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THE EXTENSION PROBLEM FOR COMPLETE UV^n-PREIMAGES

By

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Abstract. We investigate the solvability of the *extension problem* for complete preimages from the given class $\mathscr{F}$ of surjective, perfect mappings of metric spaces, which consists of representing an arbitrary mapping $f_0 : X_0 \to Y_0 \in \mathscr{F}$ as the restriction of another mapping $f : X \to Y \in \mathscr{F}$, onto the complete preimage $f_0^{-1}(Y_0) = X_0$, where Y is an arbitrary metric space, containing Y_0 as a closed subset. We prove that this problem can be solved for the class of open UV^n-mappings. Along the way, we also establish that the subset $\exp_{UV^n}(\ell_2(\tau))$ of the exponent $\exp(\ell_2(\tau))$ of the Hilbert space $\ell_2(\tau)$ of density τ, consisting of UV^n-compacta, belongs to the class of absolute retracts.

1. Introduction

Let $\mathscr{F}$ be a class of perfect surjective mappings of metric spaces. If a map $f : X \to Y$ belongs to $\mathscr{F}$ and $Y_0 \subset Y$ is any closed subset then quite often the restriction g of f onto the complete preimage $f^{-1}(Y_0)$ of the set Y_0 also belongs to the class $\mathscr{F}$. This is true for the following classes of interest:

(a) The class $\mathscr{F}_a$ of all open maps;
(b) The class $\mathscr{F}_b$ of all monotone open maps;
(c) The class $\mathscr{F}_c$ of all n-soft maps;
(d) The class $\mathscr{F}_d$ of all open UV^n-maps;
(e) The class $\mathscr{F}_e$ of all locally trivial fibrations (cf. [8]); and

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(f) The class $\mathscr{F}_f$ of all G-mappings (where Y_0 is taken to be an invariant subset of Y) (cf. [1, 2]).

In the present paper we shall be interested in the *inverse* problem, the exact meaning of which we explain below:

DEFINITION (1.1). The extension problem for complete preimages from the class $\mathscr{F}$ is said to be *solvable*, provided that for every map $g : X_0 \to Y_0$ from $\mathscr{F}$ and every closed embedding of Y_0 into the metric space Y, there exist a closed embedding of X_0 into the metric space X and a map $f : X \to Y$ from $\mathscr{F}$ such that:

(i) $f|_{X_0} = g$; and

(ii) $f(X \backslash X_0) = Y \backslash Y_0$.

It is clear that the map g is the restriction of f onto the complete preimage $f^{-1}(Y_0) = g^{-1}(Y_0) = X_0$. Consequently, this extension problem is equivalent to the problem of representing an arbitrary map $g : X_0 \to Y_0$ from $\mathscr{F}$ as the restriction of another map $f : X \to Y$ from $\mathscr{F}$ onto the complete preimage $g^{-1}(Y_0) = f^{-1}(Y_0)$, where $Y_0 \subset Y$ is an arbitrary embedding of Y_0 into the metric space Y.

Extensions of complete preimages are closely connected with the extension problem for maps into certain hyperspaces. To establish this connection let us restrict ourselves to metric spaces of a fixed weight τ. It is well-known that all such spaces are subspaces of the generalized Hilbert space $\ell_2(\tau)$ (cf. [10]). This fact allows us to represent any map $f : X \to Y$ from the class $\mathscr{F}$ as the restriction of the projection $\mathrm{pr}_1 : \ell_2(\tau) \times \ell_2(\tau) \to \ell_2(\tau)$ onto some subset of the product.

DEFINITION (1.2). A map $f : X \to Y$ between metric spaces is said to have a *kernel* Z if any of the following two equivalent conditions is satisfied:

(1) There exists a map $g : X \to Z$ such that the diagonal map $f \Delta g : X \to Y \times Z$ is a topological embedding;

(2) There exists a homeomorphism $h : X \to T$ between X and a subset $T \subset Y \times Z$ which maps every fiber $f^{-1}(y), y \in Y$, into the fiber $T \cap \{y \times Z\}$ of the map $\mathrm{pr}_Y|_T : T \to Y$, i.e. $f = (\mathrm{pr}_Y|_T) \circ h$.

PROPOSITION (1.3). *Every mapping between metric spaces of weight τ has the kernel $Z = \ell_2(\tau)$.*

Therefore, every perfect surjective map $f : X \to Y$ can be generated by a subset $T \subset \ell_2(\tau) \times \ell_2(\tau)$, which satisfies the following two conditions:

(i) For every $y \in Y \hookrightarrow \ell_2(\tau)$, the intersection $\Phi(y) = T \cap (y \times \ell_2(\tau))$ is compact; and

(ii) The map $\Phi : Y \to \exp(\ell_2(\tau))$, given by $\Phi(y) = T \cap (y \times \ell_2(\tau))$, is upper semicontinuous, i.e. for every point $y \in Y$ and every $\varepsilon > 0$, there exists a neighborhood $\mathcal{O}(y)$ such that for every $y' \in \mathcal{O}(y), \Phi(y') \subset N(\Phi(y), \varepsilon)$.

Let us list some well-known facts concerning the relationship among the classes of maps $\mathcal{F}$ and properties of the maps Φ:

(α) A map $f : X \to Y$ is open if and only if the corresponding map $\Phi : Y \to \exp(\ell_2(\tau))$ is continuous in the Hausdorff metric ρH;

(β) A map f is UV^n if and only if for every $y \in Y, \Phi(y)$ is a UV^n-set; and

(γ) A map f is n-soft if and only if for every $y \in Y, \Phi(y)$ is an $AE(n)$-set and $\Phi : Y \to \exp(\ell_2(\tau))$ is continuous in the Kuratowski metric ρK (cf. [10]).

Denote by $\exp_{UV^n} X$ the subspace of $(\exp X, \rho H)$ consisting of all UV^n-compacta. For $n = 0$, $\exp_{UV^n} X$ is better known as $\exp_c X$, the continual exponent of X, and for $n = -1$, $\exp_{UV^n} X$ is just $\exp X$. Denote by $\exp_{AE(n)} X$ the space of all $AE(n)$-compacta in X with the Kuratowski metric.

PROPOSITION (1.4). *The extension problem for complete preimages from the class $\mathcal{F}_a$ (resp. $\mathcal{F}_b, \mathcal{F}_c, \mathcal{F}_d$) is equivalent to the question whether* $\exp \ell_2(\tau) \in AE$ *(resp.* $\exp_c \ell_2(\tau) \in AE, \exp_{AE(n)} \ell_2(\tau) \in AE, \exp_{UV^n}(\ell_2(\tau)) \in AE$*).*

For complete $AE(1)$-spaces X, the exponent and the continual exponent are absolute extensors (cf. [12]). Consequently, the extension problem for complete preimages for the classes $\mathcal{F}_a$ and $\mathcal{F}_b$ is solvable. Concerning the property $\exp_{AE(n)}(\ell_2(\tau)) \in AE$, we observe that this is an important open problem of the theory of absolute extensors in finite dimensions (cf. [7]) and, consequently, the same is true for the extension problem for complete preimages for the class $\mathcal{F}_c$. The main result of the present paper concerns the last one of these four classes:

THEOREM (1.5). *For every integer $n \geq -1$, the space* $\exp_{UV^n}(\ell_2(\tau))$ *is an AE.*

As a corollary of Theorem (1.5), we deduce for $n \geq -1$ the solvability of the extension problem for complete preimages for $\mathcal{F}_a$ and $\mathcal{F}_b$:

THEOREM (1.6). *The extension problem for complete preimages for the class of open UV^n-maps is solvable, for every integer $n \geq -1$.*

QUESTION (1.7). *What can one say about compacta X such that* $\exp_{UV^n} X \in AE$*? Is it true that X must necessarily be a UV^n compactum?*

2. Preliminaries

All spaces will be assumed to be metric and all maps to be continuous. A space X is said to be an *absolute [neighborhood] extensor in dimension n*, $X \in A[N]E(n)$, provided that X has the following injectivity property: every map $\varphi : A \to X$ of a closed subset $A \subset Z$ of an n-dimensional space Z can be extended to a map $\hat{\varphi} : Z \to X$ [$\hat{\varphi} : U \to X$, for some neighborhood $U \subset Z$ of A], i.e. $\hat{\varphi}|_A = \varphi$. The map φ will be called a *partial* n-map and will be denoted by $Z \hookleftarrow A \xrightarrow{\varphi} X$.

For $n = \infty$, the class of $A[N]E(n)$-spaces (or simply, $A[N]E$-spaces) coincides with the class of *absolute [neighborhood] extensors* (cf. [5]). If a closed subset $X_0 \subset X$ of an ANE-space is an $AE(n)$, then X_0 has the following UV^{n-1}-property in X (cf. [11]):

DEFINITION (2.1). A closed subset $X_0 \subset X$ is said to have the *UV^k-property* in X, $k < \infty$, provided that for every neighborhood $U \subset X$ of X_0, there exists a neighborhood $V \subset U$ of X_0 such that embedding $i : V \hookrightarrow U$ induces the trivial homomorphism of homotopy groups $\prod_i$, for all $i \leq k$.

DEFINITION (2.2). X is said to be a *UV^k-compactum*, $k < \infty$, if for every embedding of X into an $ANE(k+1)$-space $\hat{X}, X$ has the UV^k-property in $\hat{X}$.

It is well-known that a compactum X is UV^k if and only if for some embedding of X into an $ANE(k+1)$-space $\hat{X}, X$ has the UV^k-property. Moreover, X is UV^k if and only if for every (some) embedding of X into an $ANE(k+1)$-space $\hat{X}, X$ has the following UV^k-property in $\hat{X}$:

(α) *For every neighborhood $U \subset \hat{X}$ of X, there exists a neighborhood $V \subset U$ of X such that every partial $(k+1)$-map $Z \hookleftarrow A \xrightarrow{\varphi} V$ has an extension $\hat{\varphi} : Z \to U, \hat{\varphi}|_A = \varphi$.*

PROPOSITION (2.3). *Let X be a compactum and $X_0 \subset X$ any UV^k-compactum. Suppose there exists a homotopy $H : X \times I \to X$ such that $H_0 = \mathrm{id}_X$ and $H_1(X) \subset X_0$. Then X is a UV^k-compactum.*

Therefore every cone over a compactum belongs to the UV^k-class. As a consequence, the UV^k-compacta are a wider class than the $AE(k+1)$-compacta.

DEFINITION (2.4). A map $f : X \to Y$ is said to be a *UV^k-map*, provided that every fiber of f is a UV^k-compactum.

One of the most important properties of UV^k-maps is their approximate $(k+1)$-softness (cf. [13]):

PROPOSITION (2.5). *Let $\hat{f} : \hat{Y} \times \hat{X} \to \hat{Y}$ be the projection of the product of ANE-compacta $\hat{X}$ and $\hat{Y}$ onto the first factor, $f : X \to f(X) = Y \subset \hat{Y}$ a restriction of $\hat{f}$ onto a compactum $X \subset \hat{Y} \times \hat{X}$. Then $f \in UV^k$ if and only if for every sequence of maps $\psi_i : Z \to \hat{Y}$ of a $(k+1)$-dimensional metric space Z and sequence of the partial maps $\varphi_i : A \to \hat{Y} \times \hat{X}, \mathrm{Cl}(A) = A \subset Z$, such that $\psi_i|A = \hat{f} \circ \varphi_i$, for every i, $\lim_{i\to\infty} \varphi_i(A) \subset X$, and $\lim_{i\to\infty} \psi_i(Z) \subset Y$, there exists a sequence of maps $\hat{\varphi}_i : Z \to \hat{Y} \times \hat{X}$, such that the following three conditions are satisfied*:

(1) $\lim_{i\to\infty} \mathrm{dist}(\hat{\varphi}_i|_A, \varphi_i) = 0$;

(2) $\lim_{i\to\infty} \hat{\varphi}_i(Z) \subset X$; *and*

(3) $\lim_{i\to\infty} \mathrm{dist}(\hat{f} \circ \hat{\varphi}_i, \psi_i) = 0$.

The following well-known fact from shape theory is a consequence of Proposition (2.5).

PROPOSITION (2.6). *Let $f : X \to Y$ be a UV^k-map of metric compacta. Then*

(a) *If $X \in UV^{k+1}$ then $Y \in UV^{k+1}$; and*

(b) *If $Y \in UV^k$ then $X \in UV^k$.*

In the Hilbert space $\ell_2(\tau)$ the unknotting theorem holds for Z-sets. We recall some necessary definitions:

DEFINITION (2.7). A closed subset $A \subset Z$ of a metric space is said to be a *Z-set*, provided that for every open cover $\omega \in \mathrm{cov}\, Z$, there exists a map $h : Z \to Z$ which is ω-close to id_Z and such that $A \cap \mathrm{Im}\, h = \varnothing$.

THEOREM (2.8). *Suppose that in the Hilbert space $\ell_2(\tau)$ we have a homeomorphism $h : A \to B$ of Z-sets A and B. Then there exists a homeomorphism $\hat{h} : \ell_2(\tau) \to \ell_2(\tau)$ of the entire space $\ell_2(\tau)$ such that $\hat{h}|_A = h$.*

We complete this section by some definitions and facts concerning the notion of homotopically negligible subsets:

DEFINITION (2.9). A subset $A \subset Z$ of a metric space Z is said to be *homotopically negligible* in Z, provided that there exists a homotopy $H : Z \times [0,1] \to X$ such that $H(Z \times (0,1]) \cap A = \varnothing$ and $H_0 = \mathrm{Id}$.

The following are well-known facts concerning homotopically negligible sets (cf. [15]):

PROPOSITION (2.10). *Suppose that* $Z \in A[N]E$ *and that* $A \subset Z$ *is a homotopically negligible subset of* Z. *Then* $Z \backslash A \in A[N]E$.

PROPOSITION (2.11). *Every* $A[N]E$*-space* X *can be embedded into a complete* $A[N]E$*-space* $\hat{X} \supset X$ *so that* $\hat{X} \backslash X$ *is homotopically negligible in* $\hat{X}$.

PROPOSITION (2.12). *For every metric space* X, *there exists an* $A[N]E$*-space* $\hat{X}$, *containing* X *as a closed subset and such that* X *is homotopically negligible in* $\hat{X}$.

3. Adjunction Spaces for UV^n-compacta

Let X and Y be metric spaces and let $X_0 \subset X$ be a closed subset. Any continuous map $f : X_0 \to Y$ induces a decomposition on the topological sum $Z = X \oplus Y$, if for every $y \in f(x_0)$, we shrink the set $f^{-1}(y) \cup \{y\}$ to a point. The resulting decomposition space is denoted by $X \cup_f Y$ and is called the *adjunction space* of X to Y by f. If the map f is perfect then the adjunction space $X \cup_f Y$ is metrizable. Also, if X, X_0 and Y are $A[N]E$-spaces then the adjunction space $X \cup_f Y$ is also an $A[N]E$-space (cf. [9]). We shall now prove an analogous result concerning UV^n-compacta:

THEOREM (3.1). *Let* $X \hookleftarrow X_0 \xrightarrow{f} Y$ *be a partial map such that* X *and* Y *are* UV^n*-compacta and* $X_0 \subset X$ *is a* UV^{n-1}*-compactum. Then the adjunction space* $Z = X \cup_f Y$ *is a* UV^n*-compactum.*

REMARK (3.2). Theorem (3.1) was stated without proof in [3], where theorems on adjoining $A[N]E(n)$ and n-movable spaces were also proved.

A short proof of Theorem (3.1) can be derived from Proposition (2.6): Since $X \to X/X_0$ is a UV^{n-1}-map and $X \in UV^n$, it follows that $X/X_0 \in UV^n$. Since $X \cup_f Y \to (X \cup_f Y)/Y = X/X_0$ is a UV^n-map, it follows that $X \cup_f Y \in UV^n$. As this method is not applicable to prove the adjunction theorem for n-movable spaces, we present the following expanded proof of Theorem (3.1).

PROOF. Embed Y in an ANE-compactum $\hat{Y}$ as a homotopically negligible set. Therefore there is a homotopy $H_t : \hat{Y} \to \hat{Y}$ such that $H_0 = \mathrm{Id}$ and for every

$t > 0, H_t(\hat{Y}) \cap Y = \emptyset$. Extend the map f to a map $f' : \hat{X}_0 \to \hat{Y}$, defined on some ANE-compactum $\hat{X}_0 \supset X_0$. By means of H_t we can define a new extension of f as follows:

$$\hat{f}(x) = H(f'(x), \rho(x, X_0)), \quad x \in \hat{X}_0.$$

Clearly, $\hat{f}(\hat{X}_0 \backslash X_0) \cap Y = \emptyset$.

We may assume that $\hat{X}_0$ and X intersect precisely at X_0. Embed $\hat{X}_0 \cup X$ into the ANE-compactum $\hat{X}$. It follows by the adjunction space theorem for ANE-compacta [5] that $\hat{Z} = \hat{X} \cup_{\hat{f}} \hat{Y} \in ANE$. Since the embedding of compacta into ANE is done with a great degree of freedom, it suffices, in order to verify UV^n-properties of the compactum $Z = X \cup_f Y$ in $\hat{Z}$, to take $\hat{Z}$ instead of U and to prove that there exist neighborhoods $V \subset \hat{X}$ of X and $W \subset \hat{Y}$ of Y such that:

(a) $(\hat{f})^{-1}(W) = V \cap \hat{X}_0$; and

(b) The embedding $V \cap_{\hat{f}} W \hookrightarrow \hat{Z}$ induces a trivial homomorphism of homotopy groups $\prod_i$, for all $i \leq n$.

Since $X \in UV^n$, there is a neighborhood $V_1 \subset \hat{X}$ of X such that:

(c) Every partial $(n+1)$-map $P \hookleftarrow A \xrightarrow{\varphi} V_1$ extends to a global map $P \to \hat{X}$.

We now apply the fact that $Y \in UV^n$. There exist neighborhoods $V_2 \subset V_1$ of X_0 and $W_2 \subset \hat{Y}$ of Y such that:

(d) $(\hat{f})^{-1}(W_2) = V_2 \cap \hat{X}_0$; and

(e) Every partial $(n+1)$-map $P \hookleftarrow A \xrightarrow{\varphi} V_2 \cap_{\hat{f}} W_2$ extends to a global map $P \to \hat{Z}$.

Finally, the hypothesis $X_0 \in UV^{n-1}$ implies the existence of neighborhoods $V_3 \subset V_1$ of X_0 and $W \subset W_2$ of Y such that:

(f) $(\hat{f})^{-1}(W) = V_3 \cap \hat{X}_0$; and

(g) Every partial n-map $P \hookleftarrow A \xrightarrow{\varphi} V_3$ extends to a global map $P \to V_2$.

Let $V \subset \hat{X}$ be a neighborhood of X such that $V \subset V_1$ and $V \cap \hat{X}_0 = V_3 \cap \hat{X}_0 = (\hat{f})^{-1}(W)$. We claim that V and W possess properties (a) and (b) above.

Let $\varphi : S^n \to V \cup_{\hat{f}} W$ be any n-spheroid (i.e. a continuous map of S^n into $V \cup_{\hat{f}} W$). An $(n+1)$-*membrane* spanning this n-spheroid is any extension of φ onto the ball B^{n+1} whose boundary is S^n. It is easy to find an $(n-1)$-dimensional piecewise-linear separator $F \subset S^n$ homeomorphic to S^{n-1}, which decomposes S^n into two closed subsets $A \cup B = S^n$, such that $A \cap B = F$ and

$$\varphi(F) \subset V_3 \backslash \hat{X}_0, \quad \varphi(A) \subset V \backslash \hat{X}_0, \quad \text{and} \quad \varphi(B) \subset V_2 \cup_{\hat{f}} W_2.$$

It follows by (g) above, that the partial n-map $A \hookleftarrow F \xrightarrow{\varphi} V_3$ extends to a map $\psi : A \to V_2$.

The separator F on the sphere S^n can be extended to a separator $\hat{F} \cong A$ on the ball B^{n+1} which will decompose B^{n+1} into two closed subsets $\hat{A} \cup \hat{B} = B^{n+1}$, $A \subset \hat{A}$, $B \subset \hat{B}$, and $\hat{A} \cap \hat{B} = \hat{F} \cong A$. Due to (c), the partial $(n+1)$-map

$$\hat{A} \hookleftarrow A \cup \hat{F} \xrightarrow{\varphi|_A \cup \psi} V_2 \hookrightarrow V_1$$

extends to a global map $\hat{A} \xrightarrow{\xi} \hat{X}$.

By (e) above, the partial $(n+1)$-map

$$\hat{B} \hookleftarrow B \cup \hat{F} \xrightarrow{\varphi|_B \cup \pi \circ \psi} V_2 \cup_f W_2$$

extends to a global map $\zeta : \hat{B} \to \hat{Z}$ (here, $\pi : \hat{X} \cup \hat{Y} \to \hat{X} \cup_{\hat{f}} \hat{Y} = \hat{Z}$ is the canonical projection). Gluing together maps $\pi \circ \xi$ and ζ along their common domain $\hat{F}$, we obtain the desired extension $\hat{\varphi} : B^{n+1} \to \hat{Z}$ of the n-spheroid φ. ∎

4. A Reduction to Local Connectedness

As it was also pointed out in Chapter 1, the fact that $\exp_{UV^n}(\ell_2(\tau))$ is in the class AE implies that the extension problem for complete open UV^n-preimages is solvable. Let us give a proof of this fact.

PROPOSITION (4.1). *If $\exp_{UV^n}(\ell_2(\tau)) \in AE$ then the extension problem for complete perfect open UV^n-preimages with kernel $\ell_2(\tau)$ s solvable.*

PROOF. Suppose that $f : X \to Y$ is any perfect open UV^n-map and $i : Y \hookrightarrow \hat{Y}$ is any closed embedding. Since f has the kernel $\ell_2(\tau)$, there exists a closed embedding $v : X \hookrightarrow Y \times \ell_2(\tau)$ such that $v(x) \in f(x) \times \ell_2(\tau)$, for all $x \in X$.

Denote the projection of $Y \times \ell_2(\tau)$ onto $\ell_2(\tau)$ by q. Then the formula $g(y) = q(v(f^{-1}(y)))$ defines a continuous map $g : Y \to \exp_{UV^n}(\ell_2(\tau))$ which, by hypothesis, has an extension $\hat{g} : \hat{Y} \to \exp_{UV^n}(\ell_2(\tau))$ over all of $\hat{Y}$. Inside the product $\hat{Y} \times \ell_2(\tau)$ we consider the subset $\hat{X} = \{(y, \hat{g}(y)) \mid y \in \hat{Y}\}$ which contains, in a natural way, $X \cong \{v(x) \mid x \in X\}$. The desired map $\hat{f} : \hat{X} \to \hat{Y}$ is then defined by $\hat{f}((y,x)) = y$, for every $(y,x) \in \hat{X}$. ∎

The deformation retraction $F_t : \ell_2(\tau) \to \ell_2(\tau)$, $F_t(\ell) = t \cdot \ell$, $0 \leq t \leq 1$, of the Hilbert space $\ell_2(\tau)$ to a point, induces a deformation retraction $\exp F_t$ of the space $\exp_{UV^n}(\ell_2(\tau))$ to a point. Consequently, to verify that $\exp_{UV^n}(\ell_2(\tau)) \in AE$, it suffices to prove that $\exp_{UV^n}(\ell_2(\tau))$ belongs to a wider class of ANE. But this is not all. As it follows from results of this chapter everything reduces to the local connectedness of $\exp_{UV^n}(\ell_2(\tau))$ in dimension n, the verification of which is the subject of our last chapter.

PROPOSITION (4.2). *If* $\exp_{UV^n}(\ell_2(\tau)) \in LC^n$ *then* $\exp_{UV^n}(\ell_2(\tau)) \in AE$.

First, let us see how to represent a metric space as a factor space of an n-dimensional space via a UV^{n-1}-decomposition.

PROPOSITION (4.3). *For every integer* $n \geq 1$ *and every metric space* X *there exist*:

$(\mathrm{a})_n$ *An* n*-dimensional metric space* $\hat{X}$ *of the same weight as* X; *and*

$(\mathrm{b})_n$ *An open perfect* UV^{n-1}*-surjection* $p_X : \hat{X} \to X$, *which is* n*-invertible* (*i.e. for every map* $\varphi : Z \to X$ *from an* n*-dimensional metric compactum* Z *into* X, *there exists a map* $\psi : Z \to \hat{X}$, *such that* $p_X \circ \psi = \varphi$).

PROOF. (Our argument is analogous to [6] and uses the Dranishnikov resolution [7].) Let $g : X \to Q$ be any completely 0-dimensional map into the Hilbert cube Q and let $d_n : M_n \to Q$ be the Dranishnikov resolution, from the n-dimensional Menger compactum M_n onto Q. The fiberwise product $\hat{X} = X_g \times_{d_n} M_n$ has dimension n, since the projection $g' : \hat{X} \to M_n$ is parallel to g and hence by [4], it is a completely 0-dimensional map into M_n. Therefore, the projection $d_n' : \hat{X} \to X$, parallel to d_n, is the desired open n-invertible perfect UV^{n-1}-surjection. ■

As a corollary of Proposition (4.3) we obtain a criterion for UV^n-compacta:

PROPOSITION (4.4). *Let* $p_X : \hat{X} \to X$ *be a surjection satisfying the conditions* $(\mathrm{a})_n$ *and* $(\mathrm{b})_n$, *let* X *be an* ANE *and* F *a compactum in* X. *Then* $F \in UV^n$ *if and only if the map* $p_F = p_X|_{p_F^{-1}(f)} = \hat{F} : \hat{F} \to F$ *is homotopic to the constant map inside any neighborhood of* F *in* X.

PROOF. The necessity follows by the property (α) from Chapter 2 so it remains to prove the sufficiency. Fix a neighborhood U of the compactum F. Since $X \in ANE$, there exists a smaller neighborhood $V \subset U$ such that:

(c) The map $p_V : \hat{V} \to V$ is homotopic to the constant map into U, i.e. $p_V : \hat{V} \to V \hookrightarrow U \simeq \text{const}$.

Let $\varphi : S^i \to V, i \leq n$, be an i-spheroid. Since p_X is n-invertible, there exists an i-spheroid $\hat{\varphi} : S^i \to \hat{V}$ such that $\varphi = p_V \circ \hat{\varphi}$. Finally, it follows by (c) above that $p_V \circ \hat{\varphi} \simeq \text{const}$ in U. ■

PROPOSITION (4.5). *Let* $\mathrm{Con}(\ell_2(\tau))$ *be a metric cone* $\ell_2(\tau) \times (0,1] \cup \{v\}$ *over* $\ell_2(\tau)$ *with vertex* v. *Then the following pairs of spaces are homeomorphic*:

(d) $(\ell_2(\tau) \times \ell_2(\tau), \ell_2(\tau) \times \{0\}) \cong (\ell_2(\tau) \times [0,1], \ell_2(\tau) \times \{0\})$; *and*

(e) $(\ell_2(\tau), \{0\}) \cong (\mathrm{Con}(\ell_2(\tau)), v)$.

PROOF. According to Henderson's theorem, $\mathrm{Con}(\ell_2(\tau)) \cong \ell_2(\tau)$ and $\ell_2(\tau) \times \ell_2(\tau) \cong \ell_2(\tau) \times [0,1] \cong \ell_2(\tau)$. Now, the assertion follows by the Unknotting theorem for Z-sets in $\ell_2(\tau)$. ■

PROPOSITION (4.6). *There exists a retraction*

$$r : \ell_2(\tau) \times [0,1] \to \ell_2(\tau) \times \{0\}$$

such that its restriction onto the complement $\ell_2(\tau) \times (0,1]$ *is injective.*

PROOF. Represent the index set τ as a disjoint union of a countable number of equipolent sets τ_n. Clearly, $\ell_2(\tau)$ is homeomorphic to every $\ell_2(\tau_n)$ as well as to the product $\prod_i \ell_2(\tau_i)$.

Let $\ell = (\ell_1, \ell_2, \ldots) \in \prod_i \ell_2(\tau_i)$, let $h_n : \prod_i \ell_2(\tau_i) \times [0,1] \to \ell_2(\tau_n)$ be a homeomorphism, and let $\{a_n\}$ be a monotone decreasing sequence of real numbers converging to zero. Every number $t \in [a_{n+1}, a_n]$ can be uniquely represented in the form:

$$t = a_{n+1} + s \cdot (a_n - a_{n+1}), \quad \text{where } 0 \le s \le 1.$$

The desired retraction is then defined by the following formula:

$$r(\ell, t) = (\ell_1, \ell_2, \ldots, \ell_n, s \cdot h_{n+1}(\ell, t) + (1-s) \cdot \ell_{n+1}, h_{n+2}(\ell, t), \ldots) \times \{0\}.$$ ■

PROPOSITION (4.7). *There exist retractions*

$$r : \ell_2(\tau) \times \ell_2(\tau) \to \ell_2(\tau) \times \{0\}$$

and

$$R : \ell_2(\tau) \times \mathrm{Con}(\ell_2(\tau)) \to \ell_2(\tau) \times \{v\}$$

such that

$$r|_{\ell_2(\tau) \times (\ell_2(\tau) \setminus \{0\})} \quad \textit{and} \quad R|_{\ell_2(\tau) \times (\mathrm{Con}(\ell_2(\tau)) \setminus \{v\})}$$

are injective (*here* v *is the cone point of* $\mathrm{Con}(\ell_2(\tau))$).

PROOF. This is an obvious consequence of Propositions (4.5) and (4.6). ■

Proposition (4.8). *Let W be a metric space of weight τ and $A \subseteq W$ a closed set. Then for every map $f : W \to \ell_2(\tau)$, there exists a map $g : W \to \ell_2(\tau)$ such that $g|_A = f|_A$ and $g|_{W \setminus A}$ is injective.*

Proof. Without loss of generality, we may assume that $\operatorname{diam} W < 1$ and that W lies in $\ell_2(\tau)$, via the embedding $h : W \hookrightarrow \ell_2(\tau)$. Then we can define the desired map g by $g|_A = f|_A$ and $g(x) = R(f(x), h(x), \operatorname{dist}(x, A))$, for $x \in W \setminus A$ (here the retraction $R : \ell_2(\tau) \times \operatorname{Con}(\ell_2(\tau)) \to \ell_2(\tau) \times \{v\} = \ell_2(\tau)$ is taken from Proposition (4.7)). ■

Proof of Proposition (4.2). Suppose that we have a partial map $Z \hookleftarrow A \xrightarrow{f} \exp_{UV^n}(\ell_2(\tau))$ which we wish to extend over Z. Since $w(\exp_{UV^n}(\ell_2(\tau))) = \tau$, this fact suffices to get a proof for $Z = \ell_2(\tau)$ (cf. [9]). Applying Proposition (4.3), we introduce a perfect open UV^n-surjection $p : \hat{Z} \to Z$ of the $(n+1)$-dimensional metric space $\hat{Z}, w(\hat{Z}) = \tau$. Let $\hat{A} = p^{-1}(A)$. Then the formula $\hat{f}(\hat{a}) = f(p(\hat{a}))$, $\hat{a} \in \hat{A}$, gives a partial $(n+1)$-map $\hat{Z} \hookleftarrow \hat{A} \xrightarrow{\hat{f}} \exp_{UV^n}(\ell_2(\tau))$. Due to the fact that $\exp_{UV^n}(\ell_2(\tau)) \in LC^n \cap C^n$, there exists a global extension $g : \hat{Z} \to \exp_{UV^n}(\ell_2(\tau))$, $g|_{\hat{A}} = \hat{f}$.

If we can find a map $\tilde{g} : \hat{Z} \to \exp_{UV^n}(\ell_2(\tau))$ such that $\tilde{g}|_{\hat{A}} = g|_{\hat{A}}$ and $\tilde{g}(\hat{z}) \cap \tilde{g}(\hat{z}_1) = \varnothing$, whenever $p(\hat{z}) = p(\hat{z}_1) \notin A$ and $\hat{z} \neq \hat{z}_1$, then the formula $\varphi(z) = f(z)$ if $z \in A$ and $\varphi(z) = \tilde{g}(p^{-1}(z))$ if $z \notin A$, will give the desired extension $\varphi : Z \to \exp_{UV^n}(\ell_2(\tau))$.

Let us construct such a map $\tilde{g}$. Consider the set $\tilde{W} = \{(\hat{z}, g(\hat{z})) \mid \hat{z} \in \hat{Z}\} \subset \hat{Z} \times \ell_2(\tau)$ of the weight τ, whose projection $\tilde{p} : \tilde{W} \to \hat{Z}$ is a perfect open surjection. Let $\tilde{A} = \tilde{p}^{-1}(\hat{A})$. Apply Proposition (4.8) to the projection $q : \tilde{W} \to \ell_2(\tau)$ onto the second factor and obtain the map $\tilde{q} : \tilde{W} \to \ell_2(\tau), \tilde{q}|_{\tilde{A}} = q|_{\tilde{A}}$, whose restriction onto the fiber $(\tilde{p})^{-1}(\hat{z})$, $\hat{z} \notin \hat{A}$, is injective. Then $\tilde{g}(\hat{z}) = \bigcup_{\tilde{p}(\tilde{\omega})=\hat{z}} \tilde{q}(\tilde{\omega})$ is a compactum, homeomorphic to $g(\hat{z}) \in UV^n$. ■

5. Local Connectedness of $\exp_{UV^n}(\ell_2(\tau))$

Proposition (5.1). *For every integer m, $\exp_{UV^n}(\ell_2(\tau)) \in LC^m$.*

Proof. In order to establish the local m-connectedness of $\exp_{UV^n}(\ell_2(\tau))$ let us fix a compactum $F \in UV^n$ in $\ell_2(\tau)$ and a number $\varepsilon > 0$. We must find a number $\delta > 0$ such that for every k-spheroid $\varphi : S^k \to \exp_{UV^n}(\ell_2(\tau))$, $k \leq m$, whose image $\operatorname{Im}\varphi$ is contained in $N_{\exp}(F, \delta) = \{F' \mid \rho_H(F, F') < \delta\}$, shrinks via some $(k+1)$-membrane $\hat{\varphi} : B^{k+1} \to \exp_{UV^n}(\ell_2(\tau))$ with $\operatorname{Im}\hat{\varphi} \subset N_{\exp}(F, \varepsilon)$.

Apply Proposition (4.3) to obtain an open perfect UV^{n-1}-surjection $p : T \to \ell_2(\tau)$ such that $\dim T \leq n$. Since $F \in UV^n$ it follows by Proposition (4.4) that there is a homotopy $H : \hat{F} \times [0,1] \to N(F, \varepsilon/2) = \{\ell \mid \rho(\ell, F) \leq \varepsilon/2\}$, from $H_0 = p$ to $H_1 = \text{const}$.

By the Borsuk Homotopy extension theorem, H can be slightly extended: there exist a number $\Delta > 0$ and a homotopy $G : \hat{V} \times [0,1] \to N(F, \varepsilon/2)$, where $\hat{V} = p^{-1}(V = N(F, \Delta))$, such that $G|_{\hat{F} \times [0,1]} = H, G_0 = p$, and $G_1 = \text{const}$. For δ take $\min\{\Delta/3, \varepsilon/4\}$.

LEMMA (5.2). *For every spheroid $\varphi : S^k \to \exp_{UV^n}(\ell_2(\tau))$, $k \leq m$, there exists a δ-homotopy $\varphi_t : S^k \to \exp_{UV^n}(\ell_2(\tau))$, from $\varphi_0 = \varphi$ to the k-spheroid φ_1, such that the following conditions are satisfied:*

(i) *For every $s \in S^k, \varphi(s)$ is homeomorphic to $\varphi_1(s)$;*

(ii) *For every $s \neq s', \varphi_1(s) \cap \varphi_1(s') = \emptyset$; and*

(iii) *The image $\bigcup\{\varphi_1(s) \mid s \in S^k\} = \varphi_1(S^k)$ of the spheroid φ_1 is a Z-set in $\ell_2(\tau)$.*

PROOF. Consider the graph $E = \bigcup\{(s, \varphi(S)) \mid s \in S^k\} \subset S^k \times \ell_2(\tau)$ of the multivalued map φ, which is a compactum. Then apply the Toruńczyk theory of Hilbert cube manifolds [14] and compose the projection $q : E \to \ell_2(\tau)$ onto the second factor, with a δ-homotopy of some Z-embedding of E into $\ell_2(\tau)$. We shall need a more precise result:

(iv) There exists a δ-homotopy $q_t : E \to \ell_2(\tau)$ such that $q_0 = q$ and for every $t > 0$, the map q_t is a Z-embedding.

We now define our homotopy $\varphi_t : S^k \to \exp_{UV^n}(\ell_2(\tau))$ to be $\varphi_t(s) = q_t(s, \varphi(s))$. It is easy to verify that the required properties (i)–(iii) indeed hold. ■

We continue the proof of Proposition (5.1). Observe that the image $\varphi_1(S^k)$ is a Z-set, hence the homotopy $G|_{p^{-1}(\varphi_1(S^k)) \times I}$ can be approximated by a new homotopy $G' : p^{-1}(\varphi_1(S^k)) \times I \to N(F, \varepsilon/2)$ such that

(1) $G'_0 = p = G_0$ and $G'_1 = \text{const}$;

(2) The restriction of G' onto $p^{-1}(\varphi_1(S^k)) \times (0,1)$ is an injection into $\ell_2(\tau)$; and

(3) The image $G'(p^{-1}(\varphi_1(S^k)) \times (0,1])$ does not intersect $\varphi_1(S^k)$.

Finally, let the k-spheroid $\varphi : S^k \to N_{\exp}(F, \delta) \cap \exp_{UV^n}$ be shrunk via a $(k+1)$-membrane $\hat{\varphi} : B^{k+1} \to N_{\exp}(F, \varepsilon) \cap \exp_{UV^n}$, for $k \leq n$. Since by Lemma (5.2), the homotopy φ_t is realized inside $N_{\exp}(F, 2\delta) \cap \exp_{UV^n}$, it suffices to shrink the k-spheroid

$$\varphi_1 : S^k \to N_{\exp}(F, 2\delta) \cap \exp_{UV^n} \subset N_{\exp}\left(F, \frac{\varepsilon}{2}\right)$$

by a $(k+1)$-membrane $\hat{\varphi}$. ■

LEMMA (5.3). *There exist a homotopy* $\varphi_t : S^k \to N_{\exp}(F, \varepsilon/2) \cap \exp_{UV^n}$, $1 \leq t \leq 3$, *from* φ_1 *to a spheroid* φ_3 *and a point* $\{*\}$ *such that:*
(v) $\varphi_3(s) \cap \varphi_3(s') = *$ *if and only if* $s \neq s'$; *and*
(vi) $\varphi_3(S^k) \in UV^n$.

PROOF. Define the homotopy φ_t by the formula

$$\varphi_t(s) = G'(\tilde{\varphi}(s) \times [0, t-1]), \quad \text{for } 1 \leq t \leq 2$$

and

$$\varphi_t(s) = G'(\tilde{\varphi}(s) \times [(t-2)/A, 1]), \quad \text{for } 2 \leq t \leq 3,$$

where $\tilde{\varphi}(s) = p^{-1}(\varphi_1(s)) \in UV^{n-1}$, and A is large enough number so that for all $s \in S^k$, the set $G'(\tilde{\varphi}(s) \times [1/A, a])$ lies in $N_{\exp}(F, \varepsilon/2)$.

It is clear from the formulae for φ_t that the homotopy lives in some neighborhood of $N_{\exp}(F, \varepsilon/2)$. Let us verify this for $\varphi_t(s) \in UV^n$. The compactum $\varphi_t(s)$ contracts in itself inside $G'(\tilde{\varphi}(s) \times \{0\}) = G(\tilde{\varphi}(s) \times \{0\}) = \varphi_1(s) \in UV^n$, for $1 \leq t < 2$, whereas for $2 < t \leq 3$ it contracts in itself to the point $\text{const}_1 = G'(\tilde{\varphi}(s) \times \{1\})$. Therefore by Proposition (2.3), $\varphi_t(s) \in UV^n$, for all $t \neq 2$.

For $t = 2$, the compactum $\varphi_t(s)$ is the result of the adjoining the cone $\text{Con}(\tilde{\varphi}(s)) \in UV^n$ and the UV^n-set $\varphi_1(s)$ by the partial map $\text{Con}(\tilde{\varphi}(s)) \hookleftarrow \tilde{\varphi}(s) \times \{0\} \xrightarrow{p} \varphi_1(s)$. Since $\tilde{\varphi}(s) \in UV^{n-1}$ it follows by the theorem on adjoining UV^n-sets that

$$\text{Con}(\tilde{\varphi}(s)) \cup_p \varphi_1(s) = \varphi_2(s) \in UV^n.$$

Finally, $\varphi_3(S^k) = G'(\tilde{\varphi}(S^k) \times [1/A, 1])$, where $\tilde{\varphi}(S^k) = p^{-1}(\varphi_1(S^k))$, contracts in itself to a point $* = G'(\tilde{\varphi}(S^k) \times \{1\})$, by the formula

$$G'(\tilde{\varphi}(S^k) \times [(1 - 1/A) \cdot s + 1/A, 1], \quad 0 \leq s \leq 1$$

and is therefore a UV^n-set. The property (v) follows from the property (2) of the homotopy G' and the property (ii) of the k-spheroid φ_1. ■

We can now complete the proof of Proposition (5.1). It is clear that it suffices to shrink the k-sphere φ_3 by the $(k+1)$-membrane $\hat{\varphi}$. Fix a continuous multivalued retraction $D: B^{k+1} \to S^k = \partial B^{k+1}$ from [7], such that $D(0) = S^k$, $D(r \cdot s) \cong B^k$ and $D(1 \cdot s) = s$, for $0 < r < 1, s \in S^k$.

The desired $(k+1)$-membrane $\hat{\varphi}: B^{k+1} \to N_{\exp}(F, \varepsilon/2) \cap \exp_{UV^n}$ is given by the formula $\hat{\varphi}(r \cdot s) = \varphi_3(D(r \cdot s))$. Continuity of $\hat{\varphi}$ follows from the continuity of D and φ_3. Clearly, $\hat{\varphi}(B^{k+1}) \subset N_{\exp}(F, \varepsilon/2)$. Since $\hat{\varphi}(r \cdot s)$ contracts in itself to a point $\{*\}, \hat{\varphi}(r \cdot s)$ is a UV^n-compactum, for all $r \neq 0$ and $s \in S^k$. ■

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