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PROPAGATION OF ANALYTICITY OF SOLUTIONS TO THE CAUCHY PROBLEM FOR WEAKLY HYPERBOLIC SEMI-LINEAR EQUATIONS

By

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Abstract. We consider a weakly hyperbolic operator with constant coefficients. We shall derive a priori estimates for it and by applying the estimate we prove local existence of the solution of semi-linear Cauchy problem and investigate the propagation of analyticity of the solutions.

1. Introduction

We consider the linear partial differential operator of order m with constant coefficients

$$P = P(D_t, D_x) = D_t^m + \sum_{j+|\alpha| \leq m, j<m} a_{j,\alpha} D_t^j D_x^\alpha$$

in the $n+1$ variables (t, x), where $D_t = -i\partial/\partial t$, $D_{x_k} = -i\partial/\partial x_k$ and $D_x = (D_{x_1}, \dots, D_{x_n})$. Let $\tau_{m,j}(\xi)$ be the roots of the characteristic polynomial $P(\tau, \xi) = \tau^m + \sum_{j+|\alpha|\leq m, j<m} a_{j,\alpha}\tau^j\xi^\alpha$ for $j = 1, \dots, m$.

DEFINITION 1.1. *Let $s \geq 1$. A differential operator P with a symbol $P(\tau, \xi)$ is said to be s-hyperbolic with respect to $(1, 0, 0, \dots, 0)$ if there exists a non-negative constant C such that*

$$|\mathrm{Im}\, \tau_{m,j}(\xi)| \leq C\langle\xi\rangle^{1/s} \quad \textit{for all } \xi \in \boldsymbol{R}^n,$$

where $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$. Especially, when $s = \infty$, that is $1/s = 0$, P is said to be hyperbolic with respect to t.

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E. Larsson introduced the s-hyperbolicity in [8] and solved the Cauchy problem for s-hyperbolic operators in Gevrey classes by using Laplace transformation. In this paper we shall obtain semi-group estimates of the solution to the Cauchy problem for s-hyperbolic operators and moreover by applying this estimates we can investigate propagation of analyticity of solutions to the Cauchy problem.

We consider the following $m+1$ polynomials $H_{m-k}(\tau,\xi)$, $k=0,\dots,m$, which result from m differentiation of $P(\tau,\xi)$ with respect to τ.

$$H_{m-k}(\tau,\xi)=\frac{(m-k)!}{m!}\frac{\partial^k}{\partial\tau^k}P(\tau,\xi)=\prod_{j=1}^{m-k}(\tau-\tau_{m-k,j}(\xi)),$$

for $k=0,\dots,m$, where we number the roots $\tau_{m-k,j}(\xi)$ to be continuous and let each H_{m-k} be a pseudo-differential operator with a symbol $H_{m-k}(\tau,\xi)$. Put $Hu=(H_0u,H_1u,\dots,H_{m-1}u)$. We note that H_{m-k} is s-hyperbolic if P is s-hyperbolic. From each polynomial $H_{m-k}(\tau,\xi)$ we now create $m-k$ new polynomials $P^j_{m-k-1}(\tau,\xi)$, $j=1,\dots,m-k$, of degree $m-k-1$, by crossing out one factor at a time.

$$P^j_{m-k-1}(\tau,\xi)=\prod_{l=1,l\neq j}^{m-k}(\tau-\tau_{m-k,l}(\xi)).$$

From elementary considerations it follows that

$$H_{m-k}(\tau,\xi)=\frac{1}{m-k+1}\sum_{j=1}^{m-k+1}P^j_{m-k}(\tau,\xi)$$

for $k=1,\dots,m$.

We introduce some function spaces, called Gevrey classes, and their norms. For $\rho\geq 0$, $s>1$, and $m\in\boldsymbol{R}$, we define

$$\boldsymbol{H}^m_{\rho,s}(\boldsymbol{R}^n)=\{u\in\boldsymbol{L}^2_x(\boldsymbol{R}^n);\langle\xi\rangle^m e^{\rho\langle\xi\rangle^{1/s}}\hat{u}(\xi)\in\boldsymbol{L}^2_\xi(\boldsymbol{R}^n)\},$$

where $\hat{u}(\xi)$ stands for a Fourier transform of $u(x)$ and for $\rho<0$ define $\boldsymbol{H}^m_{\rho,s}(\boldsymbol{R}^n)$ as the dual space of $\boldsymbol{H}^{-m}_{-\rho,s}(\boldsymbol{R}^n)$. If $\rho>0$, $\boldsymbol{H}^m_{\rho,s}(\boldsymbol{R}^n)$ is a Hilbert space with a norm $\|u\|_{\boldsymbol{H}^m_{\rho,s}}=\|\langle\xi\rangle^m e^{\rho\langle\xi\rangle^{1/s}}\hat{u}(\xi)\|_{\boldsymbol{L}^2}$. Put $\boldsymbol{L}^2_s(\boldsymbol{R}^n)=\bigcap_{\rho>0}\boldsymbol{H}^0_{\rho,s}(\boldsymbol{R}^n)$.

For a topological space X we denote by $C^k([0,T];X)$ the set of functions which are k times differentiable in X with respect to t in $[0,\ T]$.

THEOREM 1.1. *Let $1<s<s_0\leq\infty$. Assume that P is s_0-hyperbolic of order m. Then for arbitrary $T>0$ there are $\rho_0>0$, $\rho_1<0$ and $C>0$ such that to any $t\in(0,T)$ and $l\geq 0$,*

$$\|Hu(t,\cdot)\|_{H^l_{\rho(t),s}} \le C\left\{\sum_{k=1}^{m}\sum_{j=1}^{m-k+1}\|P^j_{m-k}u(0,\cdot)\|_{H^l_{\rho_0,s}} + \int_0^t \|Pu(t',\cdot)\|_{H^l_{\rho(t'),s}}\,dt'\right\}$$

for any $u(t,x)\in C^m([0,T];L^2_s(\boldsymbol{R}^n))$, *where* $\rho(t)=\rho_1 t+\rho_0$.

We remark that when P is ∞-hyperbolic, that is hyperbolic in the sense of Gårding, a priori estimate of P was derived by G. Peyser [10], [11]. Applying Theorem 1.1, we can solve the Cauchy problem for semi-linear equations and investigate the propagation of the analyticity of the solutions.

For $s>1$ and open set $B\subset \boldsymbol{R}^n$, we denote by $\gamma_\rho^{\{s\}}(B)$ the set of all functions satisfying the following condition: there exists a constant $C>0$ such that

$$|D_x^\alpha u(x)|\le C|\alpha|!^s\rho^{|\alpha|}$$

for any $x\in B$ and $\alpha\in \boldsymbol{N}^n$. Put $\gamma^{(s)}(B)=\bigcup_{\rho>0}\gamma_\rho^{(s)}(B)$ and $\gamma^{\{s\}}(B)=\bigcap_{\rho>0}\gamma_\rho^{(s)}(B)$.

For Ω, an open domain of $\boldsymbol{C}^m$, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions in Ω.

For an open set B in $\boldsymbol{R}^n$ and an open domain Ω in $\boldsymbol{C}^n$, we denote by $\gamma_\rho^{(s)}(B;\mathcal{O}(\Omega))$ the set of all functions which are in Gevrey class with respect to x-variables and uniformly holomorphic with respect to z-variables, in the following sense: for any $K\Subset\Omega$ there exists a constant $C_K>0$ such that

$$|D_x^\alpha f(x,z)|\le C_K\rho^{-|\alpha|}|\alpha|!^s,$$

for all $x\in B$ and $z\in K$.

We consider the following semi-linear Cauchy problem in $(0,T)\times\boldsymbol{R}^n$:

$$\begin{cases} P(D)u(t,x)=F(t,x,Hu) \\ D_t^j u(0,x)=u_j(x) \quad j=0,\dots,m-1, \end{cases} \tag{1.1}$$

where $F(t,x,z)$ is complex-valued function. Set $u^{(0)}(t,x)=\sum_{j=0}^{m-1}(it)^j u_j(x)/j!$.

The function

$$F:[0,T]\times\boldsymbol{R}^n\times\Omega\to\boldsymbol{C},$$

where Ω is open in $\boldsymbol{C}^m$ and contains the origin, is assumed to satisfy the following conditions:

$(\mathrm{A1})_s$: $F(t,x,z)$ is continuous in t, belongs to Gevrey class $\gamma_{\sigma_1}^{(s)}(\boldsymbol{R}^n)$ with respect to x and belongs to $\mathcal{O}(\Omega)$ with respect to z.

$(\mathrm{A2})_s$: There exists a constant $\sigma_2>0$ such that

$$F(t,\cdot,Hu^{(0)}(t,\cdot))\in H^l_{\sigma_2,s}(\boldsymbol{R}^n).$$

Then we get the following local existence theorem and investigate the propagation of analyticity of the solutions.

Theorem 1.2. *Let $1 < s \le s_0$ and an integer $l > 2n+1$. Assume that P be s_0-hyperbolic and $F(t,x,z)$ satisfying* $(\mathrm{A1})_s$ *and* $(\mathrm{A2})_s$. *If $u_j(x)$ belong to $\boldsymbol{H}^{l}_{\sigma_1,s}(\boldsymbol{R}^n)$ and $Hu^{(0)}(t,x)$ runs in a compact set contained by Ω, then there exist $T_2 \in (0,T)$ such that there exists a solution of the Cauchy problem* (1.1) *with $T = T_2$.*

Theorem 1.3. *Let $1 < s \le s_0$ and l be suffciently large. Assume that P is s_0-hyperbolic and $F(t,x,z)$ satisfies* $(\mathrm{A1})_1$ *and* $(\mathrm{A2})_1$, *and besides assume that there exists $u(t,x) \in C^m([0,T]; \boldsymbol{L}^2_s(\boldsymbol{R}^n))$ a solution of Cauchy problem* (1.1) *with initial data $u_j(x) \in \boldsymbol{L}^2_s(\boldsymbol{R}^n)$. Then if all initial values $u_j(x)$ are analytic, that is there exists $r > 0$ such that for $j = 0,1,\dots,m-1$,*

$$|D_x^\alpha u_j(x)| \le r^{-|\alpha|}|\alpha|! \tag{1.2}$$

for all $x \in \boldsymbol{R}^n$ and $\alpha \in \boldsymbol{N}^n$, then there exists $r' > 0$ such that

$$|D_x^\alpha u(t,x)| \le r'^{-|\alpha|}|\alpha|! \tag{1.3}$$

for any $(t,x) \in [0,T] \times \boldsymbol{R}^n$ and $\alpha \in \boldsymbol{N}^n$.

Several resluts of the propagation of analyticity are known for non-linear hyperbolic equations. S. Alinhac and G. Métivier [1] studied for strictly hyperbolic case. S. Spagnolo [12] treated a second order degenerate hyperbolic equations and M. Cicognani and L. Zanghirati treated a higher order hyperbolic equations with constant multiplicity. P. D'Ancona and S. Spagnolo [3] investigated the propagation of analyticity for non-uniformly symmetrizable systems and K. Kajitani and K. Yamaguti [7] treated uniformly symmetrizable systems.

2. Preliminaries

In this section, we mention the fundamental properties for Gevrey classes. Throughout the paper, we denote $\|\cdot\| = \|\cdot\|_{L^2(\boldsymbol{R}^n)}$ and $\|\cdot\|_{(l)} = \|\cdot\|_{H^l}$, that is Sobolev's norm. For $v(x) = (v_1(x),\dots,v_m(x))$ we denote $\|v\| = \|v_1\| + \cdots + \|v_m\|$. We introduce the semi-norms for $\gamma_p^{(s)}(B)$ and $\gamma_p^{(s)}(B;\mathcal{O}(\Omega))$ as follows: for $u \in \gamma_p^{(s)}(B)$,

$$|u|_{\rho,s,B} = \sup_{x\in B,\alpha\in N^n} \frac{|D_x^\alpha u(x)|\rho^{|\alpha|}}{|\alpha|!^s},$$

and for $f \in \gamma_p^{(s)}(B;\mathcal{O}(\Omega))$,

$$|f|_{\rho,s,B;K} = \sup_{x\in B, z\in K, \alpha\in N^n} \frac{|D_x^\alpha f(x,z)|\rho^{|\alpha|}}{|\alpha|!^s},$$

where K is a compact set of Ω. Now, we state some well-known facts of their classes.

LEMMA 2.1. (i) *Let* $a(x)\in\gamma_\rho^{(s)}(B)$. *Then for any* $\rho'\in(0,\rho)$ *and* $\alpha\in N^n$, $D_x^\alpha a(x)$ *belongs to* $\gamma_{\rho'}^{(s)}(B)$ *and there exists positive constants* C *and* $\sigma=\sigma(\rho,\rho',s)$ *such that*

$$|D_x^\alpha a|_{\rho',s,B}\le C|a|_{\rho,s,B}|\alpha|!^s\sigma^{-|\alpha|},$$

where C *is independent of* ρ,ρ' *and* a.

(ii) *Let* $f(x,z)$ *be in* $\gamma_{\sigma_1}^{(s)}(B;\mathcal{O}(\Omega))$, $v_j(x)$ *in* $\gamma_{\sigma_2}^{(s)}(B)$ *for* $j=1,\dots,m$. *Set* $v(x)=(v_1(x),\dots,v_m(x))$ *and* $|v|_{\sigma_2,s,B}=\sum_{j=1}^m|v|_{\sigma_2,s,B}$. *Assume that* $v(x)$ *runs in* K, *a compact set of* Ω, *for all* $x\in B$. *Then, there exists a constant* $\sigma_3=\sigma_3(\sigma_1,\sigma_2,\rho_K,n,|v|_{\sigma_2,s,B})$, *where* ρ_K *is the convergence radius of* $f(x,\cdot)$, *such that* $f(x,v(x))\in\gamma_{\sigma_3}^{(s)}(B)$ *and satisfies*

$$|f(\cdot,v(\cdot))|_{\sigma_3,s,B}\le C_{n,m}|f|_{\sigma_1,s,B;K},$$

where $C_{n,m}$ *depends only on the dimensions* n *and* m.

For $m\in\boldsymbol{R}$ we denote by S^m the usual symbol class of order m, and introduce the semi-norms as follows: for $a\in S^m$

$$|a|_l^{(m)} = \sup_{x,\xi\in\boldsymbol{R}^n, |\alpha+\beta|\le l} \frac{|a_{(\beta)}^{(\alpha)}(x,\xi)|}{\langle\xi\rangle^{m-|\alpha|}},$$

where $a_{(\beta)}^{(\alpha)}(x,\xi)$ means $D_x^\beta\partial_\xi^\alpha a(x,\xi)$. Next we define the symbols of Gevrey class in $\boldsymbol{R}^n$. For $s\ge 1$ and $A>0$, we denote by $\gamma_A^s S^m$ the set $\{a\in S^m$; satisfying that for any $l\in N$,

$$|a|_{A,s,l}^{(m)} = \sup_{x,\xi\in\boldsymbol{R}^n, |\alpha+\beta|\le l} \frac{|a_{(\beta)}^{(\alpha)}(x,\xi)|A^{|\beta|}}{\langle\xi\rangle^{m-|\alpha|}|\beta|!^s}<\infty\},$$

and let $\gamma^s S^m=\bigcap_{A>0}\gamma_A^s S^m$. We note that $\gamma_A^{(s)}(\boldsymbol{R}^n)$ is contained in $\gamma_A^s S^m(\boldsymbol{R}^n)$.

For $\rho>0$ we define $e^{\rho\langle D_x\rangle^{1/s}}$ by

$$e^{\rho\langle D_x\rangle^{1/s}}u(x)=(2\pi)^{-n}\int_{\boldsymbol{R}_x^n} e^{ix\xi+\rho\langle\xi\rangle^{1/s}}\hat{u}(\xi)\,d\xi$$

for $u\in\boldsymbol{H}_{\rho,s}^m$.

Let $\Lambda(t,\xi)=\rho(t)\langle\xi\rangle^{1/s}$, where $\rho(t)$ is a positive decreasing function on $[0, T]$. We denote by $e^{\Lambda}C^k([0,T];\boldsymbol{H}^l)$ the set of functions satisfying $e^{\rho(t)\langle D_x\rangle^{1/s}}u(t,x)\in C^k([0,T];\boldsymbol{H}^l)$.

LEMMA 2.2. (i) *Assume that l is large enough. Then there exisits a constant C_l such that*

$$\|uv\|_{\boldsymbol{H}^l_{\rho,s}}\le C_l\|u\|_{\boldsymbol{H}^l_{\rho,s}}\|v\|_{\boldsymbol{H}^l_{\rho,s}}$$

for any $u,v\in\boldsymbol{H}^l_{\rho,s}$, where C_l is independent of u and v.

(ii) $e^{\rho\langle D_x\rangle^{1/s}}$ *maps from $\boldsymbol{H}^l_{\rho',s}$ to $\boldsymbol{H}^l_{\rho'-\rho,s}$ continuously.*

(iii) *a pseudo-differential operator $a(x,D_x)\in\gamma^sS^m$ maps from $\boldsymbol{H}^l_{\rho,s}$ to $\boldsymbol{H}^{l-m}_{\rho,s}$ continuously.*

(iv) *Let $a_\rho(x,D_x)=e^{-\rho\langle D_x\rangle^{1/s}}a(x,D_x)e^{\rho\langle D_x\rangle^{1/s}}$ for $a\in\gamma^s_AS^m$. If $|\rho|\le(48n^{2/s})^{-1}A^{1/s}$, then $a_\rho(x,D_x)$ belongs to S^m and satisfies*

$$|a_\rho|^{(m)}_l\le C_l|a|^{(m)}_{A,s,l},$$

where C_l is independent of a.

(v) *If $|\rho|\le(48n^{2/s})^{-1}A^{1/s}$, then*

$$\|au\|_{\boldsymbol{H}^l_{\rho,s}}\le C_n|a|_{A,s,\boldsymbol{R}^n}\|u\|_{\boldsymbol{H}^l_{\rho,s}}$$

for any $a(x)\in\gamma^{(s)}_A(\boldsymbol{R}^n)$ and $u\in\boldsymbol{H}^l_{\rho,s}(\boldsymbol{R}^n)$.

The proof of this lemma is given in Proposition 2.3 of [6].

3. A Priori Estimate

We shall derive a priori estimate in Gevrey class $\boldsymbol{H}^l_{\rho,s}(\boldsymbol{R}^n)$ for s-hyperbolic equation. Since all H_{m-k} are s-hyperbolic with respect to $(1,0,0,\dots,0)$, there is a $C>0$ such that

$$|\mathrm{Im}\,\tau_{m-k,j}(\xi)|\le C\langle\xi\rangle^{1/s}\quad\text{for all }\xi\in\boldsymbol{R}^n \tag{3.1}$$

for $j=1,\dots,m-k$, and $k=0,\dots,m$.

Put $v(t,x)=e^{\rho(t)\langle D_x\rangle^{1/s}}u(t,x)$, where $\rho(t)=\rho_1t+\rho_0$ and we define $\hat{u}(t,\xi)$ by the Fourier transform of $u(t,x)$ with respect to x. Then we have

$$e^{\rho(t)\langle D_x\rangle^{1/s}}P(D_t,D_x)u(t,x)=P(D_t+i\rho_1\langle D_x\rangle^{1/s},D_x)v(t,x).$$

So,

$$\operatorname{Im}\Big\{(H_{m-k}(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)\overline{H_{m-k-1}(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)}\Big\}$$

$$= \operatorname{Im}\left\{\left[\prod_{j=1}^{m-k}(D_t+i\rho_1\langle\xi\rangle^{1/s}-\tau_{m-k,j}(\xi))\right]\hat{v}(t,\xi)\right.$$

$$\left.\times (m-k)^{-1}\sum_{l=1}^{m-k}\overline{\left[\prod_{j\neq l}(D_t+i\rho_1\langle\xi\rangle^{1/s}-\tau_{m-k,j}(\xi))\right]\hat{v}(t,\xi)}\right\}$$

$$= -\frac{1}{2}(m-k)^{-1}\frac{\partial}{\partial t}\sum_{l=1}^{m-k}\left|\left[\prod_{j\neq l}(D_t+i\rho_1\langle\xi\rangle^{1/s}-\tau_{m-k,j}(\xi))\right]\hat{v}(t,\xi)\right|^2$$

$$+ (m-k)^{-1}\sum_{l=1}^{m-k}((\rho_1\langle\xi\rangle^{1/s}-\operatorname{Im}\tau_{m-k,l}(\xi))$$

$$\times\left|\left[\prod_{j\neq l}(D_t+i\rho_1\langle\xi\rangle^{1/s}-\tau_{m-k,j}(\xi))\right]\hat{v}(t,\xi)\right|^2. \tag{3.2}$$

Since (3.1), for any $C_0>0$ there exists a negative constant ρ_1 such that to any $k=1,\dots,m$ and $j=1,\dots,m-k$,

$$\rho_1\langle\xi\rangle^{1/s}-\operatorname{Im}\tau_{m-k,j}(\xi)\le -C_0\langle\xi\rangle^{1/s}, \tag{3.3}$$

for all $\xi\in\boldsymbol{R}^n$. Put

$$K_m(t,\xi)=|P(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)|^2,$$

$$K_{m-k}(t,\xi)=(m-k+1)^{-1}\sum_{l=1}^{m-k+1}\left|\left[\prod_{j\neq l}(D_t+i\rho_1\langle\xi\rangle-\tau_{m-k+1,j}(\xi)\right]\hat{v}(t,\xi)\right|^2$$

for $k=1,\dots,m$.

We note that by virtue of Schwarz' inequality,

$$K_{m-k}(t,\xi)\ge|H_{m-k}(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)|^2 \quad (0\le k\le m). \tag{3.4}$$

From (3.2) and (3.3), we have

$$\frac{1}{2}\frac{\partial}{\partial t}\sum_{k=1}^{m}K_{m-k}(t,\xi)+mC_0\langle\xi\rangle^{1/s}\sum_{k=1}^{m}K_{m-k}(t,\xi)$$

$$\le -\operatorname{Im}\Big\{(H_{m-k}(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)\overline{H_{m-k-1}(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)}\Big\}.$$

Multiplying $\langle\xi\rangle^{2l}$ and integrating with respect to ξ over $\boldsymbol{R}^n$ both sides,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{R_\xi^n}\langle\xi\rangle^{2l}\sum_{k=1}^{m}K_{m-k}(t,\xi)\,d\xi+mC_0\int_{R^n}\langle\xi\rangle^{2l}\langle\xi\rangle^{1/s}\sum_{k=1}^{m}K_{m-k}(t,\xi)\,d\xi$$

$$\leq-\sum_{k=1}^{m}\operatorname{Im}\int_{R^n}\{\langle\xi\rangle^{2l}(H_{m-k+1}\hat{v}(t,\xi)\overline{H_{m-k}\hat{v}(t,\xi)}\}\,d\xi$$

$$\leq\int_{R^n}\langle\xi\rangle^{2l}\sum_{k=1}^{m}K_{m-k}(t,\xi)\,d\xi$$

$$-\operatorname{Im}\int_{R^n}\left\{\langle\xi\rangle^{2l}P(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)\overline{H_{m-1}(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)}\right\}d\xi.$$

Therefore, if C_0 is sufficiently large,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{R^n}\langle\xi\rangle^{2l}\sum_{k=1}^{m}K_{m-k}(t,\xi)\,d\xi$$

$$\leq\|\langle\xi\rangle^{l}P(D_t+i\rho_1\langle\xi\rangle^{1/s},\xi)\hat{v}(t,\xi)\|\int_{R^n}\langle\xi\rangle^{2l}\sum_{k=1}^{m}K_{m-k}(t,\xi)\,d\xi.$$

By virtue of Gronwall's inequality, we have Theorem 1.1.

We note that if $u_j(x)\equiv 0$, then $\sum_{k=1}^{m}\sum_{j=1}^{m-k+1}\|P_{m-k}^{j}u(0,\cdot)\|_{H_{\rho_0,s}^{l}}=0$.

COROLLARY 3.1. *Consider the following Cauchy problem in* $[0,T]\times R^n$:

$$\begin{cases}P(D)u(t,x)=f(t,x)\\ D_t^ju(0,x)=u_j(x)\quad j=0,\dots,m-1.\end{cases}\tag{3.5}$$

For any $T>0$ *there exists* $\Lambda(t,\xi)=(\rho_1t+\rho_0)\langle\xi\rangle^{1/s}$ *such that there exists a unique solution of this problem in* $e^{\Lambda}C^m([0,T];H^l(R^n))$ *for any* $f(t,x)\in e^{\Lambda}C([0,T];H^l(R^n))$ *and* $u_j(x)\in H_{\rho_0,s}^{l}(R^n)$.

4. Local Existence Theorem

In this section, we shall prove Theorem 1.2 by using standard contraction mapping method.

At first, we shall prove this theorem in the case all $u_j(x)\equiv 0$:

$$\begin{cases}P(D)u(t,x)=G(t,x,Hu)\\ D_t^ju(0,x)=0\quad j=0,\dots,m-1.\end{cases}\tag{4.1}$$

We define for $T_1 \in (0, T]$ and $M > 0$,

$$X_{T_1,M} = \Big\{ u(t,x); Hu(t,x) \in e^{\Lambda} C([0,T_1]; \boldsymbol{H}^l(\boldsymbol{R}^n)) \text{ and }$$

$$\|u\|_{X_{T_1}} = \sup_{t \in [0,T_1]} \|e^{\rho(t)\langle D_x\rangle^{1/s}} Hu(t,x)\|_{(l)} \le M \Big\},$$

where $\rho(t)$ is given by Theorem 1.1, depending on T_1.

Lemma 4.1. *Let an integer l be large enough. Assume that $G(t,x,z)$ satisfies the following conditions*:

$(\mathrm{B1})_s$: *there exsists a constant $\mu_1 > 0$ such that $G(t,x,z) \in \boldsymbol{C}([0,T_1]; \gamma^{(s)}_{\mu_1}(\boldsymbol{R}^n; \mathcal{O}(\Omega)))$, where Ω is open neighborhood of the origin in $\boldsymbol{C}^m$.*

$(\mathrm{B2})_s$: *there exists a constant $\mu_2 > 0$ such that $G(t,x,0) \in C([0,T_1]; \boldsymbol{H}^l_{\mu_2,s}(\boldsymbol{R}^n))$.*

Then there exist constants $M > 0$ and $T_1 > 0$ such that $G(t,x,w(t,x))$ belongs to $e^{\Lambda}C([0,T_1]; \boldsymbol{H}^l(\boldsymbol{R}^n))$ for any $w(t,x)$ in $X_{T_1,M}$, where $\Lambda = (\rho_1 t + \rho_0)\langle D_x\rangle^{1/s}$ is given in Theorem 1.1.

Proof. Let K be a compact neighborhood of the origin contained in Ω. Since G satisfies the conditions $(\mathrm{B1})_s$, there exisits a constant ρ_K such that for any $|z| < \rho_K$, G can expand into power series of z:

$$G(t,x,z) = G(t,x,0) + \sum_{\alpha>0} \frac{1}{\alpha!} (\partial_z^\alpha G)(t,x,0) z^\alpha.$$

By virtue of Sobolev's lemma, we pick $M > 0$ small enough, hence that $|Hw(t,x)| < \rho_K$ for any $(t,x) \in [0,T_1] \times \boldsymbol{R}^n$. Then,

$$\|e^{\Lambda} G(t,\cdot,Hw(t,\cdot))\|_{(l)} \le \|e^{\Lambda} G(t,\cdot,0)\|_{(l)} + \sum_{\alpha>0} \frac{1}{\alpha!} \|e^{\Lambda} (\partial_z^\alpha G)(t,\cdot,0) \cdot (Hw(t,\cdot))^\alpha\|_{(l)}. \tag{4.2}$$

From the assumption $(\mathrm{B2})_s$ and Lemma 2.2, we pick $\rho_0 > 0$ and $T_1 > 0$ small enough, if necessary, hence that $\|e^{\Lambda} G(t,\cdot,0)\|_{(l)}$ is bounded and moreover,

$$\begin{aligned}
\|e^{\Lambda} (\partial_z^\alpha G)(t,\cdot,0) \cdot (Hw(t,\cdot))^\alpha\|_{(l)} &\le C_n |(\partial_z^\alpha G)(t,\cdot,0)|_{\sigma_2,s,\boldsymbol{R}^n} \|e^{\Lambda}(Hw(t,\cdot))^\alpha\|_{(l)} \\
&\le C_n |G(t,\cdot,\cdot)|_{\sigma_2,s,\boldsymbol{R}^n;K} \alpha! \rho_K^{-|\alpha|} \tilde{C}_l^{|\alpha|-1} \|e^{\Lambda} Hw(t,\cdot)\|_{(l)}^{|\alpha|} \\
&\le C_{n,l} \left(\frac{\tilde{C}_l M}{\rho_K} \right)^{|\alpha|} \alpha! |G(t,\cdot,\cdot)|_{\sigma_2,s,\boldsymbol{R}^n;K}.
\end{aligned}$$

Therefore we pick M small enough again, if necessary, hence that the right hand side of (4.2) converges. Thus the proof of Lemma 4.1 is finished. ■

For $w \in X_{T_1,M}$ we denote an operator Φ from $X_{T_1,M}$ to $e^{\Lambda}C^m([0,T_1];\boldsymbol{H}^l(\boldsymbol{R}^n))$ by $\Phi(w)=u$ which is a solution of the following Cauchy problem,

$$\begin{cases} P(D)u(t,x) = G(t,x,Hw) \\ D_t^j u(0,x) = 0 \quad j=0,\dots,m-1. \end{cases} \tag{4.3}$$

From Corollary 3.1 and Lemma 4.1, we have a unique solution in $e^{\Lambda}C^m([0,T_1];\boldsymbol{H}^l(\boldsymbol{R}^n))$. Moreover,

LEMMA 4.2. *There exist* $T_2 \in (0,T_1]$ *and* $M>0$ *such that*
(i) Φ *is a mapping from* $X_{T_2,M}$ *into itself.*
(ii)

$$\|\Phi(v)-\Phi(v')\|_{X_{T_2}} \leq \frac{1}{2}\|v-v'\|_{X_{T_2}}$$

for any $v,v' \in X_{T_2,M}$.

PROOF. Let v belong to $X_{T_1,M}$ and u be $\Phi(v)$. From Theorem 1.1 and Lemma 4.1,

$$\begin{aligned}\|e^{\Lambda}Hu(t,\cdot)\|_{(l)} &\leq C_n\int_0^t \|e^{\Lambda}G(t',\cdot,Hv)\|_{(l)}\,dt' \\ &\leq C_{n,l}\int_0^t \{\|e^{\Lambda}G(t',\cdot,0)\|_{(l)} + |G(t',\cdot,\cdot)|_{\sigma_2,s,\boldsymbol{R}^n;K}\}\,dt'\end{aligned}$$

for any $t \in [0,T_1]$. Therefore we pick $T_2 \in (0,T_1]$ small enough, then $\|e^{\Lambda}G(t,x,Hv)\|_{(l)} \leq M$ for all $v \in X_{T_2,M}$, so that (i) is proved. Similarly,

$$\begin{aligned}&\|e^{\Lambda}(\Phi(v)-\Phi(v'))\|_{(l)} \\ &\quad\leq \int_0^{T_2}\left\|e^{\Lambda}\int_0^1 \nabla_y G(t',x,Hv'+\theta(Hv-Hv'))\,d\theta\cdot(Hv-Hv')\right\|_{(l)} dt' \\ &\quad\leq CT_2\|v-v'\|_{X_{T_2}},\end{aligned}$$

where C is independent of T_2. Then choose small T_2 again, if necessary, $CT_2<1/2$. Thus the proof of (ii) is finished. ■

Hence, there exists a unique solution of Cauchy problem (4.1) in $X_{T_2,M}$ by virtue of the fixed point theorem. In order to solve the general case, the Cauchy problem (1.1), we change the unknown function $w(t,x) = u(t,x) - u^{(0)}(t,x)$. Then we can reduce the problem (1.1) to (4.3) by the next Lemma. Thus the proof of Theorem 1.2 is finished.

LEMMA 4.3. *Assume that* $F(t,x,z)$ *satisfies the conditions* $(\mathrm{A1})_s$ *and* $(\mathrm{A2})_s$. *Then there exsist constants* $T' > 0$ *and* $M > 0$ *such that for any* $w(t,x) \in X_{T',M}$, $G(t,x,z) = F(t,x,z+w(t,x))$ *satisfies the conditions* $(\mathrm{B1})_s$ *and* $(\mathrm{B2})_s$ *in Lemma* 4.1.

In order to prove Lemma 4.3, we essentially use Lemma 2.1. We omit the proof of this lemma.

5. Propagation of Analyticity

We introduce semi-norms in $C([t_0,t_1];L_s^2(\boldsymbol{R}^n))$. Let an integer $N \geq 2$ and a real numbel $r \in (0,1]$. We denote

$$|u|_{r,N}^{t_0,t_1} = \sup_{t' \in (t_0,t_1],\, 2 \leq |\beta| \leq N} \frac{\|D_x^\beta u\|_{\boldsymbol{H}^l_{\rho(t'),s}} r^{|\beta|-2}}{\Gamma_2(|\beta|)}$$

for $u \in C([t_0,t_1];L_s^2(\boldsymbol{R}^n))$, where $\rho(t)$ is a positive decreasing function, $\Gamma_2(k) = \lambda_0 k! k^{-2}$ for $k \geq 1$ and $\Gamma_2(0) = \lambda_0$. We can pick λ_0 such that

$$\sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \Gamma_2(|\alpha'|+k)\Gamma_2(|\alpha-\alpha'|) \leq \Gamma_2(|\alpha|+k)$$

for any $k \in N$ and $\alpha \in N^n$. In brief we write $|u|_{r,N} = |u|_{r,N}^{t_0,t_1}$ if there is no confusion.

LEMMA 5.1. *Let* $v_i \in C([t_0,t_1];L_s^2(\boldsymbol{R}^n))$, $i = 1,\dots,n$ *and we denote* $v^\beta = v_1^{\beta_1} v_2^{\beta_2} \cdots v_n^{\beta_n}$ *for* $\beta \in N^n$. *Then there is a constant* $C_0 > 0$ *such that*

(i) *for* $2 \leq |\beta| \leq N$,

$$|v^\beta|_{r,N} \leq C_0^{|\beta|-1} \left(\sup_{t_0 \leq t' \leq t_1} \|v(t',\cdot)\|_{\boldsymbol{H}^{l+1}_{\rho(t'),s}} + r^2 |v|_{r,N} \right)^{|\beta|-1} |v|_{r,N}$$

and

(ii) *for* $|\beta| > N$,

$$|v^{\beta}|_{r,N} \le C_0^{|\beta|-1} \sup_{2\le j\le N} \left(\sup_{t_0\le t'\le t_1} \|v(t',\cdot)\|_{H^l_{\rho(t'),s}} \right)^{|\beta|-j} \times \left\{ \sum_{2\le|\alpha|\le N} \left(\sup_{t_0\le t'\le t_1} \|v(t',\cdot)\|_{H^{l+1}_{\rho(t'),s}} + r^2|v|_{r,N} \right)^{|\alpha|-1} |v|_{r,N} + \sup_{2\le j\le N} \left(\sup_{t_0\le t'\le t_1} \|v(t',\cdot)\|_{H^{l+1}_{\rho(t'),s}} \right)^j \right\}$$

where constant C_0 *depends only on the dimension* n.

(iii) *Let* $a \in \gamma^{\{1\}}(\boldsymbol{R}^n)$, *that is an entire function, and* $v \in \boldsymbol{L}^2_s(\boldsymbol{R}^n)$, *then for any* $r \in (0,1]$ *and* $N \ge 2$,

$$|a(x)v(x)|_{r,N} \le C_n |a|_{\rho',1,\boldsymbol{R}^n} |v|_{r,N},$$

where $\rho' = \max\{5r, n(\rho(0)/24)^s\}$ *and the constant* C_n *depends only on the dimension* n.

The proof of this lemma can be seen K. Kajitani and K. Yamaguti [7]. The last term in the right hand side of (iii) of Lemma 5.1 is lacked in Lemma 3.1 in [7].

Now, we shall prove Theorem 1.3. From the assumption, for any $\varepsilon > 0$ there is $\tau > 0$ such that

$$\|u(t,\cdot) - u(k\tau,\cdot)\|_{H^{l+1}_{\rho(t),s}} < \varepsilon$$

for $t \in [k\tau, (k+1)\tau]$, $k = 0, 1, \dots, [T/\tau] - 1$ and $t \in [[T/\tau], T]$, where $[x]$ stands for the greatest integer not greater than x. From the assumption (1.2), there exist constants $C > 0$ and $r_1 > 0$ such that

$$\|D^{\alpha}_x Hu^{(0)}(k\tau,\cdot)\|_{H^l_{\rho(k\tau),s}} \le C r_1^{-|\alpha|} |\alpha|!.$$

Put $v(t,x) = u(t,x) - u^{(0)}(t,x)$. Then

$$Pv(t,x) = F(t,x,Hu(t,x)) - Pu^{(0)}(t,x) = F(t,x,Hv(t,x) + Hu^{(0)}(t,x)) - Pu^{(0)}(t,x).$$

We define $G(t,x,z) = F(t,x,z + Hu^{(0)}(t,x)) - Pu^{(0)}(t,x)$, and by Lemma 4.3, $G(t,x,z)$ satisfies $(B1)_1$ and $(B2)_1$. To differentiate both sides, then we have

$$PD^{\alpha}_x v(t,x) = D^{\alpha}_x(F(t,x,Hv(t,x) + Hu^{(0)}(t,x))) - PD^{\alpha}_x u^{(0)}(t,x),$$

and we denote G_α by the right hand side. Now $D_t^j v(0,x)=0$ for $j=0,1,\ldots,m-1$, therefore from Theorem 1.1 we obtain

$$\|e^\Lambda HD_x^\alpha v(t,\cdot)\|_{(l)} \le \int_0^t \|e^\Lambda G_\alpha(t,x)\|_{(l)}\,dt' \tag{5.1}$$

for any $t\in[0,\tau]$, where $\Lambda=\rho(t)\langle D_x\rangle^{1/s}$ is given by Theorem 1.1. For simplicity we write $\|u\|_{(\rho(t))}=\|e^\Lambda u\|_{(l)}$. By virtue of Lemma 5.1, for any $2\le\alpha\le N$,

$$\begin{aligned}
&\|D_x^\alpha F(t,\cdot,Hv(t,\cdot)+Hu^{(0)}(t,\cdot))\|_{(\rho(t))}\\
&\quad\le \|D_x^\alpha F(t,\cdot,Hu^{(0)}(t,\cdot))\|_{(\rho(t))}\\
&\qquad+\sum_{\beta>0}\beta!^{-1}\|D_x^\alpha(\partial_z^\beta F(t,\cdot,Hu^{(0)}(t,\cdot))(Hv(t,\cdot))^\beta)\|_{(\rho(t))}\\
&\quad\le \Gamma_2(|\alpha|)r^{-|\alpha|+2}\Big\{|F(t,\cdot,Hu^{(0)}(t,\cdot))|_{r,N}\\
&\qquad+\sum_{\beta>0}\beta!^{-1}|(\partial_z^\beta F)(t,\cdot,Hu^{(0)}(t,\cdot))(Hv(t,\cdot))^\beta|_{r,N}\Big\}\\
&\quad\le \Gamma_2(|\alpha|)r^{-|\alpha|+2}\Big\{|F(t,\cdot,Hu^{(0)}(t,\cdot))|_{r,N}\\
&\qquad+\sum_{\beta>0}\beta!^{-1}|\partial_z^\beta F(t,\cdot,Hu^{(0)}(t,\cdot))|_{\nu_1,1,\boldsymbol{R}^n}|(Hv)^\beta|_{r,N}\Big\}\\
&\quad= \Gamma_2(|\alpha|)r^{-|\alpha|+2}\Big\{|F(t,\cdot,Hu^{(0)}(t,\cdot))|_{r,N}\\
&\qquad+C\Big\{\sum_{0<|\beta|<2}+\sum_{2\le|\beta|\le N}+\sum_{|\beta|>N}\Big\}\beta!^{-1}|\partial_z^\beta F(t,\cdot,Hu^{(0)}(t,\cdot))|_{\nu_1,1,\boldsymbol{R}^n}|(Hv)^\beta|_{r,N}\Big\}
\end{aligned}$$

where $\nu_1\ge\max\{5r,n(\rho(0)/24)^s\}$. From the assumption, for fixed t, there exists a compact set $K\subset\Omega$ such that $\{Hu^{(0)}(t,x);x\in\boldsymbol{R}^n\}\subset K$. Then by Lemma 4.3, there exists a constant $\nu_2>0$ such that

$$|\partial_z^\beta F(t,\cdot,Hu^{(0)}(t,\cdot))|_{\nu_1,1,\boldsymbol{R}^n}\le C_n|F(t,\cdot,\cdot)|_{\nu_2,1,\boldsymbol{R}^n;K}\beta!\nu_1^{-|\beta|}$$

for any $\beta\in N^m$. For sufficiently small ε, we have

$$\|D_x^\alpha F(t,\cdot,Hv(t,\cdot)+Hu^{(0)}(t,\cdot))\|_{(\rho(t))}$$

$$\leq \Gamma_2(|\alpha|)r^{-|\alpha|+2}\left\{|F(t,\cdot,Hu^{(0)}(t,\cdot))|_{r,N}\right.$$

$$\left.+C\left\{\sum_{|\beta|=1}+\sum_{2\leq|\beta|\leq N}+\sum_{|\beta|>N}\right\}v_1^{-|\beta|}|F(t,\cdot,\cdot)|_{v_2,1,\boldsymbol{R}^n;K}|(Hv)^\beta|_{r,N}\right\}$$

$$\leq \Gamma_2(|\alpha|)r^{-|\alpha|+2}\left\{|F(t,\cdot,Hu^{(0)}(t,\cdot))|_{r,N}+C_n|F(t,\cdot,\cdot)|_{v_2,1,\boldsymbol{R}^n;K}\right.$$

$$\left\{|Hv|_{r,N}+\sum_{2\leq|\beta|\leq N}v_1^{-|\beta|}C_0^{|\beta|-1}(\varepsilon+r^2|Hv|_{r,N})^{|\beta|-1}|Hv|_{r,N}\right.$$

$$\left.\left.+\sum_{|\beta|>N}v_1^{-|\beta|}C_0^{|\beta|-1}\varepsilon^{|\beta|-2}\left\{\sum_{2\leq|\gamma|\leq N}(\varepsilon+r^2|Hv|_{r,N})^{|\gamma|-1}|Hv|_{r,N}+\varepsilon^2\right\}\right\}\right\}$$

$$\leq \Gamma_2(|\alpha|)r^{-|\alpha|+2}\left\{|F(t,\cdot,Hu^{(0)}(t,\cdot))|_{r,N}+C_n'|F(t,\cdot,\cdot)|_{v_2,1,\boldsymbol{R}^n:K}\right.$$

$$\left.\times\sum_{j=0}^{N-1}(\varepsilon+r^2|Hv|_{r,N})^j|Hv|_{r,N}\right\}.$$

Here we choose $r=r(t)=r_0e^{-t}$, where $0<r_0\leq 1$. Denote

$$y_N(t)=\sup_{0\leq t'\leq t}r(t')|Hv|_{r(t'),N},$$

where

$$|v|_{r(t),N}=\sup_{0\leq t'\leq \tau,2\leq|\beta|\leq N}\{\|D_x^\beta v\|_{H'_{\rho(t'),s}}r(t')^{|\beta|-2}\Gamma_2(|\beta|)^{-1}\}.$$

Then,

$$\|D_x^\alpha F(t,\cdot,Hv(t,\cdot)+Hu^{(0)}(t,\cdot))\|_{(\rho(t))}$$

$$\leq \Gamma_2(|\alpha|)r^{-|\alpha|+2}\left\{C_1\left(1+\sum_{j=0}^{N-1}(\varepsilon+y_N(t))^j|Hv|_{r(t),N}\right)\right\},$$

where $C_1=|F(t,\cdot,Hu^{(0)}(t,\cdot))|_{r,N}+C_n'|F(t,\cdot,\cdot)|_{v_2,1,\boldsymbol{R}^n;K}$. Thus from (5.1),

$$|Hv|_{r(t),N}\leq C\int_0^t\left(1+\sum_{j=0}^{N-1}(\varepsilon+y_N(t'))^j|Hv|_{r(t'),N}\right)dt',$$

then

$$y_N(t) \le C\int_0^t \left(r(t') + \sum_{j=0}^{N-1} (\varepsilon + y_N(t'))^j y_N(t') \right) dt'.$$

From this inequality, we have $y_N(t) < \varepsilon$ for $t \in [0, \tau]$, if we choose $r_0 > 0$ small enough. In fact, assume that there is $t_1 \in [0, \tau]$ such that $y_N(t_1) = \varepsilon$ and $y_N(t) < \varepsilon$ for $t \in (0, t_1)$. Since $y_N(0) = 0$, we have $t_1 > 0$. It follows from (5.1) that

$$y_N(t) \le C\left(r_0 + \int_0^t \frac{1}{1-2\varepsilon} y_N(t') \right) dt'.$$

for $t \in [0, t_1)$. We note that the constants C, ε and r_0 can be chosen independent of N. Therefore we obtain $y_N(t) \le Cr_0 \exp(Ct/(1-2\varepsilon))$ for $t \in [0, t_1)$. This contradicts $y_N(t_1) = \varepsilon$, if we choose $r_0 > 0$ small enough.

Thus we can get $y_N(t) \le \varepsilon$ for $t \in [0, \tau]$. By induction, there is a constant $r' > 0$ such that $|D_x^\alpha v(t,x)| \le Cr'^{|\alpha|}|\alpha|!$ for $(t,x) \in [0,T] \times \boldsymbol{R}^n$ and consequently Theorem 1.3 is proved.

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