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A DUALITY ON *CMC*-1 SURFACES IN HYPERBOLIC SPACE, AND A HYPERBOLIC ANALOGUE OF THE OSSERMAN INEQUALITY

Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday

By

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Abstract. We will show the existence of a duality on *CMC*-1 surfaces in hyperbolic 3-space, and we will show an analogue of the Osserman Inequality in terms of dual surfaces. Moreover, we will show that equality holds (in this analogue) if and only if all the ends of the surface are regular and embedded.

Introduction

The total curvature of a complete immersed minimal surface $x : M^2 \to \boldsymbol{R}^3$ of finite total curvature satisfies the Osserman Inequality ([3]):

$$\frac{1}{2\pi}\int_{M^2} K\,dA \leq (\chi(M^2) - n),$$

where K is the Gaussian curvature of the surface and n is the number of ends. Furthermore, equality holds if and only if all of the ends of the surface are embedded ([2]).

CMC-1 surfaces (i.e. surfaces of constant mean curvature 1) in hyperbolic 3-space $H^3(-1)$ of constant curvature -1 have quite similar properties to minimal surfaces in $\boldsymbol{R}^3$. In fact, Bryant established an analogue of the Weierstrass representation formula for the case of *CMC*-1 surfaces in $H^3(-1)$. However, for the total curvature of these *CMC*-1 surfaces, an analogue of the Osserman Inequality does not hold directly. In a previous paper [6], the authors showed that complete immersed *CMC*-1 surfaces in $H^3(-1)$ only satisfy the

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Cohn-Vossen Inequality

$$\frac{1}{2\pi}\int_{M^2} K\,dA < \chi(M^2),$$

and equality never holds in this inequality. Such a difference mainly comes from the fact that the total curvature of CMC-1 surfaces in $H^3(-1)$ is not necessarily an integral multiple of 4π.

We will show the existence of a duality on CMC-1 surfaces and an analogue of the Osserman Inequality in terms of dual surfaces: Let M^2 be a Riemann surface and $x : M^2 \to H^3(-1)$ a complete conformal CMC-1 immersion of finite total curvature. Then its dual CMC-1 immersion $x^\sharp : \tilde{M}^2 \to H^3(-1)$, defined on the universal cover $\tilde{M}^2$ of M^2, is obtained by exchanging the hyperbolic Gauss map and the secondary Gauss map of the original CMC-1 immersion x. Though the dual CMC-1 immersion $x^\sharp$ may not be single-valued on M^2, its first fundamental form $ds^{\sharp 2}$ is defined on M^2 itself. Moreover, its total curvature on M^2 is an integral multiple of 4π. In this paper, we show the following

THEOREM. *Let M^2 be a Riemann surface and $x : M^2 \to H^3(-1)$ a complete conformal CMC-1 immersion of finite total curvature. Then the following inequality holds*: ($dA^\sharp$ *is the volume element of the dual surface* $x^\sharp$.)

$$\frac{1}{2\pi}\int_{M^2} K^\sharp\,dA^\sharp \leq (\chi(M^2) - n), \tag{1}$$

where $K^\sharp$ is the Gaussian curvature of the dual surface $x^\sharp$ and n is the number of ends of the original CMC-1 surface x. Equality holds if and only if all the ends of x are regular and embedded.

Other useful applications of the duality will be found in a forthcoming paper [4]. The authors thank Wayne Rossman for informative conversations.

Preliminaries

Let M^2 be a Riemann surface and $x : M^2 \to H^3(-1)$ a complete conformal CMC-1 immersion of finite total curvature. Then, there is a null holomorphic immersion $F : \tilde{M}^2 \to PSL(2,\mathbb{C})$ defined on the universal cover $\tilde{M}^2$ of M^2 such that $x = FF^*$. (Such an F is uniquely determined up to the ambiguity Fb for $b \in SU(2)$.) Here we use the identification (See [1,6])

$$H^3(-1) = \{X \in \mathrm{Herm}(2); \det(X) = 1, \mathrm{trace}(X) > 0\}.$$

We define a meromorphic function G by

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}, \tag{2}$$

where $F = (F_{ij})_{i,j=1,2}$. Then G is single-valued on M^2; that is, G can be considered to be a meromorphic function on M^2. The function G is called the *hyperbolic Gauss map* of x. ([1])

Since x induces a non-positively curved complete metric ds^2 of finite total curvature, there is a compact Riemann surface $\overline{M}^2$ and a finite number of points $\{p_1, \ldots, p_n\} \in \overline{M}^2$ such that $M^2 = \overline{M}^2 \setminus \{p_1, \ldots, p_n\}$. The hyperbolic Gauss map G does not necessarily extend meromorphically on $\overline{M}^2$. The end p_j is called a *regular* end if p_j is at most a pole of G. If p_j is not regular, it is called an *irregular* end. We can set

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega, \tag{3}$$

where g is a meromorphic function defined on $\tilde{M}^2$, and ω is a holomorphic 1-form defined on $\tilde{M}^2$. We call the pair (g, ω) the *Weierstrass data* of the CMC-1 immersion x. g is called the *secondary Gauss map* of x. (g also has $SU(2)$-ambiguity with respect to the choice of Fb $(b \in SU(2))$.) In terms of the Weierstrass data (g, ω), the first fundamental form ds^2 and the second fundamental form Φ are written as

$$ds^2 = (1 + |g|^2)^2 \omega \cdot \bar{\omega}, \qquad \Phi = -\omega \cdot dg - \overline{\omega \cdot dg} + ds^2, \tag{4}$$

where "$\cdot$" means the symmetric product. The holomorphic quadratic differential $Q = \omega \cdot dg$ is called the *Hopf differential* of x. By (4), Q is single-valued on M^2. Moreover, Q can be meromorphically extended on the compactification $\overline{M}^2$ of M^2 ([1]). The hyperbolic Gauss map G, the secondary Gauss map g and the Hopf differential Q satisfy the following identity ([6, 7]):

$$S(g) - S(G) = 2Q, \tag{5}$$

where $S(g) = S_z(g)\, dz^2$ and $S_z(g)$ is the Schwarzian derivative of g. The Schwarzian derivative is defined as

$$S_z(g) = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \qquad (' = d/dz).$$

Using this relation (5), the following lemma is obtained. (cf. [6; Lemma 2.3])

LEMMA 1 ([1; Prop.6]). *The end p_j is regular if and only if the order of Q at p_j is greater than or equal to -2.*

Now we set $d\sigma^2 = (-K)ds^2$, where K is the Gaussian curvature of ds^2. Then $d\sigma^2$ is a pseudometric of constant curvature 1. (See [1] or [7].) It follows from (4) that

$$d\sigma^2 = \frac{4dg \cdot d\bar{g}}{(1+|g|^2)^2}. \tag{6}$$

Hence $d\sigma^2$ is the pull back of the canonical Riemannian metric $d\sigma_0^2$ on the unit 2-sphere induced by $g : M^2 \to \boldsymbol{C} \cup \{\infty\} \cong S^2(1)$. By (4) and (6) we have

$$ds^2 \cdot d\sigma^2 = 4Q \cdot \bar{Q}. \tag{7}$$

DEFINITION ([5]). A conformal pseudometric $d\sigma^2$ on a Riemann surface $\overline{M}^2$ has a finite singularity of order β ($\beta \in \boldsymbol{R}$) at a point $p \in \overline{M}^2$ if $d\sigma^2$ has a local expression $d\sigma^2 = e^{2\omega} dz \cdot d\bar{z}$ around p such that $\omega - \beta \log|z - z(p)|$ is continuous at p. We denote the value β by $\mathrm{Ord}_p(d\sigma^2)$. When $\beta > -1$, we say that $d\sigma^2$ has a *conical singularity* at the point p.

Using this terminology, $d\sigma^2$ has a conical singularity at each end. ([1; Prop.4])

For a meromorphic function f on $\overline{M}^2$, we denote by $b_f(p)$ the branching number of f at $p \in \overline{M}^2$. We set $d\sigma_f^2 = 4df \cdot d\bar{f}/(1+|f|^2)^2$. Then one can easily check that $b_f(p) = \mathrm{Ord}_p(d\sigma_f^2)$. We denote by $\mathrm{Ord}_p(Q)$ the order of the first non-vanishing term of the Laurent expansion of the Hopf differential Q at $p \in M^2$. We prepare some lemmas.

LEMMA 2 ([1; p346]). *Let $p \in \overline{M}^2$ be a point such that $\mathrm{Ord}_p(Q) \geq -2$. Suppose that the Hopf differential Q has the Laurent expansion $Q(z) = (q_{-2}/(z-p)^2 + \cdots)dz^2$. Then the following identity holds:*

$$(b_G(p)+1)^2 - (\mathrm{Ord}_p(d\sigma^2)+1)^2 = 4q_{-2}. \tag{8}$$

PROOF. By Proposition 4 in [1], there exists a coordinate z around p such that the secondary Gauss map g is of the form

$$g = z^{\mu} \qquad (\mu > 0, \mu \in \boldsymbol{R}).$$

On the other hand, G has the following expansion:

$$G(z) \text{ or } 1/G(z) = G(p) + (z-p)^{\ell}(G_0 + G_1(z-p) + \cdots) \qquad (\ell > 0, \ell \in \boldsymbol{Z}, G_0 \neq 0).$$

So we have

$$S_z(G) = \frac{1}{2}(1-\ell^2)\frac{1}{(z-p)^2} + \cdots,$$
$$S_z(g) = \frac{1}{2}(1-\mu^2)\frac{1}{(z-p)^2} + \cdots.$$

By (5), we get the identity $\ell^2 - \mu^2 = 4q_{-2}$. By (6), we have that $\mathrm{Ord}_p(d\sigma^2) = \mu - 1$. Since $b_G(p) = \ell - 1$, this proves the lemma.

LEMMA 3. *Suppose that p_j is a regular end of x. Then the following inequality holds:*

$$b_G(p_j) - \mathrm{Ord}_{p_j}(Q) \geq 2. \tag{9}$$

Equality holds if and only if the end p_j is embedded.

PROOF. Take a coordinate z around $p = p_j$ as in the proof of Lemma 2. By Theorem 5.2 in [6], the end p_j is embedded if and only if $m = \min\{m_1, m_2\} = 1$, where m_1 and m_2 are the positive integers given by

$$m_1 = \sqrt{(\nu+1)^2 + 4q_{-2}}, \qquad m_2 = \sqrt{(2\mu+\nu+1)^2 + 4q_{-2}},$$

and

$$\nu = \mathrm{Ord}_{p_j}(\omega) = \mathrm{Ord}_{p_j}(Q/dg) = \mathrm{Ord}_{p_j}(Q) - \mu + 1.$$

Since p_j is a regular end, we have that $\mathrm{Ord}_p(Q) \geq -2$, by Lemma 1. If $\mathrm{Ord}_{p_j}(Q) = -2$, then by Lemma 2,

$$1 \leq m_1 = m_2 = \sqrt{\mu^2 + 4q_{-2}} = \ell.$$

Since $\mathrm{Ord}_p(Q) = -2$ and $b_G(p_j) = \ell - 1$, this implies that (9) holds and that equality in (9) holds if and only if $m = 1$.

Next, we assume that $\mathrm{Ord}_{p_j}(Q) > -2$. In this case $q_{-2} = 0$, and by Lemma 2, $\mu = \ell$. Thus we have

$$m_1 = |\mathrm{Ord}_{p_j}(Q) - \ell + 2|, \qquad m_2 = |\mathrm{Ord}_{p_j}(Q) + \ell + 2|.$$

Since $\mathrm{Ord}_{p_j}(Q)+2>0$, $m_1<m_2$ holds. So

$$m=m_1=|\mathrm{Ord}_{p_j}(Q)-\ell+2|. \tag{10}$$

On the other hand, by (7), we have

$$\mathrm{Ord}_{p_j}(ds^2)+\mathrm{Ord}_{p_j}(d\sigma^2)=\mathrm{Ord}_{p_j}(Q). \tag{11}$$

Since ds^2 is complete at p_j,

$$\mathrm{Ord}_{p_j}(ds^2)<-1 \tag{12}$$

holds, by Corollary 4.2 in [6].

Since $\mathrm{Ord}_{p_j}(d\sigma^2)=\mu-1=\ell-1$ is an integer, (11) and (12) yield

$$-3+\ell\geq\mathrm{Ord}_{p_j}(Q). \tag{13}$$

This implies (9). Moreover, by (10) and (13), we have

$$1\leq m=\ell-2-\mathrm{Ord}_{p_j}(Q).$$

So equality in (9) holds if and only if $m=1$. This proves the lemma.

Duality on *CMC*-1 surfaces

Let $x: M^2\to H^3(-1)$ be a complete *CMC*-1 surface of finite total curvature, as in the previous section. Let G be the hyperbolic Gauss map, (g,ω) the Weierstrass data, and $Q=\omega\cdot dg$ the Hopf differential of x. The dual surface $x^\sharp$ of x is obtained by exchanging the hyperbolic Gauss map and the secondary Gauss map of the original *CMC*-1 immersion x. We show this property from the following definition of the dual surface. (The authors thank the referee of the Bull. London Math. Soc. for suggesting the following definition.)

DEFINITION. The dual *CMC*-1 immersion $x^\sharp:\tilde{M}^2\to H^3(-1)$ associated with the Weierstrass data (g,ω) of the *CMC*-1 immersion x is defined by

$$x^\sharp=(F^{-1})(F^{-1})^*,$$

where F is the lift of x with respect to (g,ω) (namely, F satisfies (3)).

$x^\sharp$ is not necessarily single-valued on M^2. With [7; Cor. 2.4], one can easily show that $x^\sharp$ is single-valued if and only if g is single-valued on M^2. Let $(g^\sharp,\omega^\sharp)$ be a pair defined by

$$(F^\sharp)^{-1}dF^\sharp=\begin{pmatrix} g^\sharp & -(g^\sharp)^2 \\ 1 & -g^\sharp \end{pmatrix}\omega^\sharp, \tag{14}$$

where $F^\sharp = F^{-1}$. Then $(g^\sharp, \omega^\sharp)$ is the Weierstrass data of $x^\sharp$. Let $(x^\sharp)^\sharp$ be a dual CMC-1 immersion associated with $(g^\sharp, \omega^\sharp)$. Then by the definition, the relation $(x^\sharp)^\sharp = x$ is obvious. It should also be remarked that the congruent class of the dual $x^\sharp$ is independent of the choice of the Weierstrass data (g, ω) of x, but it does depend on the position of the surface x. While x and axa^* are congruent if and only if $a \in SL(2, C)$, $x^\sharp$ and $(axa^*)^\sharp$ are congruent if and only if $a \in SU(2)$.

PROPOSITION 4. *Let $x^\sharp$ be the dual CMC-1 immersion of x with respect to the Weierstrass data (g, ω). Then the hyperbolic Gauss map $G^\sharp$, the Weierstrass data $(g^\sharp, \omega^\sharp)$, and the Hopf differential $Q^\sharp$ of $x^\sharp$ are given by*

$$G^\sharp = g, \quad g^\sharp = G, \quad \omega^\sharp = -Q/dG, \quad Q^\sharp = -Q.$$

PROOF. We set $F = (F_{ij})$. Then by (2), we have

$$F'_{11}F'_{22} - F'_{12}F'_{21} = 0$$

where $' = d/dz$. Using this, the following identity is easily obtained

$$G = \frac{F'_{11}}{F'_{21}} = \frac{F'_{11}F_{22} - F'_{12}F_{21}}{F'_{21}F_{22} - F'_{22}F_{21}}. \tag{15}$$

Now we set $F^\sharp = F^{-1}$. Since $dF^{-1} = -F^{-1}(dF)F^{-1}$, we have

$$(F^\sharp)^{-1}dF^\sharp = -(dF)F^{-1} = -\begin{pmatrix} F'_{11}F_{22} - F'_{12}F_{21} & -F'_{11}F_{12} + F'_{12}F_{11} \\ F'_{21}F_{22} - F'_{22}F_{21} & -F'_{21}F_{12} + F'_{22}F_{11} \end{pmatrix}. \tag{16}$$

Hence by (14), (15), and (16), we have

$$G = g^\sharp.$$

Replacing F by $F^\sharp$, we also have

$$G^\sharp = g.$$

So by (5), we have $Q^\sharp = -Q$ and hence $\omega^\sharp = -Q/dG$. This proves the proposition.

By (4) and Proposition 4, the first fundamental form of $x^\sharp$ is given by

$$ds^{\sharp 2} = (1 + |G|^2)^2 \frac{Q}{dG} \cdot \overline{\left(\frac{Q}{dG}\right)}. \tag{17}$$

Since G and Q are single-valued on M^2, so is the metric $ds^{\sharp 2}$.

LEMMA 5. *Suppose that all the ends of x are regular. Then the induced metric $ds^{\sharp 2}$ of the dual CMC-1 immersion $x^{\sharp}$ is a complete Riemannian metric on M^2.*

PROOF. We set

$$d\sigma^{\sharp 2} = \frac{4dG \cdot d\bar{G}}{(1+|G|^2)^2}. \tag{18}$$

By (7), we have that

$$ds^2 \cdot d\sigma^2 = 4Q \cdot \bar{Q} = 4(-Q) \cdot \overline{(-Q)} = ds^{\sharp 2} \cdot d\sigma^{\sharp 2}. \tag{19}$$

Since G is single valued on M^2, so is $d\sigma^{\sharp 2}$. By (18), we have $b_G(p_j) = \mathrm{Ord}_{p_j}(d\sigma^{\sharp 2})$. So (9) is equivalent to

$$\mathrm{Ord}_{p_j}(d\sigma^{\sharp 2}) \geq \mathrm{Ord}_{p_j}(Q) + 2. \tag{20}$$

Since $x^{\sharp}$ is an immersion, $ds^{\sharp 2}$ is positive definite. By (19), we have

$$\mathrm{Ord}_{p_j}(d\sigma^{\sharp 2}) + \mathrm{Ord}_{p_j}(ds^{\sharp 2}) = \mathrm{Ord}_{p_j}(Q).$$

Combining this with (20), we have that $\mathrm{Ord}_{p_j}(ds^{\sharp 2}) \leq -2$. In particular, we have $\mathrm{Ord}_{p_j}(ds^{\sharp 2}) \leq -1$, which implies that $ds^{\sharp 2}$ is complete at p_j.

PROOF OF THE THEOREM. If x has irregular ends, G has essential singularities at those ends. By (18) and the relation $d\sigma^{\sharp 2} = (-K^{\sharp})ds^{\sharp 2}$, we see that $ds^{\sharp 2}$ has infinite total curvature on M^2. So we may assume that all the ends of x are regular. We can directly apply (20), instead of (4.3) in [6], to the proof of Theorem 4.3 in [6]. Then we have the inequality (1). Equality in (1) holds if and only if equality holds in (20); that is, if and only if

$$\mathrm{Ord}_{p_j}(d\sigma^{\sharp 2}) = \mathrm{Ord}_{p_j}(Q) + 2 \tag{21}$$

holds for each $j = 1, \cdots, n$. On the other hand, $\mathrm{Ord}_{p_j}(d\sigma^{\sharp 2}) = b_G(p_j)$, by (18). So by Lemma 3, (21) holds if and only if all the ends of x are regular and embedded. This proves the theorem.

REMARK. Let $\overline{M}^2$ be a compact Riemann surface and $x: \overline{M}^2 \setminus \{p_1, \ldots, p_n\} \to \boldsymbol{R}^3$ a complete conformal minimal immersion of finite total curvature. Then the Gauss map G and the Hopf differential Q of x are given by

$$G = \frac{\partial x_3}{\partial x_1 - i\partial x_2}, \qquad Q = (\partial x_1 - i\partial x_2) \cdot dG,$$

where $x=(x_1,x_2,x_3)$. It is well-known that G and Q can both be meromorphically extended to $\overline{M}^2$. By [2], it can be easily checked that an end p_j of x is embedded if and only if equality holds in (9). So the embedding criterion for regular ends of CMC-1 surfaces is the same as that for minimal surfaces. Finally, by our numerical experiments, we would like to propose the following:

PROBLEM. *Are any irregular ends of CMC-1 surfaces non-embedded?*

ADDED IN PROOF. Recently, Zuhuan Yu proved Lemma 5 without assuming regularity of ends ("Value distribution of hyperbolic Gauss maps", to appear in Proceedings of the American Mathematical Society). Namely, the dual of any complete CMC-1 surface is also complete.

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