normal vector field of σ), then we have

$$(3.4)\quad \langle (Dh)(x^2,y),(Dh)(x^3)\rangle+\langle (D^2h)(x^4),h(x,y)\rangle=0$$

for every $x,y\in U_pM$ *such that* $\langle x,y\rangle=0$.

PROOF. Let x and y be any orthonormal tangent vectors at p in M. We take a helix σ of order d in M with the i-th curvature λ_i satisfying that $\sigma(0)=p$, $v_1(0)=x$ and $v_2(0)=y$. By assumption, we have

$$0=v_1\langle h(v_1,v_2),(Dh)(v_1^3)\rangle|_{s=0}$$
$$=\langle (Dh)(v_1^2,v_2),(Dh)(v_1^3)\rangle|_{s=0}+\langle h(\nabla_{v_1}v_1,v_2),(Dh)(v_1^3)\rangle|_{s=0}$$
$$+\langle h(v_1,\nabla_{v_1}v_2),(Dh)(v_1^3)\rangle|_{s=0}+\langle h(v_1,v_2),(D^2h)(v_1^4)\rangle|_{s=0}$$
$$+\langle h(v_1,v_2),(Dh)(\nabla_{v_1}v_1,v_1^2)\rangle|_{s=0}+2\langle h(v_1,v_2),(Dh)(v_1^2,\nabla_{v_1}v_1)\rangle|_{s=0}$$
$$=\langle (Dh)(x^2,y),(Dh)(x^3)\rangle+\lambda_1\langle h(y^2),(Dh)(x^3)\rangle-\lambda_1\langle h(x^2),(Dh)(x^3)\rangle$$
$$+\lambda_2\langle h(x,v_3(0)),(Dh)(x^3)\rangle+\langle h(x,y),(D^2h)(x^4)\rangle$$
$$+\lambda_1\langle h(x,y),(Dh)(y,x^2)\rangle+2\lambda_1\langle h(x,y),(Dh)(x^2,y)\rangle$$

where v_3 is the second normal vector field of σ. If $d=2$, then $v_3=0$. Since x and $-y$ are orthonormal, we have (3.4). If $d \geq 3$, then we can take a unit vector $z(\in T_pM)$ satisfying that $v_3(0)=z$. Also since $x, -y$ and z are orthonormal, we have (3.4). □

Let σ be a helix of order d in M and $d \geq 2$. From (2.1), we have $\tilde{\lambda}_1 \geq \lambda_1 > 0$ where $\tilde{\lambda}_1$ (resp. λ_1) is the first curvature of $\tilde{\sigma}$ (resp. the first curvature of σ). Thus $\tilde{\sigma}$ is a 2-regular curve. Let p be a point of M, $d \geq 2$ and $\lambda_1 \cdots \lambda_{d-1}$ positive-constants. We consider the following conditions (C_2) and (C_3):

$$(\mathrm{C}_2) \quad \begin{cases} (\mathrm{C}_1) \text{ holds and the second curvature } \tilde{\lambda}_2 \text{ of } \tilde{\sigma} \text{ is constant along } \tilde{\sigma} \\ \text{for every helix } \sigma \text{ of order } d \text{ through } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \\ (1 \leq i \leq d-1), \end{cases}$$

$$(\mathrm{C}_3) \quad \begin{cases} (\mathrm{C}_2) \text{ holds and the second curvature } \tilde{\lambda}_2 \text{ of } \tilde{\sigma} \text{ is independent of the} \\ \text{choice of helix } \sigma \text{ of order } d \text{ through } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \\ (1 \leq i \leq d-1). \end{cases}$$

For $x \in UM$, we set

$$v(x) := \langle (Dh)(x^3), (Dh)(x^3) \rangle.$$

We prove the following lemma.

Lemma 3.3. *Let $d \geq 2$ and $\lambda_1, \ldots, \lambda_{d-1}$ be positive constants. Let p be a point of M satisfying* (C_2). *Then v is constant on U_pM if and only if* (3.4) *holds for every $x, y \in U_pM$ such that $\langle x, y \rangle = 0$. Moreover, we get*

$$\begin{aligned} (3.5) \qquad & 15\lambda_1^2 \langle (Dh)(x^2, y), h(x, y) \rangle + 3\lambda_1^2 \langle (Dh)(y, x^2), h(x, y) \rangle \\ & + 3\lambda_1^2 \langle (Dh)(x^3), h(y^2) \rangle + \langle (D^2h)(x^4), (Dh)(x^3) \rangle = 0 \end{aligned}$$

for every $x, y \in U_pM$ such that $\langle x, y \rangle = 0$. Moreover, if $d \geq 3$, then we have

$$(3.6) \qquad \langle (Dh)(x^3), h(x, y) \rangle = \langle (Dh)(y, x^2), h(x^2) \rangle = \langle (Dh)(x^2, y), h(x^2) \rangle = 0$$

for every $x, y \in T_pM$.

Proof. Let x and y be any orthonormal tangent vectors at p in M. We take a helix σ of order d in M with the i-th curvature λ_i satisfying that $\sigma(0)=p$,

$v_1(0) = x$ and $v_2(0) = y$ where v_1 (resp. v_2) is the tangent vector field of σ (resp. the first normal vector field of σ). Since (2.2), (3.1) and (3.2) hold and M is isotropic at p by Lemma 2.2, we obtain

$$\begin{aligned}
0 &= \langle \tilde{\lambda}_1 \tilde{w}_2, \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{w}_3 \rangle|_{s=0} \\
&= 9\lambda_1^2 \lambda_2 \langle h(x,y), h(x, v_3(0)) \rangle + 3\lambda_1 \lambda_2 \langle (Dh)(x^3), h(x, v_3(0)) \rangle \\
&\quad + 15\lambda_1^2 \langle (Dh)(x^2, y), h(x,y) \rangle + 3\lambda_1^2 \langle (Dh)(y, x^2), h(x,y) \rangle \\
&\quad + 3\lambda_1 \langle (D^2 h)(x^4), h(x,y) \rangle + 3\lambda_1^2 \langle (Dh)(x^3), h(y^2) \rangle \\
&\quad + 5\lambda_1 \langle (Dh)(x^2, y), (Dh)(x^3) \rangle + \lambda_1 \langle (Dh)(y, x^2), (Dh)(x^3) \rangle \\
&\quad + \langle (D^2 h)(x^4), (Dh)(x^3) \rangle
\end{aligned}$$

where v_3 is the second normal vector field of σ.

If $d = 2$, then $v_3 = 0$. We have

$$\begin{aligned}
(3.7) \qquad & 15\lambda_1^2 \langle (Dh)(x^2, y), h(x,y) \rangle + 3\lambda_1^2 \langle (Dh)(y, x^2), h(x,y) \rangle \\
&+ 3\lambda_1^2 \langle (Dh)(x^3), h(y^2) \rangle + 3\lambda_1 \langle (D^2 h)(x^4), h(x,y) \rangle \\
&+ 5\lambda_1 \langle (Dh)(x^2, y), (Dh)(x^3) \rangle + \lambda_1 \langle (Dh)(y, x^2), (Dh)(x^3) \rangle \\
&+ \langle (D^2 h)(x^4), (Dh)(x^3) \rangle = 0.
\end{aligned}$$

If $d \geq 3$, we can take a unit vector $z (\in T_p M)$ satisfying $v_3(0) = z$. Since x, y and $-z$ are orthonormal, we get (3.7) and

$$(3.8) \qquad 9\lambda_1^2 \lambda_2 \langle h(x,y), h(x,z) \rangle + 3\lambda_1 \lambda_2 \langle (Dh)(x^3), h(x,z) \rangle = 0.$$

In any case, we see that (3.7) holds for every $x, y \in U_p M$ such that $\langle x, y \rangle = 0$. Since x and $-y$ are orthonormal and $\lambda_1 > 0$, we obtain that (3.5) and

$$\begin{aligned}
&3(\langle h(x,y), (D^2 h)(x^4) \rangle + \langle (Dh)(x^3), (Dh)(x^2, y) \rangle) \\
&= 2\langle (Dh)(x^3), (Dh)(x^2, y) \rangle + \langle (Dh)(x^3), (Dh)(y, x^2) \rangle
\end{aligned}$$

for every $x, y \in U_p M$ such that $\langle x, y \rangle = 0$. If (3.4) holds, then we have

$$2\langle (Dh)(x^2, y), (Dh)(x^3) \rangle + \langle (Dh)(y, x^2), (Dh)(x^3) \rangle = 0$$

for every $x, y \in U_p M$ such that $\langle x, y \rangle = 0$. Hence we get ν is constant on $U_p M$. The converse is rather clear.

Here, we assume that $d \geq 3$. Since (3.8) holds for $x, -y, z$ and $\lambda_1\lambda_2 > 0$, we have $\langle (Dh)(x^3), h(x,z)\rangle = 0$ for every $x, y \in U_pM$ such that $\langle x, z\rangle = 0$. From this equation and (2.2), we have (3.6). □

Let p be a point of M. The discriminant Δ at p of the second fundamental form h is given by

$$\Delta_{xy} = \frac{\langle h(x^2), h(y^2)\rangle - \|h(x,y)\|^2}{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}$$

for linearly independent tangent vectors $x, y \in T_pM$.

We assume that p is a point of M satisfying (C_2). We take a helix σ of order d through p and put $v_1(0) = x$ and $v_2(0) = y$ where $d \geq 2$. From (2.3) and the fact that M is isotropic at p, we get

$$9\lambda_1^2\langle h(x,y), h(x,y)\rangle + 6\lambda_1\langle (Dh)(x^3), h(x,y)\rangle + v(x) + \lambda_1^2\lambda_2^2 - \tilde{\lambda}_1^2\tilde{\lambda}_2^2 = 0. \tag{3.9}$$

for $\tilde{\sigma}$. In particular, if (3.6) holds, then we get

$$9\lambda_1^2\langle h(x,y), h(x,y)\rangle + v(x) + \lambda_1^2\lambda_2^2 - \tilde{\lambda}_1^2\tilde{\lambda}_2^2 = 0. \tag{3.10}$$

Moreover, from (1.1), we get

$$\Delta_{xy} = (\tilde{\lambda}_1^2 - \lambda_1^2) - \frac{1}{3\lambda_1^2}(\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - \lambda_1^2\lambda_2^2 - v(x)). \tag{3.11}$$

From Lemma 3.2 and Lemma 3.3, we have the following theorem:

THEOREM 3.4. *Let M be an n-dimensional connected Riemannian submanifold in an m-dimensional Riemannian manifold $\tilde{M}$ isometrically immersed by f* and *$n \geq 3$. Let $d \geq 3$ and $\lambda_1, \dots, \lambda_{d-1}$ be positive constants. Suppose that the condition* (C_1) *holds at every point of M. Let p be a point of M. If the condition* (C_2) *holds at p, then v is constant on U_pM. Moreover the discriminant Δ at p is constant if and only if the condition* (C_3) *holds at p.*

In case of $d = 2$, we shall prove that (3.6) holds at p under the condition (C_3). We have the following lemma.

LEMMA 3.5. *Let $d = 2$ and λ_1 be a positive constant. Let p be a point of M satisfying* (C_3). *Then we have* (3.6) *for every $x, y \in T_pM$.* Moreover we get (3.10) *and* (3.11) *for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$.*

PROOF. We have (3.9) for any $x, y \in U_pM$ such that $\langle x, y\rangle = 0$. Since x and $-y$ are orthonormal and p is a point satisfying (C_3), we obtain $\lambda_1\langle (Dh)(x^3), h(x, y)\rangle = 0$ and (3.10). From (1.1), we obtain (3.11). Since $\lambda_1 > 0$ and (2.2) holds, we get (3.6). □

From the definition of discriminant, we have the following theorem.

THEOREM 3.6. *Let M be an n-dimensional connected Riemannian submanifold in an m-dimensional Riemannian manifold $\tilde{M}$ isometrically immersed by f and $n \geq 3$. Let $d \geq 2$ and $\lambda_1, \dots, \lambda_{d-1}$ be positive constants. Let p be a point of M satisfying the condition* (C_3). *Then v is constant on U_pM and the discriminant Δ at p is constant.*

PROOF. Let x, y, z be orthonormal in T_pM. Set $x(\theta) = \cos\theta x + \sin\theta y$. From (3.11), we get

$$v(x(\theta)) = \langle (Dh)(x(\theta)^3), (Dh)(x(\theta)^3)\rangle = \langle (Dh)(z^3), (Dh)(z^3)\rangle = v(z) \tag{3.12}$$

Differentiating (3.12) at $\theta = 0$, we see that

$$\langle (Dh)(y, x^2), (Dh)(x^3)\rangle + 2\langle (Dh)(x^2, y), (Dh)(x^3)\rangle = 0.$$

Therefore we have v is constant on U_pM. It is clear that the discriminant Δ at p is constant. □

In case of $n = 2$, from Lemma 2.2, we get the following lemma.

LEMMA 3.7. *Let $n = 2$ and $d = 2$. Let λ_1 be a positive constant and p a point of M satisfying* (C_1). *Then the discriminant Δ is constant at p and*

$$\|h(x, y)\|^2 = \frac{\tilde{\lambda}_1^2 - \lambda_1^2 - \Delta}{3} \quad \text{and} \quad \langle h(x^2), h(y^2)\rangle = \frac{\tilde{\lambda}_1^2 - \lambda_1^2 + 2\Delta}{3} \tag{3.13}$$

for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$. Thus $\|h(x, y)\|$ and $\langle h(x^2), h(y^2)\rangle$ are constant for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$.

PROOF. Let x, y be orthonormal in T_pM. Set $x(\theta) = \cos\theta x + \sin\theta y$ and $y(\theta) = -\sin\theta x + \cos\theta y$. Since M is isotropic at p, we get

$$\frac{d}{d\theta}\Delta_{x(\theta)y(\theta)} = 4\langle h(y(\theta)^2), h(x(\theta), y(\theta))\rangle - 4\langle h(x(\theta)^2), h(x(\theta), y(\theta))\rangle = 0.$$

Hence we get $\Delta_{x(\theta)y(\theta)} = \Delta_{xy}$. From the definition of Δ, we get (3.13) for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$. $\square$

From Theorem 3.6, Lemma 3.7 and Theorem 1 in [5], we get

COROLLARY 3.8. *Let $d \geq 2$ and $\lambda_1, \ldots, \lambda_{d-1}$ be positive constants. If* (C_3) *holds for every point of M and $m - n < (n+2)(n-1)/2$, then M is a totally umbilic submanifold of $\tilde{M}$. Moreover, at every point $p \in M$, we get*

$$\langle H, H\rangle = \tilde{\lambda}_1^2 - \lambda_1^2,$$

$$\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - \lambda_1^2\lambda_2^2 = \langle \nabla_x H, \nabla_x H\rangle$$

for every $x \in U_pM$ where H is the mean curvature vector field of M.

REMARK. In Corollary 3.8, we see that M is an extrinsic sphere if and only if $\tilde{\lambda}_1^2\tilde{\lambda}_2^2 = \lambda_1^2\lambda_2^2$. Then $\tilde{\lambda}_2 \leq \lambda_2$.

§4. Curves in a Riemannian manifold of constant curvature

Let M be an n-dimensional connected Riemannian submanifold in an m-dimensional Riemannian manifold $\tilde{M}$ of constant curvature c isometrically immersed by f. From the Codazzi equation, it is known that

$$R(x, y)z = c\{\langle y, z\rangle x - \langle x, z\rangle y\} + A_{h(y,z)}x - A_{h(x,z)}y, \tag{4.1}$$

$$(Dh)(x, y, z) = (Dh)(y, x, z), \tag{4.2}$$

$$R^{\perp}(x, y)\xi = h(x, A_\xi y) - h(A_\xi x, y) \tag{4.3}$$

for $x, y, z \in TM$ and $\xi \in T^{\perp}M$ where R and $R^{\perp}$ are the curvature tensor of ∇ and $\nabla^{\perp}$. From (4.2) and Lemma 2.2, we get

LEMMA 4.1. *Let p be a point of M, $d = 2$ and λ_1 a positive constant. If* (C_1) *holds at p, then we obtain* (3.6) *for every $x, y \in T_pM$.*

From Lemma 3.2, Lemma 3.3 and Lemma 4.1, we get the following theorem.

THEOREM 4.2. *Let M be an n-dimensional connected Riemannian submanifold in an m-dimensional Riemannian manifold $\tilde{M}$ of constant curvature c isometrically* *immersed by f and $n \geq 2$. Let $d = 2$ and λ_1 be a positive constant. Suppose that the condition* (C_1) *holds at every point of M. Let p a point of M. If the condition* (C_2) *holds at p, then v is constant on U_pM and the condition* (C_3) *holds at p.*

Let p be a point of M and α a constant. We define a (0,6)-tensor F by

$$\begin{aligned}F(x,y,z,u,v,w) := {} & \langle (Dh)(x,y,z), (Dh)(u,v,w)\rangle \\ & - \alpha\frac{1}{9}\{\langle y,z\rangle\langle x,u\rangle\langle v,w\rangle + \langle y,z\rangle\langle x,v\rangle\langle u,w\rangle \\ & + \langle y,z\rangle\langle x,w\rangle\langle u,v\rangle + \langle x,z\rangle\langle y,u\rangle\langle v,w\rangle \\ & + \langle x,z\rangle\langle y,v\rangle\langle u,w\rangle + \langle x,z\rangle\langle y,w\rangle\langle u,v\rangle + \langle x,y\rangle\langle z,u\rangle\langle v,w\rangle \\ & + \langle x,y\rangle\langle z,v\rangle\langle u,w\rangle + \langle x,y\rangle\langle z,w\rangle\langle u,v\rangle\}\end{aligned}$$

for $x,y,z,u,v,w \in T_pM$. We have the following Lemma 4.3. The proof of Lemma 4.3 is analogous to that of Lemma 2 in [5].

LEMMA 4.3. *Let $\tilde{M}$ be of constant curvature, p a point of M and α a constant. Then the following conditions are equivalent:*

(a) $\langle (Dh)(x,x,x), (Dh)(x,x,x)\rangle = \alpha\langle x,x\rangle^3$ *for every* $x \in T_pM$,

(b) $$\begin{aligned}& F(x,y,z,u,v,w) + F(x,y,u,v,w,z) + F(x,y,v,w,z,u) + F(x,y,w,z,u,v) \\ & + F(x,u,w,y,z,v) + F(x,z,v,y,u,w) + F(x,z,u,y,v,w) \\ & + F(x,v,w,y,z,u) + F(x,z,w,y,v,u) + F(x,v,u,y,z,w) = 0\end{aligned}$$

for $x,y,z,u,v,w \in T_pM$.

Let $n = 2$. We assume that $p \in M$ is a point satisfying all conditions of Theorem 4.2. Let $N_1(p)$ be the first normal space at p given by $\mathrm{Span}\{h(x,y) \mid x, y \in T_pM\}$. Let e_1, e_2 be an orthonormal base of T_pM. Put

$$h_{ij} := h(e_i, e_j) \quad \text{for } 1 \leq i,j \leq 2,$$

$$Dh_{ijk} := (Dh)(e_i, e_j, e_k) \quad \text{for } 1 \leq i,j,k \leq 2.$$

Since v is constant on U_qM for every point $q \in M$, we see that v is a function defined on M. We put

$$v(p) = \langle Dh_{111}, Dh_{111}\rangle.$$

From Lemma 4.3 and (3.6), we get

$$(4.4)\qquad \begin{cases} \langle Dh_{111}, Dh_{111}\rangle = \langle Dh_{222}, Dh_{222}\rangle = \nu(p), \\ \langle Dh_{111}, Dh_{112}\rangle = 0, \\ \langle Dh_{111}, Dh_{222}\rangle + 9\langle Dh_{112}, Dh_{122}\rangle = 0, \end{cases}$$

$$(4.5)\qquad \begin{cases} \langle Dh_{111}, h_{11}\rangle = \langle Dh_{222}, h_{22}\rangle = 0, \\ \langle Dh_{111}, h_{12}\rangle = \langle Dh_{112}, h_{11}\rangle = \langle Dh_{222}, h_{12}\rangle = \langle Dh_{122}, h_{22}\rangle = 0, \\ \langle Dh_{111}, h_{22}\rangle + 3\langle Dh_{112}, h_{12}\rangle = 0, \\ \langle Dh_{122}, h_{11}\rangle + \langle Dh_{112}, h_{12}\rangle = \langle Dh_{112}, h_{22}\rangle + \langle Dh_{122}, h_{12}\rangle = 0. \end{cases}$$

Let K be the Gauss curvature of M. Then $K = c + \Delta$. From Lemma 2.2 and Theorem 1 in [5], we get

$$-2(\tilde{\lambda}_1^2 - \lambda_1^2) \le \Delta(p) \le \tilde{\lambda}_1^2 - \lambda_1^2,$$

$$\dim N_1(p) = 0 \Leftrightarrow \Delta(p) = \tilde{\lambda}_1^2 - \lambda_1^2 = 0\ (\textit{i.e.}, \tilde{\lambda}_1 = \lambda_1) \Leftrightarrow p \text{ is a geodesic point},$$

$$\dim N_1(p) = 1 \Leftrightarrow \Delta(p) = \tilde{\lambda}_1^2 - \lambda_1^2 > 0 \Leftrightarrow p \text{ is a non-geodesic umbilic point},$$

$$\dim N_1(p) = 2 \Leftrightarrow \Delta(p) = -2(\tilde{\lambda}_1^2 - \lambda_1^2) < 0 \Leftrightarrow p \text{ is a non-geodesic minimal point},$$

$$\dim N_1(p) = 3 \Leftrightarrow -2(\tilde{\lambda}_1^2 - \lambda_1^2) < \Delta(p) < \tilde{\lambda}_1^2 - \lambda_1^2.$$

We shall prove the following Lemma.

LEMMA 4.4. *Let $n = 2$ and $m \le 5$. Let $d = 2$ and λ_1 be a positive constant. We assume that* (C_1) *holds at every point of M. Let p be a point of M. If* (C_2) *holds at p and $2 \le \dim N_1(p) \le 3$, then $\nu(p) = 0$ (i.e., the second fundamental form h is parallel at p).*

PROOF. We assume that $\dim N_1(p) = 2$. we obtain $N_1(p) = \mathrm{Span}\{h_{11}, h_{12}\}$. Moreover p is a minimal point of M i.e.,

$$(4.6)\qquad h_{11} = -h_{22}.$$

From (4.5) and (4.6), we have

$$\langle Dh_{111}, h_{11}\rangle = \langle Dh_{111}, h_{12}\rangle = 0,$$

$$\langle Dh_{222}, h_{12}\rangle = 0,$$

$$\langle Dh_{222}, h_{11}\rangle = -\langle Dh_{222}, h_{22}\rangle = 0,$$
$$\langle Dh_{112}, h_{11}\rangle = 0,$$
$$\langle Dh_{112}, h_{12}\rangle = -\langle Dh_{122}, h_{11}\rangle = \langle Dh_{122}, h_{22}\rangle = 0.$$

Hence we have $Dh_{111}, Dh_{222}, Dh_{112} \perp N_1(p)$. Since $\dim T_p^{\perp}M \leq 3$ and $\langle Dh_{111}, Dh_{111}\rangle = \langle Dh_{222}, Dh_{222}\rangle$ in (4.4), we have

$$Dh_{111} = \pm Dh_{222}.$$

Moreover, from (4.4), we get

$$\begin{cases} \langle Dh_{111}, Dh_{112}\rangle = 0, \\ \pm\langle Dh_{111}, Dh_{111}\rangle + 9\langle Dh_{112}, Dh_{122}\rangle = 0. \end{cases}$$

Hence we obtain $Dh_{111} = 0$.

We assume that $\dim N_1(p) = 3$. We obtain $T_p^{\perp}M = N_1(p) = \mathrm{Span}\{h_{11}, h_{12}, \xi\}$ such that $\langle \xi, \xi\rangle = 1$ and h_{11}, h_{12} and ξ are mutually orthogonal. Since $\langle Dh_{111}, h_{11}\rangle = \langle Dh_{111}, h_{12}\rangle = 0$ in (4.4), we have

$$Dh_{111} = \pm\|Dh_{111}\|\xi.$$

Suppose that $\|Dh_{111}\| \neq 0$. Since $\langle Dh_{111}, Dh_{112}\rangle = \langle Dh_{112}, h_{11}\rangle = 0$ in (4.4) and (4.5), we have $Dh_{112} = ah_{12}$ $(a \in \boldsymbol{R})$. Since $\langle Dh_{112}, h_{22}\rangle + \langle Dh_{122}, h_{12}\rangle = \langle Dh_{222}, h_{11}\rangle + 3\langle Dh_{122}, h_{12}\rangle = 0$ in (4.5) and $\langle h_{22}, h_{12}\rangle = 0$, we get

$$\langle Dh_{222}, h_{11}\rangle = \langle Dh_{122}, h_{12}\rangle = \langle Dh_{112}, Dh_{122}\rangle = 0. \tag{4.7}$$

Since $\langle Dh_{111}, Dh_{222}\rangle + 9\langle Dh_{112}, Dh_{122}\rangle = 0$ in (4.5), we have

$$\langle Dh_{222}, Dh_{111}\rangle = \langle Dh_{222}, \xi\rangle = 0. \tag{4.8}$$

From (4.7), (4.8) and $\langle Dh_{222}, h_{12}\rangle = 0$ in (4.5), we have $Dh_{222} = 0$. This contradicts the assertion $\|Dh_{111}\| \neq 0$. Hence we have $Dh_{111} = 0$. □

From Proposition 2.3 and Lemma 4.4, we get the following lemma.

LEMMA 4.5. *Let n, m d and λ_1 be as in Lemma 4.4. If* (C$_2$) *holds at every point of M, then $v \equiv 0$ on M (i.e., the second fundamental from h is parallel). Moreover $\|H\|$ is constant on M where H is the mean curvature vector field and*

$$\|H\|^2 = \frac{1}{3}(\Delta + 2(\tilde{\lambda}_1^2 - \lambda_1^2)).$$

Thus the discriminant Δ *is constant on* M *and the dimension of the first normal space is constant on* M. *Moreover, if the dimension of the first normal space is greater that two, we get*

$$\Delta = \frac{1}{4}(\tilde{\lambda}_1^2 - \lambda_1^2 - 3c). \tag{4.9}$$

Proof. Let $U := \{p \in M \mid \nu(p) > 0\}$. We shall prove that $U = \emptyset$ ($\emptyset$ is the empty set). Assume that the assertion is false. From Lemma 4.4, we see that $\dim N_1(p) \leq 1$ for every point p of U. Hence U is totally umbilic. Then we obtain that the second fundamental form is parallel because of the assumption that $\tilde{M}$ is of constant curvature and $\dim U = 2$. Hence we obtain $\nu(p) = 0$ for every point $p \in U$. This contradicts the assertion that $\nu(p) > 0$ for every point $p \in U$. Hence we have $\nu \equiv 0$ on M. Since M is constant isotropic and the second fundamental form is parallel, we obtain that $\|H\|$ is constant on M and the discriminant Δ is constant on M. From Ricci identity, (4.1), (4.2), (4.3) and the fact that M is constant isotropic, we get

$$\begin{aligned}(D^2h)(x,y,x^2) - (D^2h)(y,x^3) &= R^{\perp}(x,y)h(x^2) - 2h(R(x,y)x,x)\\ &= \{2(\tilde{\lambda}_1^2 - \lambda_1^2 + c) - 8\|h(x,y)\|^2\}h(x,y)\end{aligned}$$

for every $x, y \in UM$ such that $\langle x, y\rangle = 0$. Since $\nu \equiv 0$ on M and (3.13), we have (4.9). $\square$

From Lemma 4.5, we the following theorem.

Theorem 4.6. *Let* M *be a two-dimensional connected Riemannian submanifold in an m-dimensional Riemannian manifold* $\tilde{M}$ *of constant curvature* c *isometrically immersed by* f *and* $m \leq 5$. *Let* $d = 2$ *and* λ_1 *be a positive constant. If the condition* (C_2) *holds for every point of* M, *then the second fundamental form* h *is parallel on* M *and* M *is one of the following*:

(a) *an extrinsic sphere of constant curvature* $c + \tilde{\lambda}_1^2 - \lambda_1^2$,

(b) *a non-geodesic minimal submanifold of constant curvature* $c/3$ $(> 0, c = 3(\tilde{\lambda}_1^2 - \lambda_1^2))$,

(c) *a non-minimal submanifold of constant curvature* $(c + \tilde{\lambda}_1^2 - \lambda_1^2)/4$ $(> 0, c \neq 3(\tilde{\lambda}_1^2 - \lambda_1^2), \tilde{\lambda}_1 > \lambda_1)$.

If, for every geodesic γ in M, $f \circ \gamma$ is a helix of order $\tilde{d}$ with curvatures $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{\tilde{d}-1}$ which do not depend on γ, then f is said to be a *helical immersion of* *order* $\tilde{d}$. Let γ be a geodesic in M and v_1 the tangent vector field of γ. From (2.1), we have

$$\left\{\begin{array}{l}\tilde{\nabla}_{v_1}v_1 = h(v_1^2),\\ \tilde{\nabla}_{v_1}h(v_1^2) = -A_{h(v_1^2)}v_1 + (Dh)(v_1^3).\end{array}\right. \tag{4.10}$$

From (4.10), Proposition 2.3 and Theorem 4.6, we obtain the following fact.

COROLLARY 4.7. *Let* f, M, $\tilde{M}$, n, m d *and* λ_1 *be as in Theorem* 4.6. *Suppose that* (C_2) *holds at every point of* M. *Then* f *is a helical immersion of order at most two.*

We assume that all conditions of Theorem 4.6 hold. Let p be a point of M and σ a circle through p in M with the first curvature λ_1 and v_1, v_2 the Frenet frame fields of σ. Since $Dh = 0$, M is constant isotropic, $\tilde{\sigma}$ is a 2-regular curve and (C_2) holds, we see that

$$\tilde{\lambda}_1\tilde{w}_2 = 3\lambda_1 h(v_1, v_2), \tag{4.11}$$

$$\tilde{\lambda}_1\tilde{\lambda}_2\tilde{w}_3 = -\frac{\tilde{\lambda}_2^2}{3\lambda_1}(\tilde{\lambda}_1^2 - 3\lambda_1^2)v_2 + (\tilde{\lambda}_2^2 - 3\lambda_1^2)h(v_1^2) + 3\lambda_1^2 h(v_2^2) \tag{4.12}$$

by Lemma 3.1. Let $I_\sigma = \{s \in I \mid \tilde{w}_3(s) = 0\}$ where I is the domain of σ.

If $I_\sigma \neq \emptyset$, then we have $\tilde{\lambda}_2 = 0$ or $\tilde{\lambda}_2 = \sqrt{2}\tilde{\lambda}_1 = \sqrt{6}\lambda_1$.

In the case where $\tilde{\lambda}_2 = 0$, we obtain that $\tilde{\sigma}$ is a circle. Since $h(v_1(0), v_2(0)) = 0$ and $n = 2$, we have $h(x, y) = 0$ and $h(x^2) = h(y^2)$ for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$. Hence we see that $\tilde{\sigma}$ is a circle for every circle σ through p with the first curvature λ_1. Then it is clear that the case (a) of Theorem 4.6 holds.

In the case where $\tilde{\lambda}_2 = \sqrt{2}\tilde{\lambda}_1 = \sqrt{6}\lambda_1$, from (4.12), we obtain that $\tilde{w}_3 = \sqrt{2}H_\sigma$ where $H_\sigma = (h(v_1^2) + h(v_2^2))/2$. Since $Dh = 0$ and M is constant isotropic, we have $\tilde{\lambda}_3 = \|\tilde{w}_3\|$ is constant on I. Hence we have $\tilde{\lambda}_3 = 0$, i.e., $\tilde{\sigma}$ is a helix of order three satisfying that $\tilde{\lambda}_2 = \sqrt{2}\tilde{\lambda}_1 = \sqrt{6}\lambda_1$. Since $h(v_1^2(0)) + h(v_2(0)^2) = 0$, $\|h(v_1(0), v_2(0))\| = \|h(v_1^2(0))\|$ and $n = 2$, we have $\|h(x, y)\| = \|h(x^2)\| = \|h(y^2)\|$ for every $x, y \in U_pM$ such that $\langle x, y\rangle = 0$. Hence we see that $\tilde{\sigma}$ is a helix of order three satisfying that $\tilde{\lambda}_2 = \sqrt{2}\tilde{\lambda}_1 = \sqrt{6}\lambda_1$ for every circle σ through p with the first curvature λ_1. It is clear that the case (b) of Theorem 4.6 holds.

If $I_\sigma = \emptyset$, then $\tilde{\sigma}$ is a 4-regular curve. From (4.11), (4.12) and the fact that M is constant isotropic, we have

$$\tilde{\lambda}_3^2 = \frac{\tilde{\lambda}_1^2\tilde{\lambda}_2^2}{9\lambda_1^2} - \tilde{\lambda}_2^2 + 4\lambda_1^2. \tag{4.13}$$

From (4.13), we have $\tilde{\lambda}_3$ is constant along $\tilde{\sigma}$. Moreover, from (4.11), (4.12) and (4.13), we get

$$\tilde{\lambda}_1\tilde{\lambda}_2\tilde{\lambda}_3\tilde{\nabla}_{v_1}\tilde{v}_4 = -\tilde{\lambda}_2\tilde{\lambda}_3^2\tilde{w}_2 = \tilde{\lambda}_1\tilde{\lambda}_2\tilde{\lambda}_3(-\tilde{\lambda}_3\tilde{v}_3). \tag{4.14}$$

From (4.14), we obtain that $\tilde{\sigma}$ is a helix of order four. Then it is clear that the case (c) of Theorem 4.6 holds. Therefore, from Theorem 4.6, we have the following corollary.

COROLLARY 4.8. *Let f, M, $\tilde{M}$, n, m d and λ_1 be as in Theorem* 4.6. *Suppose that* (C_2) *holds at every point of M. Then $\tilde{\sigma}$ is one of the following*:

(a) *a circle with the first curvature $\tilde{\lambda}_1$ satisfying $\tilde{\lambda}_1 \geq \lambda_1$ for every circle σ with the first curvature λ_1,*

(b) *a helix of order three with the first curvature $\tilde{\lambda}_1$ and the second curvature $\tilde{\lambda}_2$ satisfying $\tilde{\lambda}_2 = \sqrt{2}\tilde{\lambda}_1 = \sqrt{6}\lambda_1 = \sqrt{c}(c > 0)$ for every circle σ with the first curvature λ_1,*

(c) *a helix of order four with the first curvature $\tilde{\lambda}_1$, the second curvature $\tilde{\lambda}_2$ and the third curvature $\tilde{\lambda}_3$ satisfying*

$$\tilde{\lambda}_1 > \lambda_1, \quad \tilde{\lambda}_2 = \frac{3\lambda_1\sqrt{c + \tilde{\lambda}_1^2 - \lambda_1^2}}{2\tilde{\lambda}_1^2}, \quad \tilde{\lambda}_3 = \frac{\sqrt{c + \tilde{\lambda}_1^2 - 4\tilde{\lambda}_2^2 + 15\lambda_1^2}}{2}$$

for every circle σ with the first curvature λ_1.

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Department of Mathematics
Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku
Tokyo 162-0827, Japan