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DIRICHLET-NEUMANN PROBLEM IN A DOMAIN WITH PIECEWISE-SMOOTH BOUNDARY

By

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Introduction

The Dirichlet boundary value problem has been considered for various types of partial differential operators in a domain Ω with various types of non-smooth boundaries ([1], [2], etc.).

In this paper, we assume that Ω is a smooth ρ-manifold in $\boldsymbol{R}^n$, defined in §1, whose boundary is divided into a finite number of smooth surfaces:

$$\partial\Omega = \bigcup_{i=1}^{h} \overline{\Gamma}_i = \left(\bigcup_{i \in D} \overline{\Gamma}_i\right) \cup \left(\bigcup_{i \in N} \overline{\Gamma}_i\right) \quad (D \cap N = \phi).$$

In §2, we consider an elliptic partial differential equation of 2-nd order in Ω with Dirichlet boundary conditions on Γ_i $(i \in D)$ and Neumann boundary conditions on Γ_i $(i \in N)$:

$$\text{(P)} \quad \begin{cases} Au = f & \text{in } \Omega, \\ u = g^{(i)} & \text{on } \Gamma_i \ (i \in D), \\ B_i u = h^{(i)} & \text{on } \Gamma_i \ (i \in N), \end{cases}$$

where $\{B_i\}$ are differential operators of 1-st order. We consider weak solutions, i.e. $\mathscr{H}$-weak solutions in the sense of [3]. Existence of $\mathscr{H}$-weak solutions depends on weak energy estimates for adjoint problems. Therefore our aim in this paper is to obtain weak energy estimates for adjoint problems. Weak energy estimates for (P) means

$$\|u\|_{L^2(\Omega)} \leqq C\left\{\|Au\|_{L^2(\Omega)} + \sum_{i \in D} \|u\|_{H^{\sigma-1}(\Gamma_i)} + \sum_{i \in N} \|B_i u\|_{H^{\sigma-2}(\Gamma_i)}\right\} \quad (\forall u \in H^{\sigma}(\Omega))$$

for some large integer σ.

In §3, we consider a hyperbolic partial differential equation of 2-nd order in $(0, T) \times \Omega$ with Dirichlet boundary conditions on $(0, T) \times \Gamma_i$ $(i \in D)$, Neumann boundary conditions on $(0, T) \times \Gamma_i$ $(i \in N)$ and initial conditions on $\{t = 0\} \times \Omega$.

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§1. ρ-Manifolds

Let Ω be a smooth manifold of dimension m in $\boldsymbol{R}^n$. We say that $x \in \partial\Omega$ is a *boundary point of degree* p $(1 \leqq p \leqq \rho)$, if there exist a neighborhood U of x in $\boldsymbol{R}^n$ and a neighborhood V of 0 in $\boldsymbol{R}^m$ such that

$$\Phi : \bar{\Sigma}_p \cap V \to \bar{\Omega} \cap U$$

is a smooth bijection, where

$$\Sigma_p = \{y \in \boldsymbol{R}^m \mid y_1 > 0, y_2 > 0, \dots, y_p > 0\}.$$

Let $\partial^p\Omega$ denote the set of boundary points of Ω of degree p. We say that Ω is a *ρ-manifold* $(1 \leqq \rho \leqq m)$ if

$$\partial\Omega = \bigcup_{p=1}^{\rho} \partial^p\Omega.$$

Suppose that Ω is a ρ-manifold of dimension m. Let $x \in \partial^p\Omega$, then there exists its neighbourhood $U = U(x)$ such that

$$\begin{aligned}
\partial^1\Omega \cap U = {} & \Phi(\{y \mid y_1 = 0, y_2 > 0, \dots, y_p > 0\} \cap V) \\
& \cup \Phi(\{y \mid y_1 > 0, y_2 = 0, y_3 > 0, \dots, y_p > 0\} \cap V) \\
& \cup \cdots \cup \Phi(\{y \mid y_1 > 0, y_2 > 0, \dots, y_{p-1} > 0, y_p = 0\} \cap V), \\
\partial^2\Omega \cap U = {} & \Phi(\{y \mid y_1 = y_2 = 0, y_3 > 0, \dots, y_p > 0\} \cap V) \\
& \cup \Phi(\{y \mid y_1 = 0, y_2 > 0, y_3 = 0, y_4 > 0, \dots, y_p > 0\} \cap V) \\
& \cup \cdots \cup \Phi(\{y \mid y_1 > 0, y_2 > 0, \dots, y_{p-2} > 0, y_{p-1} = y_p = 0\} \cap V), \\
\dots & \\
\partial^p\Omega \cap U = {} & \Phi(\{y \mid y_1 = y_2 = \cdots = y_p = 0\} \cap V).
\end{aligned}$$

Suppose that Ω is a bounded ρ-manifold of dimension m, then $\partial\Omega$ is covered by a finite number of subsets of $\{U(x) \mid x \in \partial\Omega\}$ with above properties, therefore, $\partial^1\Omega$ is a union of a finite number of smooth manifolds of dimension $m-1$:

$$\partial^1\Omega = \bigcup_{i \in I} \Gamma_i, \quad I = \{1, 2, \dots, h\}.$$

Moreover, we have the following two lemmas.

Lemma 1.1. *Let Ω be a bounded ρ-manifold of dimension m in $\boldsymbol{R}^n$, then it holds*

$$\partial^p \Omega = \bigcup_{v \in S_p(I)} \Gamma_v,$$

$$S_p(I) = \{v = (v_1, v_2, \dots, v_p) \in I^p \mid v_i \neq v_j \text{ if } i \neq j\},$$

where $\Gamma_{v_1 \cdots v_p}$ is a $(\rho - p)$-manifold of dimension $m - p$ such that

$$\overline{\Gamma}_{v_1 \cdots v_p} = \overline{\Gamma}_{v_1 \cdots v_{p-1}} \cap \overline{\Gamma}_{v_p}.$$

Lemma 1.2. *Let Ω be a bounded ρ-manifold of dimension m in $\boldsymbol{R}^n$ with $\partial^1 \Omega = \bigcup_{i \in I} \Gamma_i$, where $\{\Gamma_i \ (i \in I)\}$ are $(\rho - 1)$-manifolds of dimension $m - 1$. Let $D \subset I$, and suppose that $\{g_i \in H^{s+\rho-1}(\Gamma_i) \ (i \in D)\}$ $(s \geqq 0)$ satisfy*

$$(\sharp) \quad g_i = g_j \quad \textit{on } \overline{\Gamma}_i \cap \overline{\Gamma}_j \quad (i, j \in D).$$

Set

$$g_v = g_{v_1}|_{\Gamma_v} \quad (v \in S(D)),$$

then $\{g_v \ (v \in S(D))\}$ satisfy

$$(\sharp\sharp) \quad \begin{cases} \text{(i)} \quad g_{v_1 \cdots v_p} = g_{\mu_1 \cdots \mu_p} \quad \textit{if } \{v_1, \dots, v_p\} = \{\mu_1, \dots, \mu_p\}, \\ \text{(ii)} \quad g_{v_1 \cdots v_p}|_{\overline{\Gamma}_{v_1 \cdots v_p} \cap \overline{\Gamma}_j} = \begin{cases} g_{v_1 \cdots v_p} & \textit{if } j \in \{v_1, \dots, v_p\}, \\ g_{v_1 \cdots v_p j} & \textit{if } j \notin \{v_1, \dots, v_p\}, \end{cases} \end{cases}$$

and

$$\sum_{v \in S_p(D)} \|g_v\|_{H^{s+\rho-p}(\Gamma_v)} \leqq C_s \sum_{i \in D} \|g_i\|_{H^{s+\rho-1}(\Gamma_i)},$$

where

$$S_p(D) = \{v = (v_1, v_2, \dots, v_p) \in S_p(I) \mid \exists k \text{ s.t. } v_k \in D\}, \quad S(D) = \bigcup_p S_p(D).$$

First in Lemma 1.3, we consider extensions of functions in a fundamental domain

$$\Omega = \{x \in \boldsymbol{R}^m \mid x_1 > 0, \dots, x_p > 0\},$$

then

$$\Gamma_1 = \{x \in \boldsymbol{R}^m \mid x_1 = 0, x_2 > 0, \dots, x_\rho > 0\},$$
$$\dots$$
$$\Gamma_\rho = \{x \in \boldsymbol{R}^m \mid x_1 > 0, x_2 > 0, \dots, x_{\rho-1} > 0, x_\rho = 0\},$$

and moreover

$$\Gamma_v = \{x \in \boldsymbol{R}^m \mid x_{v_1} = x_{v_2} = \cdots = x_{v_p} = 0, x_j > 0 \ \ (1 \leqq j \leqq \rho, j \neq v_k)\} \quad (v \in S_p(I)).$$

Let g_v be a function defined on $\overline{\Gamma}_v$ $(v \in S(I))$, then

$$g_v = g_v(x_{v_{p+1}}, \dots, x_{v_m})$$

can be regarded as a function, defined in $\overline{\Omega}$, which is independent of variables $(x_{v_1}, \dots, x_{v_p})$.

LEMMA 1.3. *Let*

$$\Omega = \{x \in \boldsymbol{R}^m \mid x_1 > 0, \dots, x_\rho > 0\},$$

and assume that $\{g_v \in H^s(\Gamma_v), v \in S(I)\}$ $(I = \{1, 2, \dots, h\})$ *satisfy* $(\sharp\sharp)$. *Set*

$$E[\{g_v\}] = \sum_{1 \leqq i \leqq \rho} g_i - \sum_{1 \leqq i < j \leqq \rho} g_{ij} + \sum_{1 \leqq i < j < k \leqq \rho} g_{ijk} - \cdots + (-1)^{\rho-1} g_{1 \cdots \rho},$$

then $g = E[\{g_v\}]$ *satisfies*

$$g|_{\Gamma_i} = g_i \quad (i \in I),$$

and

$$\|\beta g\|_{H^s(\Omega)} \leqq C_{s\beta} \sum_{v \in S(I)} \|g_v\|_{H^s(\Gamma_v)},$$

where $\beta \in \mathscr{D}(\boldsymbol{R}^m)$.

PROOF. From $(\sharp\sharp)$, we have

$$\sum_{1 \leqq i \leqq \rho} g_i|_{\Gamma_1} = g_1 + \sum_{2 \leqq i \leqq \rho} g_{1i},$$

$$\sum_{1 \leqq i < j \leqq \rho} g_{ij}|_{\Gamma_1} = \sum_{2 \leqq j \leqq \rho} g_{1j} + \sum_{2 \leqq i < j \leqq \rho} g_{1ij}, \dots,$$

therefore we have

$$
\begin{aligned}
g(0, x_2, \dots, x_m) \\
&= \left\{ g_1 + \sum_{2 \leqq i \leqq \rho} g_{1i} \right\} - \left\{ \sum_{2 \leqq j \leqq \rho} g_{1j} + \sum_{2 \leqq i < j \leqq \rho} g_{1ij} \right\} \\
&\quad + \left\{ \sum_{2 \leqq j < k \leqq \rho} g_{1jk} + \sum_{2 \leqq i < j < k \leqq \rho} g_{1ijk} \right\} - \cdots + (-1)^{\rho-1} g_{1 \cdots \rho} \\
&= g_1.
\end{aligned}
$$

In the same way, we have

$$
g|_{\Gamma_i} = g_i \quad (i \in I). \qquad \square
$$

Next in general, we have

Lemma 1.4. *Let Ω be a bounded ρ-manifold of dimension m in $\boldsymbol{R}^n$ with*

$$
\partial^p \Omega = \bigcup_{v \in S_p(I)} \Gamma_v \quad (1 \leqq p \leqq \rho, I = \{1, \dots, h\}),
$$

where Γ_v $(v \in S_p(I))$ are $(\rho - p)$-manifolds of dimension $m - p$. Let $\{g_v \in H^s(\Gamma_v)$, $v \in S(I)\}$ be given functions satisfying (♯♯). Then there exists a function g defined in Ω such that

$$
g = g_v \quad \text{on } \Gamma_v \quad (v \in S(I))
$$

and

$$
\begin{aligned}
\sum_{i \in I} \sum_{|\alpha| \leqq s} \|\partial_x^\alpha g\|_{L^2(\Gamma_i)} + \|g\|_{H^s(\Omega)} \\
\leqq C_s \sum_{v \in S(I)} \|g_v\|_{H^s(\Gamma_v)}.
\end{aligned}
$$

Proof. Since Ω is a bounded ρ-manifold of dimension m in $\boldsymbol{R}^n$, there exist open sets $\{U_1, \dots, U_J\}$ in $\boldsymbol{R}^n$ such that

$$
\partial\Omega \subset \bigcup_{j=1}^{J} U_j, \quad \Phi_j(\Sigma_{p_j} \cap V) = \Omega \cap U_j.
$$

Depending on open sets $\{U_1, \dots, U_J\}$, there exist smooth functions $\{\beta_1, \dots, \beta_J\}$ such that

$$
\operatorname{supp}[\beta_j(x)] \subset U_j, \quad \sum_{j=1}^{J} \beta_j(x)^2 = 1 \quad \text{near } \partial\Omega.
$$

For fixed j, set

$$\tilde{g}_\nu^{(j)} = \beta_j g_\nu \quad \text{on } \Gamma_\nu, \quad h_\nu^{(j)} = \tilde{g}_\nu^{(j)} \circ \Phi_j$$

for $\nu \in S(I)$. Since (♯♯) is satisfied by $\{h_\nu^{(j)}, \nu \in S(I)\}$, there exists

$$h_j = E[\{h_\nu^{(j)}\}]$$

from Lemma 1.3. Then

$$g(x) = \sum_{j=1}^{J} \beta_j(x)(h_j \circ \Phi_j^{-1})(x)$$

satisfies the required properties. □

PROPOSITION 1.1. *Let Ω be a bounded ρ-manifold of dimension m in $\boldsymbol{R}^n$, with*

$$\partial^1\Omega = \bigcup_{i \in I} \Gamma_i,$$

where $\{\Gamma_i \ (i \in I)\}$ are $(\rho - 1)$-manifolds of dimension $m - 1$. Suppose that $\{g^{(i)} \in H^{s+\rho-1}(\Gamma_i) \ (i \in D)\}$ satisfy

$$(\sharp) \quad g^{(i)} = g^{(j)} \quad \textit{on } \overline{\Gamma}_i \cap \overline{\Gamma}_j \quad (i, j \in D).$$

Then there exists a function g defined in $\overline{\Omega}$ such that

$$g = g^{(i)} \quad \textit{on } \Gamma_i \quad (i \in D)$$

and

$$\sum_{i \in I} \sum_{|\alpha| \leqq s} \|\partial_x^\alpha g\|_{L^2(\Gamma_i)} + \|g\|_{H^s(\Omega)}$$
$$\leqq C_s \sum_{i \in D} \|g^{(i)}\|_{H^{s+\rho-1}(\Gamma_i)}.$$

PROOF. Set

$$g_\nu = g_{\nu_1}|_{\Gamma_\nu} \quad (\nu \in S_p(D)),$$

and apply Lemma 1.2 to $\{g_\nu, \nu \in S(D)\}$, then we have

$$\sum_{\nu \in S_p(D)} \|g_\nu\|_{H^{s+\rho-p}(\Gamma_\nu)} \leqq C_s K,$$

where

$$K = \sum_{i \in D} \|g_i\|_{H^{s+\rho-1}(\Gamma_i)}.$$

Hence we have

$$\sum_{\nu\in S(D)} \|g_\nu\|_{H^s(\Gamma_\nu)} \leqq C_s K.$$

In case when $S(D) \neq S(I)$, set $S_p' = S_p(I) - S_p(D)$. Define

$$g_\nu = 0 \quad \text{for } \nu \in S_\rho' \quad (\text{if } S_\rho' \neq \phi).$$

Then we can define $\{g_\mu \ (\mu \in S_{\rho-1}')\}$ satisfying

$$\|g_\mu\|_{H^s(\Gamma_\mu)} \leqq C_s \sum_{\nu\in S_\rho(D)} \|g_\nu\|_{H^s(\Gamma_\nu)} \leqq C_s K \quad (\mu \in S_{\rho-1}')$$

by Lemma 1.4, because $\Gamma_\mu \ (\mu \in S_{\rho-1}')$ is 1-manifold of dimension $m-\rho+1$. Next, we consider $\Gamma_\mu \ (\mu \in S_{\rho-2}')$, which is 2-manifold of dimension $m-\rho+2$ and

$$\partial^1 \Gamma_\mu = \bigcup_{\nu \in A_\mu} \Gamma_\nu \quad (A_\mu \subset S_{\rho-1}(I)).$$

Since $\{g_\nu \ (\nu \in A_\mu)\}$ satisfy ($\sharp$), we can define g_μ on Γ_μ by Lemma 1.2 and Lemma 1.4, satisfying

$$g_\mu|_{\Gamma_\nu} = g_\nu \quad (\nu \in A_\mu),$$

$$\begin{aligned} \|g_\mu\|_{H^s(\Gamma_\mu)} &\leqq C_s \sum_{\nu\in S_\rho(I)\cup S_{\rho-1}(I)} \|g_\nu\|_{H^s(\Gamma_\nu)} \\ &\leqq C_s' \sum_{\nu\in S_\rho(D)\cup S_{\rho-1}(D)} \|g_\nu\|_{H^s(\Gamma_\nu)} \quad (\mu \in S_{\rho-2}'). \end{aligned}$$

Repeating these constructions, we can define a function g with required properties. □

Proposition 1.2. *Let Ω be a bounded connected ρ-manifold of dimension n in $\boldsymbol{R}^n$, with*

$$\partial^1\Omega = \bigcup_{i\in I} \Gamma_i,$$

where $\{\Gamma_i \ (i \in I)\}$ are $(\rho-1)$-manifolds of dimension $n-1$ and

$$I = D \cup N, \quad D \cap N = \phi, \quad D \neq \phi.$$

Then it holds that

$$\begin{aligned} &\sum_{i\in N} \|u\|_{L^2(\Gamma_i)}^2 + \|u\|_{L^2(\Omega)}^2 \\ &\qquad \leqq C\left\{\sum_{i\in D} \|u\|_{L^2(\Gamma_i)}^2 + \|\partial_x u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\right\} \quad (\forall u \in H^1(\Omega)), \end{aligned}$$

where

$$\|\partial_x u\|^2 = \sum_{j=1}^{n} \|\partial_j u\|^2, \quad \partial_j = \partial_{xj}.$$

PROOF. Let $a \in \Gamma_a$ $(\alpha \in D \neq \phi)$ be fixed.

1) Let $b \in \Gamma_\beta$. Draw a smooth line L in Ω such that

$$L : x(t) \quad (0 \leqq t \leqq 1), \quad x(0) = a, \quad x(1) = b,$$

where $x'(0)$ is not tangent to Γ_α and $x'(1)$ is not tangent to Γ_β, then we have

$$u(b)^2 = u(a)^2 + \int_0^1 \sum_{j=1}^{n} \partial_j \{u(x(t))^2\} x_j'(t)\, dt.$$

Moreover, we have continuous deformations of L:

$$L_y : x(t, y) = x(t, y_1, \dots, y_{n-1}) \quad (0 \leqq t \leqq 1), \quad x(0, y) \in \Gamma_\alpha, \quad x(1, y) \in \Gamma_\beta$$

for $y \in V$, where $L_0 = L$ and V is a neighborhood of 0 in $\boldsymbol{R}^{n-1}$. Then we have

$$\begin{aligned} &\int_V u(x(1, y))^2\, dy - \int_V u(x(0, y))^2\, dy \\ &\qquad = \int_V dy \int_0^1 \sum 2\partial_j u(x(t, y)) u(x(t, y)) (\partial_t x_j)(t, y)\, dt, \end{aligned}$$

therefore we have

$$\begin{aligned} &\int_V u(x(1, y))^2\, dy \\ &\qquad \leqq \int_V u(x(0, y))^2\, dy + C \int_V dy \int_0^1 |\partial_x u(x(t, y))|\, |u(x(t, y))|\, dt. \end{aligned}$$

In the same way, we have

$$\begin{aligned} &\int_V u(x(s, y))^2\, dy \\ &\qquad \leqq \int_V u(x(0, y))^2\, dy + C \int_V dy \int_0^s |\partial_x u(x(t, y))|\, |u(x(t, y))|\, dt \quad (0 < s < 1). \end{aligned}$$

Integrating the both sides with respect to s, we have

$$\int_0^1 dt \int_V u(x(t,y))^2\,dy$$
$$\leqq \int_V u(x(0,y))^2\,dy + C\int_0^1 dt \int_V |\partial_x u(x(t,y))|\,|u(x(t,y))|\,dy.$$

2) Let $b \in \partial^p \Gamma_\beta$, then there exists a family of lines $\{L_y \mid y \in V \cap \Sigma_p\}$ such that

$$L_y : x(t,y) = x(t, y_1, \dots, y_{n-1}) \in \Omega \quad (0 < t < 1),$$
$$x(0,y) \in \Gamma_\alpha, \quad x(1,y) \in \Gamma_\beta,$$

where V is a neighborhood of 0 in $\boldsymbol{R}^{n-1}$ and

$$\Sigma_p = \{ y \in \boldsymbol{R}^{n-1} \mid y_1 > 0, \dots, y_p > 0 \}.$$

Then we have, in the same way as in 1),

$$\int_{V\cap\Sigma_p} u(x(1,y))^2\,dy$$
$$\leqq \int_{V\cap\Sigma_p} u(x(0,y))^2\,dy + C\int_{V\cap\Sigma_p} dy \int_0^1 |\partial_x u(x(t,y))|\,|u(x(t,y))|\,dt.$$

and

$$\int_0^1 dt \int_{V\cap\Sigma_p} u(x(t,y))^2\,dy$$
$$\leqq \int_{V\cap\Sigma_p} u(x(0,y))^2\,dy + C\int_0^1 dt \int_{V\cap\Sigma_p} |\partial_x u(x(t,y))|\,|u(x(t,y))|\,dy.$$

By finite sums of inequalities in 1) and 2), we obtain the required inequalities. □

§ 2. Elliptic Dirichlet-Neumann Problem

We assume, hereafter, Ω is a bounded connected ρ-manifold of dimension n in $\boldsymbol{R}^n$, with boundary

$$\partial\Omega = \bigcup_{i\in I} \overline{\Gamma}_i = \left(\bigcup_{i\in D} \overline{\Gamma}_i\right) \cup \left(\bigcup_{i\in N} \overline{\Gamma}_i\right),$$

where $\{\Gamma_i\}$ are $(\rho-1)$-manifolds of dimension $n-1$, $I = D \cup N$, $D \cap N = \phi$ and $D \neq \phi$.

Let A, B_i be defined as follows:

$$A = \sum_{p,q=1}^{n} \partial_p a_{pq}(x)\partial_q + \sum_{q=1}^{n} b_q(x)\partial_q + c(x)$$

and

$$B_i = (d/dn_A^{(i)}) + \sigma^{(i)}(x), \quad (d/dn_A^{(i)}) = \sum_{p,q=1}^{n} n_p^{(i)} a_{pq}(x)\partial_q \quad (i \in I),$$

where $n^{(i)}$ is the unit outer normal to Γ_i and $a_{pq}(x)$, $b_q(x)$, $c(x)$, $\sigma^{(i)}(x)$ are smooth real valued functions. Here we assume

$$\sum_{p,q=1}^{n} a_{pq}(x)\xi_p\xi_q \geqq \delta|\xi|^2 \quad (\forall x \in \Omega, \forall \xi \in \boldsymbol{R}^n)$$

where $a_{qp}(x) = a_{pq}(x)$, $\delta > 0$.

Let $u \in H^{2+\rho}(\Omega)$ be a real valued function satisfying

$$(\mathrm{P}) \quad \begin{cases} Au = f & \text{in } \Omega, \\ u = g^{(i)} & \text{on } \Gamma_i \ (i \in D), \\ B_i u = h^{(i)} & \text{on } \Gamma_i \ (i \in N), \end{cases}$$

then our aim is to obtain energy estimates for (P) in §2. Moreover, we use the following notations through in §2:

$$\|\cdot\|_s = \|\cdot\|_{H^s(\Omega)}, \quad \|\cdot\| = \|\cdot\|_0, \quad (\cdot\,,\cdot) = (\cdot\,,\cdot)_{L^2(\Omega)},$$

$$\langle\,\cdot\,\rangle_{(i),s} = \|\cdot\|_{H^s(\Gamma_i)}, \quad \langle\,\cdot\,\rangle_{(i)} = \langle\,\cdot\,\rangle_{(i),0}, \quad \langle\cdot\,,\cdot\rangle = \langle\cdot\,,\cdot\rangle_{L^2(\Gamma_i)}.$$

LEMMA 2.1. *Let $u \in H^{2+\rho}(\Omega)$ satisfy* (P). *Then there exists a function g such that*

$$g = g^{(i)} \quad \textit{on} \ \Gamma_i \ (i \in D)$$

and

$$\sum_{i=1}^{\rho} \sum_{j=0}^{s} \langle (d/dn^{(i)})^j g \rangle_{(i),s-j} + \|g\|_s \leqq C_s \sum_{i \in D} \langle g^{(i)} \rangle_{(i),s+(\rho-1)} \quad (s = 1, 2).$$

Set $v = u - g$, then v satisfies

$$(\mathrm{P}_0) \quad \begin{cases} Av = \tilde{f} & \textit{in } \Omega, \\ v = 0 & \textit{on } \Gamma_i \ (i \in D), \\ B_i v = \tilde{h}^{(i)} & \textit{on } \Gamma_i \ (i \in N), \end{cases}$$

where $\tilde{f} = f - Ag$, $\tilde{h}^{(i)} = h^{(i)} - B_i g|_{\Gamma_i}$ *satisfy*

$$\|\tilde{f}\| \leqq C\left\{\|f\| + \sum_{i \in D} \langle g^{(i)} \rangle_{(i), 2+(\rho-1)}\right\},$$

and

$$\langle \tilde{h}^{(i)} \rangle_{(i)} \leqq C\left\{\langle h^{(i)} \rangle_{(i)} + \sum_{i \in D} \langle g^{(i)} \rangle_{(i), 1+(\rho-1)}\right\}.$$

PROOF. Since $u \in H^{2+\rho}(\Omega)$ satisfy (P), $\{g^{(i)}$ on Γ_i $(i \in D)\}$ satisfy ($\sharp$). Therefore, there exists g satisfying above conditions, from Proposition 1.1. □

REMARK. Energy inequalities for (P) follows from energy inequalities for (P_0).

Using the integration by parts, we have

LEMMA 2.2 (Green's formula (I)). *It holds*

$$(Au, u) = \sum_{i=1}^{h} \langle (d/dn_A^{(i)})u, u \rangle_{(i)} - \sum_{p,q=1}^{n} (a_{pq}\partial_q u, \partial_p u) + \frac{1}{2}\sum_{i=1}^{h} \langle b^{(i)} u, u \rangle_{(i)} - \frac{1}{2}(bu, u) + (cu, u),$$

for any real $u \in H^2(\Omega)$, where

$$b = \sum_{q=1}^{n} \partial_q(b_q), \quad b^{(i)} = \sum_{q=1}^{n} n_q^{(i)} b_q|_{\Gamma_i}.$$

From Green's formula (I), we have

$$(Av, v) = \sum_{i \in N} \langle B_i v, v \rangle_{(i)} - \sum_{i \in N} \left\langle \left(\sigma^{(i)} - \frac{b^{(i)}}{2}\right)v, v \right\rangle_{(i)} - \sum_{p,q=1}^{n} (a_{pq}\partial_q v, \partial_p v) + \left(\left(c - \frac{b}{2}\right)u, u\right).$$

Assume that $\sigma^{(i)} - \dfrac{b^{(i)}}{2} \geqq 0$ $(i \in N)$ and $c - \dfrac{b}{2} \leqq 0$, then we have

$$\delta\|\partial_x v\|^2 \leqq \|\tilde{f}\|\,\|v\| + \sum_{i\in N}\langle\tilde{h}^{(i)}\rangle_{(i)}\langle v\rangle_{(i)}.$$

From Proposition 1.2, we have

$$\sum_{i\in N}\langle v\rangle_{(i)}^2 + \|v\|^2 \leqq C\|\partial_x v\|^2,$$

therefore

$$\sum_{i\in N}\langle v\rangle_{(i)}^2 + \|v\|_1^2 \leqq C\left\{\|\tilde{f}\|^2 + \sum_{i\in N}\langle\tilde{h}^{(i)}\rangle_{(i)}^2\right\}.$$

Hence we have

PROPOSITION 2.1. *Assume that* $\sigma^{(i)} - \dfrac{b^{(i)}}{2} \geqq 0$ $(i\in N)$ *and* $c - \dfrac{b}{2} \leqq 0$. *Then it holds that*

$$\langle u\rangle^2 + \|u\|_1^2 \leqq C\left\{\|f\|^2 + \sum_{i\in D}\langle g^{(i)}\rangle_{(i),1+\rho}^2 + \sum_{i\in N}\langle h^{(i)}\rangle_{(i)}^2\right\}$$

for any $u\in H^{2+\rho}(\Omega)$ *satisfying* (P), *where*

$$\langle u\rangle^2 = \sum_{i\in D\cup N}\langle u\rangle_{(i)}^2.$$

By the integration by parts, we have

LEMMA 2.3 (Green' formula (II)). *It holds*

$$\begin{aligned}&(Au, v) - (u, A^*v)\\ &\quad = \sum_{i\in D\cup N}\{\langle(d/dn_A^{(i)})u, v\rangle_{(i)} - \langle u, (d/dn_A^{(i)})v\rangle_{(i)} + \langle b^{(i)}u, v\rangle_{(i)}\}\end{aligned}$$

for any $u, v\in H^{2+\rho}(\Omega)$, *where*

$$A^* = \sum_{p,q=1}^{n}\partial_p a_{pq}\partial_q - \sum_{q=1}^{n}\partial_q b_q + c = \sum_{p,q=1}^{n}\partial_p a_{pq}\partial_q - \sum_{q=1}^{n} b_q\partial_q + (c-b).$$

From Green's formula (II), setting

$$B_i' = (d/dn_A^{(i)}) + \sigma^{(i)} - b^{(i)} \quad (i\in D\cup N),$$

we have

$$(Au, v) - (u, A^*v) = \sum_{i \in D \cup N} \{\langle B_i, u, v\rangle_{(i)} - \langle u, B_i'v\rangle_{(i)}\}.$$

Hence we have an adjoint problem for (P):

$$(\mathrm{P}') \quad \begin{cases} A^*v = f & \text{in } \Omega, \\ v = g^{(i)} & \text{on } \Gamma_i \ \ (i \in D), \\ B_i'v = h^{(i)} & \text{on } \Gamma_i \ \ (i \in N). \end{cases}$$

Since the type of (P′) is the same as that of (P), we have from Proposition 2.1

Proposition 2.2. *Assume that* $\sigma^{(i)} - \dfrac{b^{(i)}}{2} \geqq 0$ $(i \in N)$ *and* $c - \dfrac{b}{2} \leqq 0$. *Then it holds*

$$\langle v\rangle^2 + \|v\|_1^2 \leqq C\left\{\|f\|^2 + \sum_{i \in D} \langle g^{(i)}\rangle^2_{(i), 1+\rho} + \sum_{i \in N} \langle h^{(i)}\rangle^2_{(i)}\right\}$$

for any $v \in H^{2+\rho}(\Omega)$ *satisfying* (P′).

According to Theorem I in [3], we have from Proposition 2.2

Theorem 1. *Assume that* $\sigma^{(i)} - \dfrac{b^{(i)}}{2} \geqq 0$ $(i \in N)$ *and* $c - \dfrac{b}{2} \leqq 0$. *Let* $f \in L^2(\Omega)$, *then there exists a* $\mathscr{H}$*-weak solution* $u \in L^2(\Omega)$ *for*

$$(\mathrm{P}) \quad \begin{cases} Au = f & \textit{in } \Omega, \\ u = 0 & \textit{on } \Gamma_i \ \ (i \in D), \\ B_i u = 0 & \textit{on } \Gamma_i \ \ (i \in N), \end{cases}$$

where $\mathscr{H}$*-weak solution is defined as follows.*

$\mathscr{H}$ *is a Hilbert space defined by the completion of* $H^{2+\rho}(\Omega)$ *by the norm*:

$$\|u\|_{\mathscr{H}}^2 = \|A^*u\|^2 + \sum_{i \in D} \langle u\rangle^2_{(i), 1+\rho} + \sum_{i \in N} \langle B_i'u\rangle^2_{(i)}.$$

We say that $u \in L^2(\Omega)$ *is* $\mathscr{H}$*-weak solution of the problem* (P), *if there exists* $w \in \mathscr{H}$ *such that* $u = A^*w$ *and*

$$[w, v]_{\mathscr{H}} = (f, v)_{L^2(\Omega)} \quad (v \in \mathscr{H}),$$

where $[w, v]_{\mathscr{H}}$ *is the inner product of* $\mathscr{H}$, *which is derived from the norm* $\|\cdot\|_{\mathscr{H}}$.

§ 3. Dirichlet-Neumann Problems for $\partial_t^2 - A$

We consider the following initial boundary value problem in § 3:

$$
(\mathrm{P}) \quad \begin{cases} (\partial_t^2 - A)u = f & \text{in } (0,T)\times\Omega, \\ u = g^{(i)} & \text{on } (0,T)\times\Gamma_i \ \ (i\in D), \\ B_i u = h^{(i)} & \text{on } (0,T)\times\Gamma_i \ \ (i\in N), \\ u = u_0, \partial_t u = u_1 & \text{on } \{t=0\}\times\Omega, \end{cases}
$$

where Ω, A, B_i are the same ones as in § 2. Since Ω is a ρ-manifold of dimension n in $\boldsymbol{R}^n$, $(0,T)\times\Omega$ is a $(\rho+1)$-manifold of dimension $n+1$ in $\boldsymbol{R}^{n+1}$. Following notations of norms and inner products are used in § 3:

$$\|\cdot\|_s = \|\cdot\|_{H^s((0,T)\times\Omega)},$$

$$\|\cdot\| = \|\cdot\|_0, \quad (\cdot\,,\cdot) = (\cdot\,,\cdot)_{L^2((0,T)\times\Omega)},$$

$$\langle\cdot\rangle_{(i),s} = \|\cdot\|_{H^s((0,T)\times\Gamma_i)},$$

$$\langle\cdot\rangle_{(i)} = \langle\cdot\rangle_{(i),0}, \quad \langle\cdot\,,\cdot\rangle_{(i)} = (\cdot\,,\cdot)_{L^2((0,T)\times\Gamma_i)},$$

$$[\cdot]_{(t),s} = \|\cdot\|_{H^s(\{t=t\}\times\Omega)},$$

$$[\cdot]_{(t)} = [\cdot]_{(t),0}, \quad [\cdot\,,\cdot]_{(t)} = (\cdot\,,\cdot)_{L^2(\{t=t\}\times\Omega)},$$

$$\langle\!\langle\cdot\rangle\!\rangle_{(i,t),s} = \|\cdot\|_{H^s(\{t=t\}\times\Gamma_i)},$$

$$\langle\!\langle\cdot\rangle\!\rangle_{(i,t)} = \langle\!\langle\cdot\rangle\!\rangle_{(i,t),0}, \quad \langle\!\langle\cdot\,,\cdot\rangle\!\rangle_{(i,t)} = (\cdot\,,\cdot)_{L^2(\{t=t\}\times\Gamma_i)},$$

Lemma 3.1. *Let $u \in H^{3+\rho}(\Omega)$ satisfy* (P). *Then there exists a function g such that*

$$g = g^{(i)} \quad \text{on } (0,T)\times\Gamma_i \quad (i\in D)$$

and

$$\sum_{j=0}^{s}[\partial_t^j g]_{(0),s-j} + \sum_{i=1}^{\rho}\sum_{j=0}^{s}\langle (d/dn^{(i)})^j g\rangle_{(i),s-j} + \|g\|_s \leqq C_s \sum_{i\in D}\langle g_i\rangle_{(i),s+\rho} \quad (s=1,2).$$

Set $v = u - g$, then v satisfies

$$
(\mathrm{P}_0) \quad \begin{cases} (\partial_t^2 - A)v = \tilde f & \text{in } (0,T)\times\Omega, \\ v = 0 & \text{on } (0,T)\times\Gamma_i \ \ (i\in D), \\ B_i v = \tilde h^{(i)} & \text{on } (0,T)\times\Gamma_i \ \ (i\in N), \\ v = \tilde u_0, \partial_t v = \tilde u_1 & \text{on } \{t=0\}\times\Omega, \end{cases}
$$

where

$$\tilde{f} = f - (\partial_t^2 - A)g, \quad \tilde{h}^{(i)} = h^{(i)} - B_i g, \quad \tilde{u}_0 = u_0 - g|_{t=0}, \quad \tilde{u}_1 = u_1 - \partial_t g|_{t=0},$$

which satisfy

$$\|\tilde{f}\| \leqq C\left\{ \|f\| + \sum_{i \in D} \langle g^{(i)} \rangle_{(i), 2+\rho} \right\},$$

$$\langle \tilde{h}^{(i)} \rangle_{(i), 1} \leqq C\left\{ \langle h^{(i)} \rangle_{(i), 1} + \sum_{i \in D} \langle g^{(i)} \rangle_{(i), 2+\rho} \right\},$$

$$[\tilde{u}_0]_{(0), 1} \leqq C\left\{ [u_0]_{(0), 1} + \sum_{i \in D} \langle g^{(i)} \rangle_{(i), 1+\rho} \right\},$$

$$[\tilde{u}_1]_{(0)} \leqq C\left\{ [u_1]_{(0)} + \sum_{i \in D} \langle g^{(i)} \rangle_{(i), 1+\rho} \right\}.$$

Proof. Since $u \in H^{3+\rho}((0,T) \times \Omega)$ satisfies (P), $\{g^{(i)}$ on $(0,T) \times \Gamma_i \ (i \in D)\}$ satisfy ($\sharp$). Therefore, from Proposition 1.1, there exists a function g satisfying above properties. $\square$

Setting

$$V = e^{-\gamma t} v, \quad F = e^{-\gamma t} \tilde{f}, \quad H^{(i)} = e^{-\gamma t} \tilde{h}^{(i)}, \quad U_0 = \tilde{u}_0, \quad U_1 = \tilde{u}_1 \quad (\gamma > 0),$$

(P_0) is transformed to

$$(\mathrm{P}_0)_\gamma \quad \begin{cases} ((\partial_t + \gamma)^2 - A)V = F & \text{in } (0,T) \times \Omega, \\ V = 0 & \text{on } (0,T) \times \Gamma_i \ (i \in D), \\ B_i V = H^{(i)} & \text{on } (0,T) \times \Gamma_i \ (i \in N), \\ V = U_0, (\partial_t + \gamma)V = U_1 & \text{on } \{t = 0\} \times \Omega. \end{cases}$$

Therefore energy estimates for (P) follow from those for $(\mathrm{P}_0)_\gamma$.

Now we try to get energy estimates for $(\mathrm{P}_0)_\gamma$. By the integration by parts, we have

Lemma 3.2 (Green's formula (III)). *It holds*

$$(((\partial_t + \gamma)^2 - A)V, (\partial_t + \gamma)V)$$

$$= (1/2)\left\{ [(\partial_t + \gamma)V]_{(T)}^2 + \sum_{p,q} [a_{pq} \partial_q V, \partial_p V]_{(T)} \right\}$$

$$- (1/2)\left\{[(\partial_t+\gamma)V]^2_{(0)} + \sum_{p,q}[a_{pq}\partial_q V, \partial_p V]_{(0)}\right\}$$

$$+ \gamma\left\{\|(\partial_t+\gamma)V\|^2 + \sum_{p,q}(a_{pq}\partial_q V, \partial_p V)\right\}$$

$$- \sum_{i\in D\cup N}\langle (d/dn_A^{(i)})V, (\partial_t+\gamma)V\rangle_{(i)}$$

$$- \left(\sum_q b_q\partial_q V, (\partial_t+\gamma)V\right) - (cV, (\partial_t+\gamma)V)$$

for any real $V \in H^2((0,T)\times\Omega)$.

Applying Green's formula (III) for V satisfying $(P_0)_\gamma$, we have

$$\|F\|\,\|(\partial_t+\gamma)V\|$$

$$\geqq (1/2)\{[(\partial_t+\gamma)V]^2_{(T)} + \delta[\partial_x V]^2_{(T)}\}$$

$$- (1/2)\{[U_1]^2_{(0)} + C[\partial_x U_0]^2_{(0)}\} + \gamma\{\|(\partial_t+\gamma)V\|^2 + \delta\|\partial_x V\|^2\}$$

$$- \sum_{i\in N}|\langle H^{(i)}, (\partial_t+\gamma)V\rangle_{(i)}| - \sum_{i\in N}|\langle \sigma^{(i)}V, (\partial_t+\gamma)V\rangle_{(i)}|$$

$$- C(\|\partial_x V\| + \|V\|)\|(\partial_t+\gamma)V\|.$$

Now we consider

$$I_i = \langle H^{(i)}, (\partial_t+\gamma)V\rangle_{(i)} \quad \text{and} \quad J_i = \langle \sigma^{(i)}V, (\partial_t+\gamma)V\rangle_{(i)}.$$

Since

$$I_i = \langle(-\partial_t+\gamma)H^{(i)}, V\rangle_{(i)} + \langle\!\langle H^{(i)}, V\rangle\!\rangle_{(i,T)} - \langle\!\langle H^{(i)}, V\rangle\!\rangle_{(i,0)},$$

we have

$$|I_i| \leqq \{\langle(-\partial_t+\gamma)H^{(i)}\rangle_{(i)} + \langle\!\langle H^{(i)}\rangle\!\rangle_{(i,T)} + \langle\!\langle H^{(i)}\rangle\!\rangle_{(i,0)}\}$$

$$\times \{\langle V\rangle_{(i)} + \langle\!\langle V\rangle\!\rangle_{(i,T)} + \langle\!\langle V\rangle\!\rangle_{(i,0)}\}.$$

Since

$$J_i = \gamma\langle\sigma^{(i)}V, V\rangle_{(i)} - (1/2)\langle(\partial_t\sigma^{(i)})V, V\rangle_{(i)}$$

$$+ (1/2)\langle\!\langle\sigma^{(i)}V, V\rangle\!\rangle_{(i,T)} - (1/2)\langle\!\langle\sigma^{(i)}V, V\rangle\!\rangle_{(i,0)},$$

we have

$$|J_i| \leqq C\{\gamma\langle V\rangle_{(i)}^2 + \langle\!\langle V\rangle\!\rangle_{(i,T)}^2 + \langle\!\langle V\rangle\!\rangle_{(i,0)}^2\} \quad (\gamma \geqq 1).$$

From Proposition 1.2, we have

$$\langle\!\langle V\rangle\!\rangle_{(i,t)}^2 \leqq C[\partial_x V]_{(t)}[V]_{(t)} \leqq C\gamma^{-1/2}\{[\partial_x V]_{(t)}^2 + \gamma[V]_{(t)}^2\}.$$

Integrating both sides with respect to t, we have

$$\langle V\rangle_{(i)}^2 \leqq C\gamma^{-1/2}\{\|\partial_x V\|^2 + \gamma\|V\|^2\}.$$

Hence we have

$$\begin{aligned}
&|I_i| + |J_i| \\
&\quad \leqq \langle(-\partial_t + \gamma)H^{(i)}\rangle_{(i)}^2 + \langle\!\langle H^{(i)}\rangle\!\rangle_{(i,T)}^2 + \langle\!\langle H^{(i)}\rangle\!\rangle_{(i,0)}^2 \\
&\qquad + C\gamma^{-1/2}\{\gamma(\|\partial_x V\|^2 + \gamma\|V\|^2) \\
&\qquad\qquad + ([\partial_x V]_{(T)}^2 + \gamma[V]_{(T)}^2) + ([\partial_x V]_{(0)}^2 + \gamma[V]_{(0)}^2\} \quad (\gamma \geqq 1).
\end{aligned}$$

By the way,

Lemma 3.3. *It holds*

$$\begin{aligned}
&\{[\partial_t V]_{(T)}^2 + \gamma^2[V]_{(T)}^2\} + \gamma\{\|\partial_t V\|^2 + \gamma^2\|V\|^2\} \\
&\qquad \leqq 3\{[(\partial_t + \gamma)V]_{(T)}^2 + \gamma\|(\partial_t + \gamma)V\|^2 + \gamma^2[V]_{(0)}^2\}
\end{aligned}$$

for any $V \in H^2((0,T)\times\Omega)$.

Proof. Since

$$[(\partial_t + \gamma)V]_{(T)}^2 \geqq (1/3)[\partial_t V]_{(T)}^2 - (1/2)\gamma^2[V]_{(T)}^2$$

and

$$\begin{aligned}
\gamma\|(\partial_t + \gamma)V\|^2 &= \gamma\{\|\partial_t V\|^2 + \gamma^2\|V\|^2\} + \gamma^2\{(\partial_t V, V) + (V, \partial_t V)\} \\
&= \gamma\{\|\partial_t V\|^2 + \gamma^2\|V\|^2\} + \gamma^2\{[V]_{(T)}^2 - [V]_{(0)}^2\},
\end{aligned}$$

we have

$$\begin{aligned}
&[(\partial_t + \gamma)V]_{(T)}^2 + \gamma\|(\partial_t + \gamma)V\|^2 \\
&\qquad \geqq \gamma\{\|\partial_t V\|^2 + \gamma^2\|V\|^2\} + (1/3)[\partial_t V]_{(T)}^2 + (1/2)\gamma^2[V]_{(T)}^2 - \gamma^2[V]_{(0)}^2. \quad \square
\end{aligned}$$

Owing to Lemma 3.3, we have

$$\gamma\{\|\partial_t V\|^2 + \|\partial_x V\|^2 + \gamma^2\|V\|^2\} + \{[\partial_t V]^2_{(T)} + [\partial_x V]^2_{(T)} + \gamma^2[V]^2_{(T)}\}$$
$$\leqq C\left\{\gamma^{-1}\|F\|^2 + \sum_{i\in N}\langle\partial_t H^{(i)}\rangle^2_{(i)} + \sum_{i\in N}\gamma^2\langle H^{(i)}\rangle^2_{(i)}\right.$$
$$\left. + [U_1]^2_{(0)} + [\partial_x U_0]^2_{(0)} + \gamma^2[U_0]^2_{(0)}\right\} \quad (\gamma \geqq \gamma_0).$$

Hence we have

$$\{\|\partial_t v\|^2 + \|\partial_x v\|^2 + \|v\|^2\} + \{[\partial_t v]^2_{(T)} + [\partial_x v]^2_{(T)} + [v]^2_{(T)}\}$$
$$\leqq C\left\{\|\tilde{f}\|^2 + \sum_{i\in N}\langle\partial_t\tilde{h}^{(i)}\rangle^2_{(i)} + \sum_{i\in N}\langle\tilde{h}^{(i)}\rangle^2_{(i)} + [\tilde{u}_1]^2_{(0)} + [\partial_x\tilde{u}_0]^2_{(0)} + [\tilde{u}_{(0)}]^2_{(0)}\right\},$$

that is,

$$\|v\|_1^2 + \{[\partial_t v]^2_{(T)} + [v]^2_{(T),1}\}$$
$$\leqq C\left\{\|\tilde{f}\|^2 + \sum_{i\in N}\langle\tilde{h}^{(i)}\rangle^2_{(i),1} + [\tilde{u}_1]^2_{(0)} + [\tilde{u}_0]^2_{(0),1}\right\}$$
$$\leqq C'\left\{\|f\|^2 + \sum_{i\in N}\langle h^{(i)}\rangle^2_{(i),1} + [u_1]^2_{(0)} + [u_0]^2_{(0),1} + \sum_{i\in D}\langle g^{(i)}\rangle_{(i),2+\rho}\right\}.$$

Here we have

Proposition 3.1. *It holds that*

$$\|u\|_1^2 + \{[\partial_t u]^2_{(T)} + [u]^2_{(T),1}\}$$
$$\leqq C\left\{\|f\|^2 + \sum_{i\in D}\langle g^{(i)}\rangle_{(i),2+\rho} + \sum_{i\in N}\langle h^{(i)}\rangle^2_{(i),1} + [u_1]^2_{(0)} + [u_0]^2_{(0),1}\right\}$$

for any $u \in H^{3+\rho}((0,T)\times\Omega)$ *satisfying* (P).

By the integration by parts, we have

Lemma 3.4 (Green's formula (IV)). *It holds that*

$$
\begin{aligned}
&((\partial_t^2 - A)u, v) - (u, (\partial_t^2 - A^*)v) \\
&\quad = \{[\partial_t u, v]_{(T)} - [u, \partial_t v]_{(T)}\} - \{[\partial_t u, v]_{(0)} - [u, \partial_t v]_{(0)}\} \\
&\qquad - \sum_{i \in D \cup N} \{\langle (d/dn_A^{(i)})u, v\rangle_{(i)} - \langle u, ((d/dn_A^{(i)}) - b^{(i)})v\rangle_{(i)}\}
\end{aligned}
$$

for any $u, v \in H^2((0, T) \times \Omega)$.

Set

$$B_i' = (d/dn_A^{(i)}) + \sigma^{(i)} - b^{(i)},$$

then we have from Green's formula (IV)

$$
\begin{aligned}
&((\partial_t^2 - A)u, v) - (u, (\partial_t^2 - A^*)v) \\
&\quad = \{[\partial_t u, v]_{(T)} - [u, \partial_t v]_{(T)}\} - \{[\partial_t u, v]_{(0)} - [u, \partial_t v]_{(0)}\} \\
&\qquad - \sum_{i \in D \cup N} \{\langle B_i u, v\rangle_{(i)} - \langle u, B_i' v\rangle_{(i)}\}.
\end{aligned}
$$

Hence we have an adjoint problem for (P):

$$
(\mathrm{P}') \quad \begin{cases}
(\partial_t^2 - A^*)v = f & \text{in } (0, T) \times \Omega, \\
v = g^{(i)} & \text{on } (0, T) \times \Gamma_i \quad (i \in D), \\
B_i' v = h^{(i)} & \text{on } (0, T) \times \Gamma_i \quad (i \in N), \\
v = u_0, \partial_t v = u_1 & \text{on } \{t = T\} \times \Omega.
\end{cases}
$$

By the variable transformation: $t = T - t'$, (P′) becomes the same type problem as (P). Therefore we have from Proposition 3.1

Proposition 3.2. *It holds that*

$$
\begin{aligned}
&\|v\|_1^2 + \{[\partial_t v]_{(0)}^2 + [v]_{(0),1}^2\} \\
&\quad \leqq C\left\{\|f\|^2 + \sum_{i \in D} \langle g^{(i)}\rangle_{(i), 2+\rho} + \sum_{i \in N} \langle h^{(i)}\rangle_{(i),1}^2 + [u_1]_{(T)}^2 + [u_0]_{(T),1}^2\right\}
\end{aligned}
$$

for any $v \in H^{3+\rho}((0, T) \times \Omega)$ *satisfying* (P′).

According to Theorem I in [3], we have from Proposition 3.2

THEOREM 2. *Let $f \in L^2((0,T)\times\Omega)$, then there exists a $\mathscr{H}$-weak solution $u \in L^2((0,T)\times\Omega)$ of*

$$(\mathrm{P}) \quad \begin{cases} (\partial_t^2 - A)u = f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\Gamma_i \quad (i \in D), \\ B_i u = 0 & \text{on } (0,T)\times\Gamma_i \quad (i \in N), \\ u = \partial_t u = 0 & \text{on } \{t=0\}\times\Omega, \end{cases}$$

where $\mathscr{H}$-weak solution is defined as follows.

$\mathscr{H}$ is a Hilbert space defined by the completion of $H^{3+\rho}((0,T)\times\Omega)$ by the norm:

$$\|u\|_{\mathscr{H}}^2 = \|(\partial_t^2 - A^*)u\|^2 + \sum_{i\in D}\langle u\rangle_{(i),2+\rho}^2 + \sum_{i\in N}\langle B_i' u\rangle_{(i),1}^2 + [\partial_t u]_{(T)}^2 + [u]_{(T),1}^2.$$

We say that $u \in L^2((0,T)\times\Omega)$ is $\mathscr{H}$-weak solution of the problem (P), *if there exists $w \in \mathscr{H}$ such that $u = A^* w$ and*

$$[w,v]_{\mathscr{H}} = (f,v)_{L^2((0,T)\times\Omega)} \quad (\forall v \in \mathscr{H}),$$

where $[w,v]_{\mathscr{H}}$ is the inner product of $\mathscr{H}$, which is derived from the norm $\|\cdot\|_{\mathscr{H}}$.

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