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A NOTE ON THE WAVE PACKET TRANSFORMS

Dedicated to Professor Kunihiko Kajitani on the occasion of his sixtieth birthday

By

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1. Introduction

The notion of wave front sets of distributions is important in the theory of partial differential equations since it was introduced by M. Sato and L. Hörmander in both the analytic category and the C^{∞} one. The FBI transformation, introduced by Bros-Iagolnitzer and Sjöstrand, is very useful to give their characterization in these two categories.

On the other hand, G. B. Folland ([2]) introduced the notion of wave packet transforms $\mathscr{P}_{\phi}$ corresponding to each member ϕ of the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$. In this context the standard FBI transformation can be viewed as the wave packet transform corresponding to the Gaussian function $e^{-|x|^2/2}$. In his book he developed a certain symbol calculus related to ϕ when ϕ is an arbitrary nontrivial even function. Furthermore, as its application, he proved that if ϕ is an arbitrary nontrivial even function, wave front sets can be characterized by the wave packet transforms. He raised there an open question whether it is necessary to assume that ϕ is even or not for the symbol calculus.

Our aim is to present a different type of sufficient condition on ϕ. This condition means that ϕ is not necessarily even. Furthermore, we also discuss the H^s wave front sets in terms of wave packet transforms.

2. Wave Packet Transforms

First of all, we recall the definition of the wave front set of distributions, which can give a precise description of the local smoothness properties of distributions. Let $\Omega \subset \boldsymbol{R}^n$ be open and $u \in \mathscr{D}'(\Omega)$. Consider a couple $(x_0, \xi_0) \in \Omega \times \dot{\boldsymbol{R}}^n$. Here and

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in what follows, $\dot{R}^n$ denotes $R^n\backslash\{0\}$. For a function f belonging to the Schwartz space $\mathscr{S}(R^n)$, $\hat{f}$ denotes its Fourier transformation:

$$\hat{f}(\xi) = \int_{R^n} e^{-ix\cdot\xi} f(x)\, dx.$$

DEFINITION 2.1. *We will say that* (x_0, ξ_0) *does not belong to the wave front set of* u, *denoted by* $WF(u)$, *if and only if*

$$\exists f \in C_0^\infty(\Omega) \text{ with } f(x_0) \neq 0, \quad \exists \text{ an open cone } \Gamma \ni \xi_0$$

such that

$$\forall N \in N, \quad \exists C > 0 / \forall \xi \in \Gamma, \quad |\widehat{fu}(\xi)| \leq C(1 + |\xi|)^{-N}.$$

Next, we recall the definition of wave packet transforms, introduced by G. B. Folland [2].

Let q and p be in R^n, and let u be a measurable function on R^n. i denotes the imaginary unit $\sqrt{-1}$. We define the function $\rho(p, q)f$ on R^n by

$$(\rho(p,q)f)(x) = e^{iq\cdot x + (i/2)q\cdot p} f(x+p), \quad x \in R^n. \tag{2.1}$$

It is easily verified that $\rho(p,q) : L^2(R^n) \to L^2(R^n)$ is a unitary operator for all p and q in R^n and that

$$\rho(p,q)^{-1} = \rho(-p, -q).$$

Given a nonzero function $\phi \in \mathscr{S}(R^n)$, we set

$$\phi^\lambda(x) = \lambda^{n/4}\phi(\lambda^{1/2}x),$$

and define the wave packet transform of u as

$$\mathscr{P}_\phi^\lambda u(x,\xi) = \int u(y)\overline{e^{(i/2)x\cdot\xi - iy\cdot\xi}\phi^\lambda(y-x)}\, dy = \int u(y)\overline{\rho(-x,\xi)\phi^\lambda(y)}\, dy.$$

Using the unitary transformation (2.1), we rewrite it as

$$\mathscr{P}_\phi^\lambda u(x,\xi) = (u, \rho(-x,\xi)\phi^\lambda).$$

Let u be a tempered distribution on R^n. The FBI transformation of u is the function on $C^n \times [0, +\infty)$ defined by

$$Tu(z,\lambda) = \int e^{-(\lambda/2)(z-y)^2} u(y)\, dy,$$

where $(z-y)^2 = \sum_{j=1}^n (z_j - y_j)^2$. It is an entire function of the complex variable z, real analytic with respect to the parameter λ. If u is a compactly supported distribution, it is of finite order. Thus, there exist an integer N and a constant $C > 0$ such that

$$|Tu(z,\lambda)| \leq C(1+\lambda+|\mathrm{Im}\, z|)^N \exp\{\lambda(\mathrm{Im}\, z)^2/2\}$$

for $z \in \boldsymbol{C}^n$ and $\lambda \in [0,\infty)$. From the simple identity

$$-\frac{1}{2}(x-i\xi-y)^2 = -\frac{1}{2}|x-y|^2 + i\xi(x-y) + \frac{1}{2}|\xi|^2,$$

it is obvious that

$$e^{-\lambda|\xi|^2/2} Tu(x-i\xi,\lambda) = e^{i\lambda x\cdot\xi/2} P_\phi^\lambda u(x,\lambda\xi)$$

where $\phi(x)$ is the Gaussian function $e^{-|x|^2/2}$.

Now, we can state a characterization of the wave front set of a tempered distribution u.

THEOREM 2.2. *Suppose that $\phi \in \mathscr{S}(\boldsymbol{R}^n)$ satisfies*

$$\int_{\boldsymbol{R}^n} x^\alpha \phi(x)\, dx \neq 0. \tag{2.2}$$

for some $\alpha \in (\boldsymbol{N} \cup \{0\})^n$. Let Ω be an open subset of $\boldsymbol{R}^n$, let (x_0,ξ_0) be a point of $\Omega \times \dot{\boldsymbol{R}}^n$ and let u be a compactly supported distribution defined in Ω. Then, (x_0,ξ_0) does not belong to the wave front set $WF(u)$ if and only if there is a conic neighborhood V of (x_0,ξ_0) such that for all a, $N \geq 1$,

$$|\mathscr{P}_\phi^\lambda u(x,\lambda\xi)| \leq C_{a,N}\lambda^{-N} \quad \text{for } \lambda \geq 1,\ a^{-1} \leq |\xi| \leq a \text{ and } (x,\xi) \in V.$$

REMARK 2.1. This gives a generalization of the result by G. B. Folland, who proved the similar result under the restriction that ϕ is an even non-trivial function.

As for the H^s wave front sets, we have

THEOREM 2.3. *Suppose that $\int_{\boldsymbol{R}^n} \phi(x)\, dx \neq 0$ and u is a compactly supported distribution defined in Ω. Then $(x_0,\xi_0) \in \Omega \times (\boldsymbol{R}^n \backslash \{0\})$ does not belong to the H^s wave front set $WF_s(u)$ if and only if there is a relatively compact neighborhood $V \times \Gamma$ of (x_0,ξ_0) such that*

$$\int_1^\infty \lambda^{3n/2-1+2s}\, d\lambda \int_\Gamma |\{\psi(x)\mathscr{P}_\phi^\lambda(u)\}^\wedge(\lambda\xi/2,\lambda\xi)|^2\, d\xi < \infty$$

for a smooth function $\psi \in C_0^\infty(V)$ *satisfying* $\psi(x) = 1$ *near* x_0. *Here,* ^ *denotes the Fourier transform with respect to the first variables* x.

THEOREM 2.4. *Suppose that* $\phi \in \mathscr{S}(\boldsymbol{R}^n)$ *satisfies*

$$\int_{\boldsymbol{R}^n} \phi(x)\, dx \neq 0.$$

Let $(x_0, \xi_0) \in \Omega \times \dot{\boldsymbol{R}}^n$ *and* u *a compactly supported distribution defined in* Ω. *Then,* (x_0, ξ_0) *does not belong to the* H^s *wave front set* $WF_s(u)$ *if there is a relatively compact neighborhood* V *of* (x_0, ξ_0) *such that*

$$\int_1^\infty d\lambda \int_V \lambda^{n-1+2s} |\mathscr{P}_\phi^\lambda(u)(x, \lambda\xi)|^2\, dx d\xi < \infty. \tag{2.3}$$

Conversely if $(x_0, \xi_0) \notin WF_s(u)$, *then there is a relatively compact neighborhood* V *of* (x_0, ξ_0) *such that*

$$\int_1^\infty d\lambda \int_V \lambda^{n-1+2s-\varepsilon} |\mathscr{P}_\phi^\lambda u(x, \lambda\xi)|^2\, dx d\xi < \infty \tag{2.4}$$

for all $\varepsilon > 0$.

REMARK 2.2. There is a small gap between (2.3) and (2.4). P. Gérard gave a similar characterization of the H^s wave front set in terms of the FBI transformation without this gap ($\varepsilon = 0$).

REMARK 2.3. Under the more general condition (2.2) on ϕ, the similar conclusion to that of Theorem 2.4 can be verified in the same manner if we allow an additional gap.

REMARK 2.4. It is well known that $\rho \in \dot{T}^*(\Omega)$ does not belong to $WF_s(u)$ if and only if there exists a function $v \in H^s(\Omega)$ such that $\rho \notin WF(u - v)$. Therefore, the action of a local diffeomorphism χ on the H^s wave front sets follows from the one on the usual C^∞ wave front sets.

3. Proof of Theorem 2.2

In what follows, we shall use the notion of the wave front set of u with respect to ϕ. Namely, we say that $(x_0, \xi_0) \in \Omega \times \dot{\boldsymbol{R}}^n$ does not belong to $WF_\phi(u)$ if and only if there is a conic neighborhood V of (x_0, ξ_0) such that for all a, $N \geq 1$,

$$|\mathscr{P}_\phi^\lambda u(x, \lambda\xi)| \leq C_{a,N} \lambda^{-N} \quad \text{for } \lambda \geq 1,\ a^{-1} \leq |\xi| \leq a \text{ and } (x, \xi) \in V.$$

The half of the result of Theorem 2.2 is actually proved in Theorem 3.22 of [2] without any additional assumption on ϕ.

THEOREM 3.1 ([2]). *Under the same assumption as in Theorem* 2.2, *it holds that*

$$WF(u) \supset WF_{\phi}(u).$$

Therefore, the essential part of Theorem 2.2 is the converse relation:

THEOREM 3.2. *Under the same assumption as in Theorem* 2.2, *it holds that*

$$WF(u) \subset WF_{\phi}(u).$$

We introduce the lexicographic order relation in $(\boldsymbol{N} \cup \{0\})^n$. Namely, let us consider any pair α and β of $(\boldsymbol{N} \cup \{0\})^n$. We say $\alpha \prec \beta$ if and only if there exists ℓ such that $\alpha_j = \beta_j$ for any $0 \le j \le \ell$ and $\alpha_{\ell+1} < \beta_{\ell+1}$. Let δ be the minimum element of all multi-indices α for which $\int x^{\alpha}\phi(x)\,dx \neq 0$. Let $\tilde{M} = (\tilde{M}_1, \dots, \tilde{M}_n)$ be a vector of positive entries satisfying the inequalities

$$\tilde{M}_j > \sum_{k=j+1}^{n} \delta_k \tilde{M}_k, \quad \forall j = 1, \dots, n. \tag{3.1}$$

For the sake of this choice, we can easily verify the following important property.

LEMMA 3.3. *For any* $\alpha \succ \delta$, *it holds that*

$$\tilde{M} \cdot \alpha > \tilde{M} \cdot \delta.$$

Let $M_j = \tilde{M}_j - 1/2$ and define an auxiliary function $w(x, \lambda)$ as

$$w(x, \lambda) = \sum_{j=1}^{n} x_j \lambda^{-M_j}.$$

We are going to show an asymptotic expansion. By a translation, we may assume that $x_0 = 0$. Let $\xi_0 \in \dot{\boldsymbol{R}}^n$ and $\chi \in C_0^{\infty}(\Omega)$ be any nonnegative function supported in a neighborhood of the origin such that $\chi(x) = 1$ near 0.

PROPOSITION 3.4. *Let* $M_n > 0$ *be a large enough and* $\psi \in C_0^{\infty}(\boldsymbol{R}^n)$ *be a non-negative function equal to one in a neighborhood of the origin. Then, there exists a sequence* $\{d_{\alpha,\beta}\}$ *of positive numbers such that for any* $N \in \boldsymbol{N}$,

$$\text{(3.2)} \qquad \int \psi(x) e^{-(i/2)x\cdot\lambda\xi} e^{iw(x,\lambda)} \mathscr{P}_\phi^\lambda(\chi u)(x, \lambda\xi)\, dx$$

$$= \sum_{|\alpha|=0}^{N-1} \sum_{|\beta|=0}^{N-1} \Phi_\beta d_{\alpha,\beta} \lambda^{-n/4 - M\cdot\alpha - \tilde{M}\cdot\beta} (\partial_\xi^\alpha \widehat{\chi u})(\lambda\xi) + O(\lambda^{-N})$$

as $\lambda \to \infty$. *Here,* $\Phi_\beta = \int_{\boldsymbol{R}^n} x^\beta \bar{\phi}(x)\, dx$

PROOF. First, we use the following identity.

LEMMA 3.5.

$$\mathscr{P}_\phi^\lambda f(x, \xi) = (2\pi)^n e^{(i/2)x\cdot\xi} \hat{f} * \zeta_{x,\lambda}(\xi) = (2\pi)^n e^{(i/2)x\cdot\xi} \int \hat{f}(\xi - \eta) \zeta_{x,\lambda}(\eta)\, d\eta,$$

where

$$\zeta_{x,\lambda}(\eta) = e^{-ix\cdot\eta} \lambda^{-n/4} \overline{\hat{\phi}(-\lambda^{-1/2}\eta)}.$$

PROOF. This can be easily verified in view of

$$e^{-(i/2)x\cdot\xi} \mathscr{P}_\phi^\lambda f(x, \xi) = \{f(y)\phi^\lambda(y - x)\}^\wedge(\xi). \qquad \square$$

Applying Lemma 3.5 for $f = \chi u$, we obtain

$$\text{(3.3)} \qquad e^{-(i/2)x\cdot\lambda\xi} \mathscr{P}_\phi^\lambda(\chi u)(x, \lambda\xi) = \lambda^{-n/4} \int e^{-ix\cdot\eta} \overline{\hat{\phi}(-\lambda^{-1/2}\eta)} \widehat{\chi u}(\lambda\xi - \eta)\, d\eta$$

$$= \lambda^{3n/4} \int e^{-ix\cdot\lambda\eta} \overline{\hat{\phi}(-\lambda^{1/2}\eta)} \widehat{\chi u}(\lambda\xi - \lambda\eta)\, d\eta.$$

According to the Taylor theorem, we have the following expansion:

$$\text{(3.4)} \qquad \widehat{\chi u}(\lambda\xi - \lambda\eta) = \sum_{|\alpha|<N} \frac{(-\lambda\eta)^\alpha}{\alpha!} (\partial_\xi^\alpha \widehat{\chi u})(\lambda\xi) + \sum_{|\alpha|=N} \frac{(-\lambda\eta)^\alpha}{\alpha!} (\partial_\xi^\alpha \widehat{\chi u})(\lambda\xi - \theta\lambda\eta)$$

for some $0 < \theta < 1$.

In view of the identity

$$(-\lambda\eta)^\alpha e^{-ix\cdot\lambda\eta} = \left(\frac{1}{i}\partial_x\right)^\alpha e^{-ix\cdot\lambda\eta},$$

an integration by parts in x implies that

LEMMA 3.6.

$$\lambda^{3n/4}\sum_{|\alpha|<N}\alpha!^{-1}(\partial_\xi^\alpha\widehat{\chi u})(\lambda\xi)\int e^{-ix\cdot\lambda\eta}\psi(x)e^{iw(x,\lambda)}\overline{\hat{\phi}(-\lambda^{1/2}\eta)}(-\lambda\eta)^\alpha\,dxd\eta \tag{3.5}$$

$$=\lambda^{-n/4}\sum_{|\alpha|<N}\frac{(2\pi)^n}{\alpha!}(\partial_\xi^\alpha\widehat{\chi u})(\lambda\xi)\int(i\partial_x)^\alpha\{\psi(x)e^{iw(x,\lambda)}\}\bar{\phi}(\lambda^{1/2}x)\,dx.$$

PROOF. We see that

$$\lambda^{3n/4}\int\psi(x)e^{iw(x,\lambda)}e^{-ix\cdot\lambda\eta}\overline{\hat{\phi}(-\lambda^{1/2}\eta)}\frac{(-\lambda\eta)^\alpha}{\alpha!}\,dxd\eta \tag{3.6}$$

$$=\frac{1}{\alpha!}\lambda^{3n/4}\int(i\partial_x)^\alpha\{\psi(x)e^{iw(x,\lambda)}\}e^{-ix\cdot\lambda\eta}\overline{\hat{\phi}(-\lambda^{1/2}\eta)}\,dxd\eta$$

$$=\frac{1}{\alpha!}(2\pi)^n\lambda^{n/4}\int(i\partial_x)^\alpha\{\psi(x)e^{iw(x,\lambda)}\}\bar{\phi}(\lambda^{1/2}x)\,dx$$

At the last equality, we have used a consequence of the Fourier inversion formula:

$$\int e^{-ix\cdot\lambda\eta}\overline{\hat{\phi}(-\lambda^{1/2}\eta)}\,d\eta=(2\pi)^n\bar{\phi}(\lambda^{1/2}x)\lambda^{-n/2}. \qquad\square$$

Since $\psi(x)\in\mathscr{S}$ is equal to one near the origin, it holds that for any $N>1$,

$$\int x^\beta\psi(x)\bar{\phi}(\lambda^{1/2}x)\,dx=\int x^\beta\bar{\phi}(\lambda^{1/2}x)\,dx+\mathcal{O}(\lambda^{-N})$$

and

$$x^\beta\partial_x^\alpha\psi(x)\bar{\phi}(\lambda^{1/2}x)=\mathcal{O}(\lambda^{-N}),\quad\forall|\alpha|>0. \tag{3.7}$$

Thus, it is seen that

$$(\lambda^{1/2}\partial_x)^\alpha e^{iw(\lambda^{-1/2}x,\lambda)}=i^{|\alpha|}\lambda^{-M\cdot\alpha}e^{iw(\lambda^{-1/2}x,\lambda)} \tag{3.8}$$

and

$$e^{iw(\lambda^{-1/2}x,\lambda)}=\prod_{j=1}^n e^{ix_j\lambda^{-M_j-1/2}}=\prod_{j=1}^n\sum_{k_j=1}^\infty\frac{(ix_j\lambda^{-M_j-1/2})^{k_j}}{k_j!}.$$

Inserting it into (3.5), we arrive at the formal expansion as in the statement of Proposition 3.4 because the corresponding terms in the right hand side of (3.6) when $|\alpha|>0$ is rapidly decreasing as $\lambda\to\infty$.

To complete the proof, it suffices to estimate the remainder terms.

LEMMA 3.7. *Let $u \in H^s(\boldsymbol{R}^n)$ for some $s \in \boldsymbol{R}$ and let $g(x)$ a function on $\boldsymbol{R}^n$ belonging to $C_0^\infty(\boldsymbol{R}^n)$ and equal to one in a neighborhood of the origin. Set*

$$F_\alpha(\xi, \lambda) = \int g(x)(\lambda\eta)^\alpha e^{-ix\cdot\lambda\eta} e^{iw(x,\lambda)} \bar{\hat{\phi}}(-\lambda^{1/2}\eta)(\partial_\xi^\alpha \widehat{\chi u})(\lambda\xi - \theta\lambda\eta)\, dxd\eta$$

with $0 < \theta < 1$. Then, for any compact subset K of $\boldsymbol{R}^n$, there exists a positive number ε,

$$\sup_{\xi \in K} |F_\alpha(\xi, \lambda)| = O(\lambda^{M_0 - \varepsilon|\alpha|})$$

for some real number M_0 which is independent of α.

PROOF. We note that there exist a real numbers m_0 and m_1 such that

$$|\widehat{\chi u}(\lambda(\xi - \theta\eta))| \le C\lambda^{m_0}(1 + |\eta|)^{m_1}$$

for all ξ in any compact subset of $\boldsymbol{R}^n$ and for all $0 < \theta < 1$. It holds that

$$F_\alpha(\xi, \lambda) = \int (\partial_x)^\alpha \{g(x) e^{iw(x,\lambda)}\} e^{-ix\cdot\lambda\eta} \bar{\hat{\phi}}(-\lambda^{1/2}\eta)(\partial_\xi^\alpha \widehat{\chi u})(\lambda(\xi - \theta\eta))\, dxd\eta. \tag{3.9}$$

In view of (3.7) and (3.8), we arrive at the conclusion of Lemma 3.7 if we take a positive number ε such that $\varepsilon < \min\{M_j\}$. Indeed, if $\operatorname{supp}\chi \cap \operatorname{supp}\nabla g = \varnothing$, then we can use

$$(-\Delta_{\lambda\eta}/|x - \theta y|^2)^N e^{-i(x-\theta y)\cdot\lambda\eta} = e^{-i(x-\theta y)\cdot\lambda\eta}.$$

Hence, an integration by parts in η shows that

$$\sup_{x \in \operatorname{supp}(1-g)} \left| \iint \bar{\hat{\phi}}(-\lambda^{1/2}\eta) e^{-ix\cdot\lambda\eta - iy\cdot\lambda(\xi - \theta\eta)} (-iy)^\alpha \chi u(y)\, dyd\eta \right| = \mathcal{O}(\lambda^{-N}). \qquad \square$$

Now, we are in a position to prove $WF(u) \subset WF_\phi(u)$ (Theorem 3.2). We assume that for some $s_0 \in \boldsymbol{R}$ and a positive number h,

$$\sup_{\xi \in B_h(\xi_0)} |\widehat{\chi u}(\lambda\xi)| = \mathcal{O}(\lambda^{s_0}).$$

Thus, it is seen that

$$\sup_{\xi \in B_h(\xi_0)} |(\partial_\xi^\alpha \widehat{\chi u})(\lambda\xi)| = \mathcal{O}(\lambda^{s_0}).$$

Let (x_0, ξ_0) be in the complement of $WF_\phi(u)$ and δ the minimum element of all multi-indices α for which $\int x^\alpha \phi(x)\,dx \neq 0$.

Let χ be chosen so that its support is small enough such that

$$\operatorname{supp}\chi \cap \operatorname{supp}(1-\psi) = \varnothing.$$

Then from Lemma 3.3 and Proposition 3.4, it follows that if ξ is in a neighborhood of ξ_0,

$$(3.10)\quad \left|\sum_{|\alpha|<N} \lambda^{-n/4-M\cdot\alpha-\tilde{M}\delta-|\delta|/2} f_\alpha(\lambda)(\partial_\xi^\alpha \widehat{\chi u})(\lambda\xi)\right|$$

$$\leq \mathcal{O}(\lambda^{-N+s}) + C\left\{\int_{\operatorname{supp}\psi} |\mathscr{P}_\phi^\lambda(\chi u)(x,\lambda\xi)|^2\,dx\right\}^{1/2}, \quad \forall N > 1.$$

Here, $f_\alpha(\lambda)$ denotes the function

$$\sum_{\{\beta:\tilde{M}\cdot\beta<N-M\cdot\alpha\}} \Phi_\beta d_{\alpha,\beta}\lambda^{-\tilde{M}\cdot(\beta-\delta)},$$

so that

$$f_\alpha(\lambda) = \Phi_\delta + \mathcal{O}(\lambda^{-\kappa}), \quad \exists\kappa > 0.$$

In conclusion, there exists a positive number κ such that

$$(3.11)\quad |\widehat{\chi u}(\lambda\xi)| \leq \sum_{0<|\alpha|\leq N-1} C_\alpha \lambda^{-\kappa-M\cdot\alpha}|\partial_\xi^\alpha \widehat{\chi u}(\lambda\xi)|$$

$$+ C \sup_{x\in\operatorname{supp}\psi} |\mathscr{P}_\phi^\lambda(\chi u)(x,\lambda\xi)|\lambda^{n/4+\tilde{M}\cdot\delta} + O(\lambda^{-N+n/4+s_0+\tilde{M}\cdot\delta})$$

if ξ is in a neighborhood Γ of ξ_0.

On the other hand, from the stability property (Lemma 3.9) of $\mathscr{P}_\phi^\lambda(u)$ under multiplication $(u \mapsto \chi u)$, it follows that if $0 < r' < r$ and $\bar{\Gamma}' \subset \Gamma$,

$$\sup_{|x-x_0|<r',\xi\in\Gamma'} |\mathscr{P}_\phi^\lambda(\chi u)(x,\lambda\xi)| = \mathcal{O}(\lambda^{-N}), \quad \forall N > 1$$

since

$$\sup_{|x-x_0|<r,\xi\in\Gamma} |\mathscr{P}_\phi^\lambda(u)(x,\lambda\xi)| = \mathcal{O}(\lambda^{-N}), \quad \forall N > 1.$$

Finally, we obtain

$$|\widehat{\chi u}(\lambda\xi)| \le \sum_{0<|\alpha|\le N-1} C_\alpha \lambda^{-\kappa - M\cdot\alpha} |\partial_\xi^\alpha \widehat{\chi u}(\lambda\xi)| + \mathcal{O}(\lambda^{-N}), \quad \forall N \ge 1, \ \xi \in \Gamma.$$

We would like to use an iteration procedure to conclude that

$$|\widehat{\chi u}(\lambda\xi)| = \mathcal{O}(\lambda^{-N}), \quad \forall N > 1$$

while ξ is in a neighborhood of ξ_0.

To make use of the iteration process, we require the analogue of the local stability of wave front set.

Lemma 3.8. *Let Γ be a cone of $\boldsymbol{R}^n$, let s, $N_0 \in \boldsymbol{R}$ and let $u \in H^{-N_0}(\boldsymbol{R}^n)$. Suppose that for all closed conic sets $\Gamma' \subset \Gamma$,*

$$\sup_{\Gamma'} |(1 + |\xi|)^s \hat{u}(\xi)| < \infty.$$

Then, for any $\zeta \in C_0^\infty(\boldsymbol{R}^n)$ and any closed conic $\Gamma' \subset \Gamma$,

$$\sup_{\Gamma'} |(1 + |\xi|)^s \widehat{\zeta u}(\xi)| < \infty.$$

Proof. Choose $\varepsilon > 0$ so small that $\xi - \eta \in \Gamma$ when $\xi \in \Gamma'$ and $|\eta| < \varepsilon|\xi|$. We write

$$(3.12) \qquad (2\pi)^n \widehat{\zeta u}(\xi) = \int_{|\eta|<\varepsilon|\xi|} \hat{\zeta}(\eta)\hat{u}(\xi - \eta)\, d\eta + \int_{|\eta|\ge\varepsilon|\xi|} \hat{\zeta}(\eta)\hat{u}(\xi - \eta)\, d\eta$$
$$= I_1 + I_2.$$

When $|\eta| < \varepsilon|\xi|$, there exist positive constants C and C' such that

$$C'\langle\xi\rangle \ge \langle\xi - \eta\rangle \ge C\langle\xi\rangle.$$

Thus, for $\xi \in \Gamma'$,

$$|I_1| \le C_N \int_{|\eta|<\varepsilon|\xi|} |\hat{\zeta}(\eta)| \langle\xi - \eta\rangle^{-s}\, d\eta \le C_N' \langle\xi\rangle^{-s} \int |\hat{\zeta}(\eta)|\, d\eta.$$

On the other hand, $|\hat{u}(\eta)| \le C\langle\eta\rangle^K$ for some K, and when $|\eta| \ge \varepsilon|\xi|$ we have $\langle\xi - \eta\rangle \le C\langle\eta\rangle$. Therefore, if $s \ge 0$, then

$$(3.13) \qquad |I_2| \le C\langle\xi\rangle^{-|s|} \int_{|\eta|\ge\varepsilon|\xi|} |\hat{\zeta}(\eta)| (\varepsilon^{-1}\langle\eta\rangle)^{|s|} (\xi - \eta)^K\, d\eta$$
$$\le C\langle\xi\rangle^{-|s|} \int |\hat{\zeta}(\eta)| \langle\eta\rangle^{|s|+K}\, d\eta.$$

If $s < 0$, then

$$(3.14) \qquad |I_2| \leq C\langle\xi\rangle^{-s} \int_{|\eta| \geq \varepsilon|\xi|} |\hat{\zeta}(\eta)|(\xi - \eta)^K \, d\eta \leq C\langle\xi\rangle^{-s} \int |\hat{\zeta}(\eta)| \langle\eta\rangle^K \, d\eta. \quad \square$$

Similarly,

LEMMA 3.9. *Let $V \times \Gamma$ be a relatively compact neighborhood of (x_0, ξ_0) of $\boldsymbol{R}^n$, let s, $N_0 \in \boldsymbol{R}$ and let $u \in H^{-N_0}(\boldsymbol{R}^n)$. Suppose that for all closed set $V' \times \Gamma' \subset \Gamma$,*

$$\sup_{V' \times \Gamma'} |\lambda^s \mathscr{P}_\phi^\lambda(u)(x, \lambda\xi)| < \infty.$$

Then, for any $\zeta \in C_0^\infty(\boldsymbol{R}^n)$ and any closed set $V' \times \Gamma' \subset \Gamma$,

$$\sup_{V' \times \Gamma'} |\lambda^s \mathscr{P}_\phi^\lambda(\zeta u)(x, \lambda\xi)| < \infty.$$

PROOF.

$$(2\pi)^n e^{-(i/2)x\cdot\xi} \mathscr{P}_\phi^\lambda(\zeta u)(x, \xi) = \int \hat{\zeta}(\eta) \int e^{-iy\cdot(\xi - \eta)} u(y) \overline{\phi^\lambda}(y - x) \, dy d\eta$$

We divide the above integral in η into two parts as in the proof of the previous lemma. The same reasoning arrive at the conclusion. $\square$

We now continue the proof of Theorem 3.2. We note that

$$(3.15) \qquad (\partial_\xi^\alpha \widehat{\chi u})(\lambda\xi) = (y^\alpha \chi(y) u(y))^\wedge(\lambda\xi).$$

Now, we choose $M_n, M_{n-1}, \ldots, M_1$ so small that $\varepsilon > \tilde{M} \cdot \delta - |\delta|/2 > 0$ for any $|\beta| \neq 0$.

Applying Lemma 3.8 to (3.15), we see that the inequality (3.11) implies

$$\sup_{\xi \in B_h(\xi_0)} |\widehat{\chi u}(\lambda\xi)| = \mathcal{O}(\lambda^{\max\{s_0 - \kappa, -N + n/4 + s_0 + \tilde{M}\cdot\delta\}})$$

We repeat this procedure ℓ times to arrive at the conclusion

$$\sup_{\xi \in B_h(\xi_0)} |\widehat{\chi u}(\lambda\xi)| = \mathcal{O}(\lambda^{\max\{s - \ell\kappa, -N + n/4 + s_0 + \tilde{M}\cdot\delta\}}).$$

This completes the proof of Theorem 3.2.

4. Proof of Theorem 2.4

In what follows if a function $f(\lambda)$ defined in $(1,\infty)$ satisfies that for all $N>1$ there exists a positive constant C_N such that

$$|f| \le C_N \lambda^{-N}, \quad \forall \lambda > 1,$$

then we use the notation $f = \mathcal{O}(\lambda^{-\infty})$.

The key lemma to deal with H^s wave front sets is given by

LEMMA 4.1. *Let $\chi, \psi \in C_0^\infty$ be supported in a sufficiently small neighborhood of (x_0, ξ_0) such that*

$$\operatorname{supp}(1-\psi(x)) \cap \operatorname{supp} \chi = \emptyset. \tag{4.1}$$

Then, there exist neighborhoods $V_\pm$ $(V_- \subset V \subset V^+)$ of (x_0, ξ_0) such that

$$\lambda^{-(1-2\sigma)n/2} \int_V d\xi \int |\chi(x)\mathscr{P}_\phi^\lambda(\psi u)(x, \lambda\xi)|^2\, dx \le C \int_{V_+} |(\widehat{\psi u})(\lambda\xi)|^2\, d\xi + \mathcal{O}(\lambda^{-\infty})$$

and

$$\int_V d\xi \int |\chi(x)\mathscr{P}_\phi^\lambda(\psi u)(x, \lambda\xi)|^2\, dx \ge C\lambda^{-n} \int_{V_-} |(\widehat{\psi u})(\lambda\xi)|^2\, d\xi + \mathcal{O}(\lambda^{-\infty}).$$

PROOF. Plancherel's theorem implies

(4.2)

$$\lambda^{-n}\pi^{-n} \int |\chi(x)\mathscr{P}_\phi^\lambda(\psi u)(x, \lambda\xi)|^2\, dx = \int d\eta \left| \int e^{-i\lambda x\cdot\eta/2} \chi(x) \mathscr{P}_\phi^\lambda(\psi u)(x, \lambda\xi)\, dx \right|^2 dx.$$

We split the last integral in η into two parts.

$$\int d\eta\{\cdots\} = \int_{|\eta-\xi| \ge \lambda^{-\sigma}} |\cdots|^2\, d\eta + \int_{|\eta-\xi| \le \lambda^{-\sigma}} |\cdots|^2\, d\eta = I_1 + I_2.$$

Suppose that $0 \le \sigma < 1/2$. Since

$$e^{-i\lambda x\cdot\eta/2} \mathscr{P}_\phi^\lambda(\psi u)(x, \lambda\xi) = e^{-i\lambda x\cdot(\eta-\xi)/2} \int e^{-i\lambda y\cdot\xi} \overline{\phi^\lambda}(y-x)\psi(y)u(y)\, dy,$$

it is easily verified that for any compact set K of $\dot{\boldsymbol{R}}^n$

$$\sup_{\xi \in K} |I_1(\xi)| = \mathcal{O}(\lambda^{-\infty}). \tag{4.3}$$

To see this, use integration by parts in x with aid of

$$\partial_x^\alpha e^{-i\lambda x\cdot(\eta-\xi)/2} = (-i\lambda(\eta-\xi)/2)^{|\alpha|} e^{-i\lambda x\cdot(\eta-\xi)/2}$$

and

$$\partial_x^\alpha \phi^\lambda(y-x) = \lambda^{|\alpha|/2}((-\partial_x)^\alpha \phi^\lambda)(y-x).$$

Applying Leibniz's rule to $\partial_x^\alpha\{\chi(x)\phi^\lambda(y-x)\}$, we arrive at (4.3) if supp $\nabla\chi \cap \psi = \varnothing$.

On the other hand, since

$$I_2 = \int_{|\eta-\xi|\le\lambda^{-\sigma}} d\eta \left| \int dx \chi(x) e^{-i\lambda x\cdot(\eta-\xi)/2} \int e^{-i\lambda y\cdot\xi} \overline{\phi^\lambda}(y-x)\psi(y)u(y)\, dy \right|^2,$$

it follows from Lemma 3.5 that

$$I_2 = ((2\pi)^n \lambda^{-n/4})^2 \int_{|\eta-\xi|\le\lambda^{-\sigma}} d\eta \left| \int dx \chi(x) e^{-i\lambda x\cdot(\eta-\xi+2\lambda^{-1}\zeta)/2} \bar{\hat{\phi}}(-\lambda^{-1/2}\zeta)\widehat{\psi u}(\lambda\xi-\zeta)\, d\zeta \right|^2.$$

Note

(4.4)

$$\iint \chi(x) e^{-i\lambda x\cdot(\eta-\xi+2\lambda^{-1}\zeta)/2} \widehat{\psi u}(\lambda\xi-\zeta)\bar{\hat{\phi}}(-\lambda^{-1/2}\zeta)\, dx d\zeta$$

$$= \iint \chi(x) e^{-i\lambda x\cdot(\eta-\xi+2\lambda^{-1}\zeta)/2} \widehat{\psi u}(\lambda(\xi-\eta-2\lambda^{-1}\zeta)/2 + \lambda(\eta+\xi)/2)\bar{\hat{\phi}}(-\lambda^{-1/2}\zeta)\, dx d\zeta.$$

We shall use the Taylor expansions of $\widehat{\psi u}$ at $\lambda(\eta+\xi)/2$ and the relation

$$\{\lambda(\xi-\eta-2\lambda^{-1}\zeta)\}^\alpha e^{-i\lambda x\cdot(\eta-\xi+2\lambda^{-1}\zeta)/2} = (2i\partial_x)^\alpha e^{-i\lambda x\cdot(\eta-\xi+2\lambda^{-1}\zeta)/2}.$$

It is seen that

$$\text{(4.5)} \qquad \int \{(2i\partial_x)^\alpha e^{-i\lambda x\cdot(\eta-\xi+2\lambda^{-1}\zeta)}\}\chi(x)\widehat{\psi u}(\lambda(\eta+\xi)/2)\bar{\hat{\phi}}(-\lambda^{-1/2}\zeta)\, dx d\zeta$$

$$= (2\pi)^n \lambda^{n/2} \widehat{\psi u}(\lambda(\eta+\xi)/2) \int (-2i\partial_x)^\alpha \{\chi(x)\bar{\phi}(\lambda^{1/2}x)\}\, dx.$$

Thus, after performing the same procedure as before for the terms arising from the remainder in the Taylor expansion, we get

$$I_2 \le C\lambda^{-n/2-\sigma n} \sup_{|\xi-\eta|\le\lambda^{-\sigma}} |\widehat{\psi u}(\lambda(\xi+\eta)/2)|^2 + \mathcal{O}(\lambda^{-N})$$

for any $N > 1$. Thus, for any relatively compact neighborhood V of ξ_0 there exists another neighborhood $\tilde{V} \supset V$ such that

$$\int_V I_2 \, d\xi \leq C\lambda^{-\sigma n} \int_{\tilde{V}} |\widehat{\psi u}(\lambda\xi)|^2 \, d\xi + \mathcal{O}(\lambda^{-N}). \tag{4.6}$$

We shall use the property that for any relatively compact neighborhood U of ξ_0, there exists a positive constant C and conic neighborhoods $\Gamma_\pm$ of ξ_0 such that

$$C^{-1} \int_{\Gamma_-} |v(\eta)|^2 \, d\eta \leq \int_1^\infty d\lambda \int_U \lambda^{n-1} |v(\lambda\xi)|^2 \, d\xi \leq C \int_{\Gamma^+} |v(\eta)|^2 \, d\eta \tag{4.7}$$

for all $v(\eta) \in L^2(\Gamma_+)$. Thus (4.6) and (4.7) give the second part of the result in Theorem 2.4 since we can take σ to be any close number to $1/2$.

Now we are going to prove the first part of Theorem 2.4. We must estimate I_2 from below.

$$I_2 = \int_{|\xi-\eta| \leq \rho\lambda^{-1/2}} |\cdots|^2 \, d\eta + \int_{\rho\lambda^{-1/2} \leq |\xi-\eta| \leq \lambda^{-\sigma}} |\cdots|^2 \, d\eta = I_{21} + I_{22} \geq I_{21},$$

which is equal to

$$I_{21} = (2\pi)^{2n}\lambda^{-n/2} \int_{|\xi-\eta| \leq \rho\lambda^{-1/2}} \left| \int \chi(\lambda^{-1/2}x)\overline{\phi(x)} e^{-i(\lambda^{1/2}/2)(\eta-\xi)\cdot x} \, dx \widehat{\psi u}(\lambda\xi) \right|^2 d\eta + \mathcal{O}(\lambda^{-\infty}).$$

If $|\zeta| \leq \rho\lambda^{-1/2}$ and $\rho > 0$ is small enough, then there exists a positive constant C_0 independent of λ such that

$$\left| \int \chi(\lambda^{-1/2}x)\bar{\phi}(x) e^{-i\lambda^{1/2}\zeta\cdot x/2} \, dx \right| = |\overline{\hat{\phi}(-\lambda^{1/2}\zeta/2)} + \mathcal{O}(\lambda^{-\infty})| \geq C_0$$

for the sake of our assumption $\hat{\phi}(0) \neq 0$. Therefore, if we shrink the support of ψ so small that (4.1) is fulfilled, then there exists a neighborhood V_- of (x_0, ξ_0) and a positive constant C such that

$$\int_V I_{21} \, d\xi \geq \lambda^{-n} C \int_{V_-} |\widehat{\psi u}(\lambda\xi)|^2 \, d\xi - \mathcal{O}(\lambda^{-\infty})$$

in the same manner as in the previous section. From the identity (4.2) and the property (4.7), we arrive at the conclusion. □

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