TSUKUBA J. MATH.
Vol. 25 No. 2 (2001), 413–442

RIGID SPACES AND THE AR-PROPERTY

By

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Abstract. A rigid space is a topological vector space whose endomorphisms are all simply scalar multiples of the identity. A rigid space can be constructed so as to admit compact operators [14]. This paper proves that the rigid space admitting compact operators constructed in [14] can be modified to be an AR, and hence is homeomorphic to the Hilbert space ℓ_2.

§ 1. Introduction

Rigid spaces, which appeared for the first time in [16] and then in [6] [7] [14], are among the most operator-poor of spaces in the class of linear metric spaces. In fact, these spaces do not have any endomorphisms other than scalar multiples of the identity map. Nevertheless, rigid spaces can share some nice topological properties with the richest of spaces in functional analysis: Hilbert spaces. For instance, in [11] it was shown that a rigid space can be constructed to be homeomorphic to the Hilbert space ℓ_2. Thus, rigid spaces may look poor from the point of view of functional analysis, yet look rather wealthy from the point of view of topology.

In this paper, we continue our investigation on the AR-property for rigid spaces. The AR-propery for linear metric spaces is of special interest, since infinite dimensional separable complete linear metric spaces with the AR-property are homeomorphic to Hilbert space, see [4].

Observe that Cauty [3] constructed a σ-compact linear metric space which is not an AR. By a theorem of Torunczyk [15], the completion of any non-AR-linear metric space is still a non-AR-space. Therefore the completion of Cauty's example provides a separable complete linear metric space which is not an AR.

1991 Mathematics Subject Classification: Primary 46A16, 54F65; Secondary 46C05, 54G15.
Key Words and Phrases: Linear Metric Space, Rigid Space, AR-Space, Hilbert Space.
Received July 3, 2000.

It should also be observed that while Cauty showed the existence of non-AR-linear metric spaces, it is difficult to use his argument to obtain an intuitive picture of such a space. In fact, Cauty's example is based on some rather deep facts from infinite dimensional topology and a more self-contained example of a non-AR-linear metric space would be much appreciated. Naturally, it is hoped that such an example should be found among pathological objects in linear metric spaces.

We also hope that our investigation on the AR-property for rigid spaces will shed light on the following question which is one of the most outstanding open problems in infinite dimensional topology:

QUESTION. Is every compact convex set in a linear metric space an AR? Does every compact convex set have the fixed point property? The second part of the above question, known as "Schauder's Conjecture", was posed by Schauder in early 1930's, but is still open today.

The result obtained in this paper is much harder than the result obtained in [11], where a similar theorem was established.

NOTATION AND CONVENTIONS. In this paper, all maps are assumed to be continuous. By a linear metric space we mean a topological space which is metrizable. The zero element of X is denoted by θ. The space X will be equipped with an F-norm $\|\cdot\|$ (see [13]); that is, a function $\|\cdot\| : X \to [0, \infty)$ such that

(a) $\|x\| = 0$ if and only if $x = 0$,
(b) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$,
(c) $\|\lambda x\| \leq \|x\|$ for every $x \in X$ and $\lambda \in \boldsymbol{R}$ with $|\lambda| \leq 1$,
(d) $\|\alpha x\| \to 0$ whenever $|\alpha| \to 0$.

Let A be a subset of a linear metric space X. By span A we mean the linear subspace of X spanned by A, and by conv A we mean the convex hull of A in X. We also use the following notation:

$$\|x - A\| = \inf\{\|x - y\| : y \in A\} \quad \text{for } x \in X;$$

$$\operatorname{diam} A = \sup\{\|x - y\| : x, y \in X\}.$$

Let $\{(X_\alpha, \|\cdot\|_\alpha)\}$ be a collection of F-normed vector spaces, and let $X = \operatorname{span}\{X_\alpha\}$. For $x \in X$, let

$$\|x\| = \inf\left\{\sum_{i=1}^{n} \|x_{\alpha_i}\|_{\alpha_i} : x = \sum_{i=1}^{n} x_{\alpha_i}; x_{\alpha_i} \in X_{\alpha_i}; n \in N\right\}.$$

The F-norm $\|\cdot\|$ defined as above will be referred to as inf-norm $\{(X_\alpha, \|\cdot\|_\alpha)\}$ and will be used frequently throughout this paper.

For undefined notation, see [1] [2] and [13].

§2. A Rigid Space Admitting Compact Operators

In this section, we describe the rigid space admitting compact operators constructed in [14]. This space is the main object of our investigation.

Let W be a finite dimensional linear space with a basis $\{w_1, w_2, \ldots, w_n\}$. For $p, \beta \in (0,1)$ we define an F-norm $|\cdot|^0$, which will be called the (p,β)-norm on W, as follows: for $x \in W$ with $x = \sum_{i=1}^n x_i w_i \in W$, let

(1) $$|x|^1 = \sum_{i=1}^n |x_i|;$$

(2) $$|x|^2 = \beta \sum_{i=1}^n |x_i|^p;$$

(3) $$|x|^0 = \text{inf-norm}\{|x|^1, |x|^2\}.$$

Observe that the (p,β)-norm $|\cdot|^0$ defined by (1) (2) (3) is an F-norm, not a norm.

Now we are going to describe the rigid space which was constructed in [14]. Let V denote the space of all finitely non-zero valued sequences. Let

(4) $$A = \{e_1 + e_n\}_{n=2}^{\infty} \cup \{e_1 - e_n\}_{n=2}^{\infty} \cup \{e_1\}$$

where e_n is the sequence with a 1 in the n-th slot and zeros elsewhere. Let $\{a_n\}$ be a sequence in A such that for each $a \in A$, $a = a_n$ for infinitely many n. Let $\{p_n\}$ be a sequence of positive numbers such that

(5) $$0 < p_1 < p_2 < \cdots < p_n < \cdots < 1 \quad \text{and}$$

(6) $$\lim_{n \to \infty} p_n = 1.$$

Let $\{V_n\}$ be a sequence of finite dimensional spaces of V, with $\dim V_n = \ell(n)$, such that

(7) Each V_n has a basis of the form $\{e_1^n, \ldots, e_{\ell(n)}^n\}$ with $a_n = [\ell(n)]^{-1}(e_1^n + \cdots + e_{\ell(n)}^n)$.

(8) If $a_n \in A$ and $a_n \notin V_1 + \cdots + V_{n-1}$, then $V_n = \boldsymbol{R}a_n$. Otherwise

(9) $$V_n \cap (V_1 + \cdots + V_{n-1}) = \boldsymbol{R}a_n.$$

For any $n \in N$, let $|\cdot|_n$ denote the (p_n, β_n)-norm on V_n. Let

$$E_n = V_1 + \cdots + V_n \tag{10}$$

and define $\|\cdot\|_n$ on E_n by

$$\|\cdot\|_n = \text{inf-norm}\{(V_1, |\cdot|_1), \ldots, (V_n, |\cdot|_n)\}. \tag{11}$$

Let

$$E = \bigcup_{n=1}^{\infty} E_n; \quad \text{and} \quad |||\cdot|||_n = 4^{n-1}\|\cdot\|_n. \tag{12}$$

The space E will be equipped with the F-norm

$$|||\cdot||| = \text{inf-norm}\{(E_n, |||\cdot|||_n)\}. \tag{13}$$

Observe that in [14] the F-norm $\|\cdot\|_n$ defined by (11) was chosen to satisfy the condition

$$\|\cdot\|_n \geq \frac{1}{2}\|\cdot\|_{n-1} \quad \text{on } E_{n-1}.$$

Therefore from (12) we get

$$|||\cdot|||_n \geq 2|||\cdot|||_{n-1} \quad \text{on } E_{n-1}. \tag{14}$$

Let X denote the completion of $(E, |||\cdot|||)$. It was proved in [14] that for certain choice of sequences $\{p_n\}$, $\{\beta_n\}$ satisfying conditions (5) (6) and $\{\ell(n)\} \subset N$, the resulting space X will be a rigid space admitting compact operators. Our aim is to demonstrate:

THEOREM 1. *X is an AR.*

From Theorem 1 and from Theorem A we obtain

MAIN THEOREM. *X is homeomorphic to the Hilbert space ℓ_2.*

§3. Some Properties of the (p, β)-Norm

Let W be a finite dimensional linear space with a basis $\{w_1, \ldots, w_m\}$ equipped with a (p, β)-norm defined by (1)–(3), where $p \in (0, 1)$ and $\beta > 0$. For every $x \in W$, $x = \sum_{i=1}^{m} x_i w_i$, let

$$I(x) = \{i : |x_i| \leq \beta^{1/(1-p)}\} \quad \text{and} \quad J(x) = \{i : |x_i| > \beta^{1/(1-p)}\}. \tag{15}$$

Then $I(x) \cup J(x) = \{1, \dots, m\}$. Let

$$x^1 = \sum_{i \in I(x)} x_i w_i \quad \text{and} \quad x^2 = \sum_{i \in J(x)} x_i w_i.$$

Then $x^1 + x^2 = x$. We claim that

Lemma 1. $|x|^0 = |x^1|^1 + |x^2|^2$, *see* (1) (2) (3).

For the proof of Lemma 1 we need the following simple fact.

Claim 1. If $p \in (0,1)$, $\beta > 0$, and $|a| > \beta^{1/(1-p)}$, then

$$|x| + \beta|a - x|^p \geq \beta|a|^p.$$

Proof. We prove the claim for $a > \beta^{1/(1-p)}$. The proof for the case $a < -\beta^{1/(1-p)}$ is similar. Consider the following cases:

Case 1. $x > a$. Then $x > \beta^{1/(1-p)}$. Therefore $x > \beta x^p > \beta a^p$ and the claim follows.

Case 2. $x < 0$. Then $a - x > a$. Therefore

$$|x| + \beta|a - x|^p > \beta|a - x|^p > \beta a^p$$

and the claim follows.

Case 3. $x \in [0, a]$. Consider the function

$$\varphi(x) = x + \beta(a - x)^p.$$

Then we have

$$\varphi'(x) = 1 - \beta p(a - x)^{p-1} \quad \text{for every } x \in (0, a).$$

Hence

$$\varphi'(x) = 0 \quad \text{for } x = a - (\beta p)^{1/(1-p)}.$$

Observe that φ is increasing on $[0, a - (\beta p)^{1/(1-p)}]$ and is decreasing on $[a - (\beta p)^{1/(1-p)}, a]$. Hence

$$\varphi(x) = x + \beta(a - x)^p \geq \varphi(0) = \beta a^p \quad \text{for every } x \in [0, a - (\beta p)^{1/(1-p)}],$$

and

$$\varphi(x) = x + \beta(a - x)^p \geq \varphi(a) = a > \beta a^p \quad \text{for every } x \in [a - (\beta p)^{1/(1-p)}, a].$$

It follows that

$$\varphi(x) = x + \beta(a-x)^p \geq \beta a^p \quad \text{for every } x \in [0,a].$$

The claim is proved.

PROOF OF LEMMA 1. By (3), $|x|^0 \leq |x^1|^1 + |x^2|^2$. We shall show that $|x|^0 \geq |x^1|^1 + |x^2|^2$.

Assume to the contrary that $|x|^0 < |x^1|^1 + |x^2|^2$. Then there exist $y^j \in W$,

$$y^j = \sum_{i=1}^{m} y_i^j w_i, \quad j = 1,2, \quad \text{with } y^1 + y^2 = x,$$

such that

$$|y^1|^1 + |y^2|^2 < |x^1|^1 + |x^2|^2. \tag{16}$$

Then we have

$$\sum_{i=1}^{m}(|y_i^1| + \beta|y_i^2|^p) < \sum_{i \in I(x)} |x_i| + \sum_{i \in J(x)} \beta|x_i|^p. \tag{17}$$

Therefore there exists at least one i, say $i = 1$, such that

$$|y_1^1| + \beta|y_1^2|^p < |x_1| \quad \text{if } |x_1| \leq \beta^{1/(1-p)}, \tag{18}$$

$$|y_1^1| + \beta|y_1^2|^p < \beta|x_1|^p \quad \text{if } |x_1| > \beta^{1/(1-p)}. \tag{19}$$

Observe that $y_i^1 + y_i^2 = x_i$ for every $i = 1, \ldots, m$. In particular, $y_1^1 + y_1^2 = x_1$. Consider the two cases:

CASE 1. $|x_1| \leq \beta^{1/(1-p)}$. From (19) it follows that

$$|y_1^2| < |x_1| < \beta^{1/(1-p)}.$$

Therefore $|y_1^2| < \beta|y_1^2|^p$. Since $x_1 = y_1^1 + y_1^2$, we get

$$|x_1| \leq |y_1^1| + |y_1^2| < |y_1^1| + \beta|y_1^2|^p$$

which contradicts (18).

CASE 2. $|x_1| > \beta^{1/(1-p)}$. Then by Claim 1 we get

$$|y_1^1| + \beta|x_1 - y_1^1|^p \geq \beta|x_1|^p.$$

Since $x_1 - y_1^1 = y_1^2$, we have

$$|y_1^1| + \beta|y_1^2|^p \geq \beta|x_1|^p$$

which contradicts (19). Thus, the lemma is proved.

From Lemma 1 we get

COROLLARY 1. *For every* $x \in W$, $x = \sum_{i=1}^m x_i w_i$, *we have*

$$|x|^0 = \sum_{i \in I(x)} |x_i| + \sum_{i \in J(x)} \beta|x_i|^p.$$

where $I(x)$ *and* $J(x)$ *were defined by* (15).

§4. Some Algebraic Properties

LEMMA 2. *Let* $\{V_n\}$ *denote a sequence of finite dimensional linear spaces of* V *satisfying conditions* (7)–(9). *Then for every* $n \in N$, $\{a_n, e_i^n, i = 1, \dots, \ell(n) - 1\}$ *is a linearly independent subset in* V, *hence is a basis for* V_n.

PROOF. Assume that $\lambda a_n + \sum_{i=1}^{\ell(n)-1} \lambda_i e_i^n = \theta$. Then we have

$$\frac{\lambda}{\ell(n)}(e_1^n + \cdots + e_{\ell(n)}^n) + \sum_{i=1}^{\ell(n)-1} \lambda_i e_i^n = \theta.$$

It follows that

$$\sum_{i=1}^{\ell(n)-1} \left(\lambda_i + \frac{\lambda}{\ell(n)}\right) e_i^n + \frac{\lambda}{\ell(n)} e_{\ell(n)}^n = \theta. \tag{20}$$

Since $\{e_i^n, i = 1, \dots, \ell(n)\}$ is a basis of V_n, we get

$$\frac{\lambda}{\ell(n)} = 0 \quad \text{and} \quad \lambda_i + \frac{\lambda}{\ell(n)} = 0 \quad \text{for } i = 1, \dots, \ell(n) - 1.$$

Therefore $\lambda = 0$ and $\lambda_i = 0$ for $i = 1, \dots, \ell(n) - 1$. The lemma is proved.

Let

$$S_n = \{e_i^k, i = 1, \dots, \ell(k) - 1, k = 1, \dots, n\} \quad \text{and} \quad S = \bigcup_{n=1}^{\infty} S_n. \tag{21}$$

LEMMA 3. $\operatorname{span} S \cap \operatorname{span}\{a_n : n \in N\} = \{\theta\}$.

PROOF. It suffices to show that

$$\operatorname{span} S_n \cap \operatorname{span}\{a_i, i = 1, \dots, n\} = \{\theta\} \quad \text{for every } n \in N. \tag{22}$$

We prove (22) by induction. If $n=1$, then $\ell(1)=1$, see (8). Therefore $\ell(1)-1=0$ and so $S_1=\varnothing$ and span $S_1=\{\theta\}$ and the claim is true.

Assume that (22) has been proved up to n. Let

$$x = x_1 + x_2 = a + \lambda a_{n+1} \in \operatorname{span} S_{n+1} \cap \operatorname{span}\{a_1, \dots, a_{n+1}\} \tag{23}$$

where

$$x_1 \in \operatorname{span} S_n, \quad x_2 \in \operatorname{span}\{e_1^{n+1}, \dots, e_{\ell(n+1)-1}^{n+1}\} \quad \text{and} \quad a \in \operatorname{span}\{a_i, i = 1, \dots, n\}.$$

Observe that

$$x_2 = \sum_{i=1}^{\ell(n+1)-1} \mu_i e_i^{n+1}, \quad a_{n+1} = [\ell_{(n+1)}]^{-1}(e_1^{n+1} + \cdots + e_{\ell(n+1)}^{n+1}). \tag{24}$$

From (23) we get

$$\begin{aligned} x_2 - \lambda a_{n+1} = a - x_1 &\in \operatorname{span}(S_n \cup \{a_1, \dots, a_n\}) \cap V_{n+1} \\ &= (V_1 + \cdots + V_n) \cap V_{n+1}. \end{aligned} \tag{25}$$

Then by (8) (9)

$$x_2 - \lambda a_{n+1} = \alpha a_{n+1} \quad \text{for some } \alpha \in \boldsymbol{R}. \tag{26}$$

Hence

$$x_2 - (\lambda + \alpha) a_{n+1} = \theta.$$

Since $x_2 \in \operatorname{span}\{e_1^{n+1}, \dots, e_{\ell(n+1)-1}^{n+1}\}$ and by Lemma 2, $\{a_n, e_i^n, i = 1, \dots, \ell(n) - 1\}$ is linearly independent independent we have $x_2 = 0$ and $\lambda + \alpha = 0$. Therefore

$$\lambda + \alpha = 0 \quad \text{and} \quad \mu_i = 0 \quad \text{for } i = 1, \dots, \ell(n+1) - 1. \tag{27}$$

Hence from (23) and (24) we get $x = x_1 \in \operatorname{span} S_n$. Consider the two cases:

CASE 1. $a_{n+1} \in \operatorname{span}\{a_1, \dots, a_n\}$. Then from (23) we get

$$x = a + \lambda a_{n+1} \in \operatorname{span} S_n \cap \operatorname{span}\{a_1, \dots, a_n\}.$$

By the inductive assumption we get $x = \theta$.

CASE 2. $a_{n+1} \notin \operatorname{span}\{a_1, \dots, a_n\}$.

Then by (8), $V_{n+1} \cap (V_1 + \cdots + V_n) = \{\theta\}$. Therefore from (25) (26) we get $\alpha = 0$ and so by (27), $\lambda = 0$. Consequently from (23) we obtain

$$x \in \operatorname{span} S_n \cap \operatorname{span}\{a_1, \dots, a_n\}.$$

By the inductive assumption, $x = \theta$. The lemma is proved.

LEMMA 4. *The set S defined by* (21) *is a linearly independent subset of V.*

PROOF. It suffices to show that S_n is linearly independent for every $n \in N$, see (21). We will prove this by induction.

For $n = 1$, we get $\ell(1) = 1$, see (8). Therefore $S_1 = \varnothing$, see (21).

Assume that the claim has been proved up to n. Observe that

$$S_{n+1} = S_n \cup \{e_i^{n+1}, i = 1, \dots, \ell(n+1) - 1\}.$$

Let

$$\lambda_1 s_1 + \cdots + \lambda_m s_m + \lambda_{m+1} s_{m+1} + \cdots + \lambda_k s_k = \theta \tag{28}$$

where $s_i = S_n$ for $i = 1, \dots, m$ and $s_i \in \{e_j^{n+1}, j = 1, \dots, \ell(n+1) - 1\}$ for $i = m+1, \dots, k$. We may assume that

$$k - m = \ell(n+1) - 1 \quad \text{and} \quad s_{m+i} = e_i^{n+1} \quad \text{for } i = 1, \dots, \ell(n+1) - 1.$$

Then

$$\lambda_1 s_1 + \cdots + \lambda_m s_m = -\lambda_{m+1} e_1^{n+1} - \cdots - \lambda_{m+\ell(m+1)-1} e_{\ell(n+1)-1}^{n+1}. \tag{29}$$

Let

$$x = -\lambda_{m+1} e_1^{n+1} - \cdots - \lambda_{m+\ell(n+1)-1} e_{\ell(n+1)-1}^{n+1}. \tag{30}$$

Then $x \in V_{n+1} \cap (V_1 + \cdots + V_n)$. Therefore by (8) (9)

$$x = \lambda a_{n+1} \quad \text{for some } \lambda \in \boldsymbol{R}.$$

Since by Lemma 2, $\{a_{n+1}, e_1^{n+1}, \dots, e_{\ell(n+1)-1}^{n+1}\}$ is linearly independent, from (30) we get

$$\lambda = 0 \quad \text{and} \quad \lambda_{m+i} = 0 \quad \text{for } i = 1, \dots, \ell(n+1) - 1.$$

Consequently from (28) we get $\lambda_1 s_1 + \cdots + \lambda_m s_m = \theta$. Since $s_i \in S_n$ for $i = 1, \dots, m$, and by the inductive assumption S_n is linearly independent, we get $\lambda_1 = \cdots = \lambda_m = 0$.

The lemma is proved.

§5. X Is a Quotient of an AR-Space

In this section we shall show that the rigid space X constructed in Section 2 is a quotient of an AR-linear metric space.

Recall that V denotes the linear space of all finitelly non-zero valued sequences. Let

$$\{u_i^n, i = 1, \dots, \ell(n), n = 1, 2, \dots\} \tag{31}$$

be a linearly independent sequence in V. Let

$$(32)\qquad U_n = \operatorname{span}\{u_1^n, \dots, u_{\ell(n)}^n\}; \quad F_n = U_1 + \cdots + U_n; \quad U = \bigcup_{n=1}^{\infty} F_n.$$

Using the sequences $\{p_n\}$ and $\{\beta_n\}$, see (5) (6), we define an F-norm $|||\cdot|||$ on U in the same way as the definition of the F-norm on E. In fact, first let $|.|_n$ denote the (p_n, β_n)-norm on U_n and define $\|.\|_n$ on F_n by the formula (11). Then define $|||\cdot|||$ on U by (12) (13). Observe that the spaces U and E are very much similar. The only difference between U and E is that $\{u_i^n, i = 1, \dots, \ell(n), n = 1, 2, \dots\}$ are linearly independent, while $\{e_i^n, i = 1, \dots, \ell(n), n = 1, 2, \dots\}$ are not linearly independent. Let Z denote the completion of $(U, |||\cdot|||)$. We shall prove

THEOREM 2. *Z is an AR.*

The proof of Theorem 2 will be given in the last section.

Our aim is to show that the space X constructed in Section 2 is a quotient space of Z. First we prove

LEMMA 5. *If $x \in \bigcup_{k=1}^{\infty} E_k$, say $x \in E_n$, then*

$$|||x||| = \inf\left\{\sum_{k=1}^{n} 4^{k-1}|x^k|_k : x^k \in V_k, \sum_{k=1}^{n} x^k = x\right\}.$$

PROOF. First observe that for any $x^k \in V_k$, $k = 1, \dots n$, with $x^1 + \cdots + x^n = x$ we have $|||x||| \le \sum_{k=1}^{n} 4^{k-1}|x^k|_k$. We shall prove that for every $\varepsilon > 0$ there exists an expression $x = x^1 + \cdots + x^n$ such that $|||x||| > \sum_{k=1}^{n} 4^{k-1}|x^k|_k - \varepsilon$.

We need the following fact:

CLAIM 2. Let $x \in E_n$, $n \ge 2$. Then for every $\varepsilon > 0$ there exist $x^{n-1} \in E_{n-1}$ and $x^n \in V_n$ such that $x^{n-1} + x^n = x$ and

$$|||x|||_n > |||x^{n-1}|||_{n-1} + 4^{n-1}|x^n|_n - \varepsilon.$$

PROOF. Let $x \in E_n$, $n \ge 2$. By the definition of inf-norm for every $\varepsilon > 0$ there exist $x_i \in V_i$, $i = 1, \dots, n$, such that $x_1 + \cdots + x_n = x$ and

$$|||x|||_n = 4^{n-1}\|x\|_n > 4^{n-1}(|x_1|_1 + \cdots + |x_n|_n) - \varepsilon.$$

Let $x^{n-1} = x_1 + \cdots + x_{n-1}$ and $x^n = x_n$. Then $x^{n-1} + x^n = x$ and

$$|x_1|_1 + \cdots + |x_{n-1}|_{n-1} \ge \|x^{n-1}\|_{n-1}.$$

Therefore

$$\begin{aligned}|||x||| &> 4^{n-1}(\|x^{n-1}\|_{n-1} + |x^n|_n) - \varepsilon \\ &= 4^{n-1}\|x^{n-1}\|_{n-1} + 4^{n-1}|x^n|_n - \varepsilon \\ &\geq 4^{n-2}\|x^{n-1}\|_{n-1} + 4^{n-1}|x^n|_n - \varepsilon \\ &= |||x^{n-1}|||_{n-1} + 4^{n-1}|x^n|_n - \varepsilon.\end{aligned}$$

The claim is proved.

CLAIM 3. For every $n \in N$ and for every $\varepsilon > 0$, there exist $x_k \in E_k$, $k = 1, \ldots, n$, such that $x_1 + \cdots + x_n = x$, and

$$|||x||| > |||x_1|||_1 + |||x_2|||_2 + \cdots + |||x_n|||_n - 2^{-n}\varepsilon.$$

PROOF. Observe that, given $n \in N$, and $\varepsilon > 0$, by the definition of $|||.|||$ there exist $x_k \in E_k$, $k = 1, \ldots, m$, such that $x_1 + \cdots + x_m = x$, and

$$|||x||| > |||x_1|||_1 + |||x_2|||_2 + \cdots + |||x_m|||_m - 2^{-n}\varepsilon.$$

Therefore if $m \leq n$, then the claim is proved. Assume that $m > n$. Since $x \in E_n$ and $n \leq m - 1$ we have $x_m = x - (x_1 + \cdots + x_{m-1}) \in E_{m-1}$. Therefore, from (14) we get

$$\begin{aligned}|||x||| &> \sum_{k=1}^{m} |||x_k|||_k - 2^{-n}\varepsilon \\ &= \sum_{k=1}^{m-2} |||x_k|||_k + |||x_{m-1}|||_{m-1} + |||x_m|||_m - 2^{-n}\varepsilon \\ &\geq \sum_{k=1}^{m-2} |||x_k|||_k + |||x_{m-1}|||_{m-1} + 2|||x_m|||_{m-1} - 2^{-n}\varepsilon \\ &\geq \sum_{k=1}^{m-2} |||x_k|||_k + |||x_{m-1}|||_{m-1} + |||x_m|||_{m-1} - 2^{-n}\varepsilon \\ &\geq \sum_{k=1}^{m-2} |||x_k|||_k + |||x_{m-1} + x_m|||_{m-1} - 2^{-n}\varepsilon \\ &= \sum_{k=1}^{m-1} |||y_k|||_k - 2^{-n}\varepsilon,\end{aligned}$$

where $y_k = x_k$ for $k = 1, \dots, m-2$ and $y_{m-1} = x_{m-1} + x_m$. Consequently, the claim is proved by induction.

Now we are able to complete the proof of Lemma 5. By Claim 3,

$$|||x||| > |||x_1|||_1 + |||x_2|||_2 + \cdots + |||x_n|||_n - 2^{-n}\varepsilon. \tag{33}$$

By Claim 2 for every $k = 2, \dots, n$ there exist $x_k^k \in V_k$ and $y^{k-1} \in E_{k-1}$ such that $y^{k-1} + x_k^k = x_k$ and

$$|||x_k|||_k > |||y^{k-1}|||_{k-1} + 4^{k-1}|x_k^k|_k - 2^{-2n}\varepsilon.$$

Applying Claim 2 again for y^{k-1} and for $2^{-2n}\varepsilon$, so on, we obtain

$$\begin{aligned}
|||x|||_k &> |||y^{k-2}|||_{k-2} + 4^{k-2}|x_k^{k-1}|_{k-1} + 4^{k-1}|x_k^k| - 2^{-2n+1}\varepsilon \\
&> \cdots \\
&> |||y^1|||_1 + 4|x_k^2|_2 + 4^2|x_k^3|_3 + \cdots + 4^{k-1}|x_k^k| - 2^{-n}\varepsilon \\
&= |x_k^1|_1 + 4|x_k^2|_2 + 4^2|x_k^3|_3 + \cdots 4^{k-1}|x_k^k| - 2^{-n}\varepsilon \\
&\quad (\text{where } x_k^1 = y^1).
\end{aligned}$$

Therefore from (33) we get

$$\begin{aligned}
|||x||| &> |||x_1|||_1 + |||x_2|||_2 + \cdots + |||x_n|||_n - 2^{-n}\varepsilon \\
&> |x_1|_1 + (|x_1^2|_1 + 4|x_2^2| - 2^{-n}\varepsilon) + \cdots \\
&\quad + (|x_1^n|_1 + 4|x_2^n|_2 + \cdots + 4^{n-1}|x_n^n|_n - 2^{-n}\varepsilon) \\
&= (|x_1|_1 + |x_1^2|_1 + \cdots + |x_1^n|_1) \\
&\quad + 4(|x_2^2|_2 + |x_2^3|_2 + \cdots + |x_2^n|_2) + \cdots \\
&\quad + 4^{n-2}(|x_{n-1}^{n-1}|_{n-1} + |x_{n-1}^n|_{n-1}) + 4^{n-1}|x_n^n|_n - n2^{-n}\varepsilon.
\end{aligned}$$

Let

$$\begin{aligned}
x^1 &= x_1 + x_1^2 + \cdots + x_1^n \in V_1; \\
x^2 &= x_2^2 + x_2^3 + \cdots + x_2^n \in V_2; \\
&\cdots \\
x^{n-1} &= x_{n-1}^{n-1} + x_{n-1}^n \in V_{n-1}; \\
x^n &= x_n^n \in V_n.
\end{aligned}$$

Then we have

$$|||x||| > |x^1|_1 + 4|x^2|_2 + \cdots + 4^{n-2}|x^{n-1}|_{n-1} + 4^{n-1}|x^n|_n - \varepsilon.$$

The lemma is proved.

Corollary 2. *For every* $x \in Z$, $x = \sum_{n=1}^{\infty} x^n$, $x^n \in U_n$, *we have*

$$|||x||| = \sum_{n=1}^{\infty} 4^{n-1}|x^n|_n.$$

Proof. Since $\{u_i^n, i = 1, \ldots, \ell(n), n = 1, 2, \ldots\}$ are linearly independent, for every $x \in Z$, the expresion $x = \sum_{n=1}^{\infty} x^n$, $x^n \in U_n$, is unique and the assertion follows.

From Corollaries 1 and 2 we get

Corollary 3. *For every* $x \in Z$, $x = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n u_i^n$, *we have*

$$|||x||| = \sum_{n=1}^{\infty} 4^{n-1} \left(\sum_{i \in I_n(x)} |x_i^n| + \sum_{i \in J_n(x)} \beta_n |x_i^n|^{p_n} \right),$$

where

$$I_n(x) = \{i : |x_i^n| \le \beta_n^{1/(1-p_n)}\} \quad \text{and} \quad J_n(x) = \{i : |x_i^n| > \beta_n^{1/(1-p_n)}\}.$$

Proof. For every $x \in Z$, $x = \sum_{n=1}^{\infty} x^n$, $x^n \in U_n$. From Corollary 2 we get

$$|||x||| = \sum_{n=1}^{\infty} 4^{n-1}|x^n|_n.$$

Observe that

$$x^n = \sum_{i=1}^{\ell(n)} x_i^n e_i^n \in U_n \quad \text{for every } n \in N.$$

Therefore the assertion follows from Corollary 1.

Now we define $g : U \to E$ to be the "natural" projection from U onto E, that is

$$g(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n e_i^n \quad \text{for every } x = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n u_i^n \in U,$$

(with only finitely many x_i^n are non-zero).

From Lemma 5 and from Corollary 2 we get

$$|||g(x)||| \leq |||x||| \quad \text{for every } x \in U.$$

Therefore g can be extended to a continuous linear map, which is still denoted by g, from Z into X. We claim that

PROPOSITION 1. *The quotient map $g^*: Z/g^{-1}(\theta) \to X$ is an isometric embedding.*

The proof of Proposition 1 will be given in the next section.

By Proposition 1 $g^*(Z/g^{-1}(\theta))$ is complete. Since $g^*(Z/g^{-1}(\theta)) \supset E$, and since E is dense in X, we have $g^*(Z/g^{-1}(\theta)) = X$. It follows that $g(Z) = g^*(Z/g^{-1}(\theta)) = X$.

Consequently, X is a quotient space of Z and the assertion is established.

§6. The Kernel of g and Proof of Proposition 1

In main result of this section, Lemma 6, describes the kernel $g^{-1}(\theta)$ of the map g defined in Section 5. This fact will be used in the proofs of Proposition 1 and Theorem 1.

First we define the sequence $\{b_k\}_{k=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ as follows. Let $b_1 = a_1$. Assume that $b_1, \dots, b_{k-1}$ have been selected. Let $n \in N$ denote the smallest number such that $a_n \notin V_1 + \cdots + V_{k-1}$. We define $b_k = a_n$.

For each $k \in N$, denote

$$N(k) = \{n : a_n = b_k\}. \tag{34}$$

Then by the definition of $\{a_n\}$, $N(k)$ is infinite for every $k \in N$, and

$$N(k) \cap N(k') = \phi \quad \text{for } k \neq k' \quad \text{and} \quad \bigcup_{n=1}^{\infty} N(k) = N.$$

Let

$$F_k = \operatorname{span}\{a_n : n \in N(k)\}.$$

$$B_k = \{u_n : n \in N(k)\}, \quad \text{where } u_n = [\ell(n)]^{-1}(u_1^n + \cdots + u_{\ell(n)}^n). \tag{35}$$

$$G_k = \{\lambda_1 u_{n(1)} + \cdots + \lambda_p u_{n(p)} : n(i) \in N(k), i = 1, \dots, p \text{ and } \lambda_1 + \cdots + \lambda_p = 0\}. \tag{36}$$

$$G = \overline{\bigoplus_{k=1}^{\infty} G_k} \subset Z. \tag{37}$$

Then we get

$$F_k \cap F_{k'} = \{\theta\} \quad \text{for every } k \neq k'.$$

We prove

Lemma 6. $g^{-1}(\theta) = G$.

Proof. We first claim that

$$G_k \subset g^{-1}(\theta) \quad \text{for every } k \in N. \tag{38}$$

In fact if $x \in G_k$, then

$$x = \sum_{i=1}^{p} \lambda_i u_{n(i)}, \quad \text{where } u_{n(i)} \in B_k, \text{ see (35)}, \ i = 1, \dots, p \text{ and } \sum_{i=1}^{p} \lambda_i = 0.$$

Then we have

$$g(x) = \sum_{i=1}^{p} \lambda_i a_{n(i)} = \left(\sum_{i=1}^{p} \lambda_i \right) b_k = 0 \ b_k = \theta.$$

Therefore $x \in g^{-1}(\theta)$ and the claim is proved.

From (37) (38) we get $G \subset g^{-1}(\theta)$. To prove $g^{-1}(\theta) \subset G$, let $x \in U$ such that $g(x) = \theta$. Then we have

$$x = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n u_i^n \quad \text{(with only finitely many } x_i^n \text{ are non-zero)}. \tag{39}$$

Write

$$\sum_{i=1}^{\ell(n)} x_i^n u_i^n = \sum_{i=1}^{\ell(n)} x_{\ell(n)}^n u_i^n + \sum_{i=1}^{\ell(n)-1} (x_i^n - x_{\ell(n)}^n) u_i^n.$$

Let

$$\lambda_n = \ell(n) x_{\ell(n)}^n \quad \text{and} \quad y_i^n = x_i^n - x_{\ell(n)}^n, \quad i = 1, \dots, \ell(n) - 1. \tag{40}$$

Then we get, see (35)

$$\sum_{i=1}^{\ell(n)} x_i^n u_i^n = \lambda_n u_n + \sum_{i=1}^{\ell(n)-1} y_i^n u_i^n.$$

Therefore

$$x = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n u_i^n = \sum_{n=1}^{\infty} \lambda_n u_n + \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)-1} y_i^n u_i^n,$$

(with only finitely many x_i^n and λ_n are non-zero). Hence

$$g(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n a_i^n = \sum_{n=1}^{\infty} \lambda_n a_n + \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)-1} y_i^n e_i^n,$$

(with only finitely many x_i^n and λ_n are non-zero). Since $g(x) = \theta$ from Lemma 3 we get

$$\sum_{n=1}^{\infty} \lambda_n a_n = -\sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)-1} y_i^n e_i^n \in \operatorname{span} S \cap \operatorname{span}\{a_n\} = \{\theta\},$$

(with only finitely many x_i^n and λ_n are non-zero). Thus

$$\sum_{n=1}^{\infty} \lambda_n a_n = \theta \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)-1} y_i^n e_i^n = \theta.$$

(with only finitely many x_i^n and λ_n are non-zero). By Lemma 4 we get

$$y_i^n = 0 \quad \text{for every } i = 1, \dots, \ell(n+1) - 1, n = 1, 2, \dots$$

Consequently

$$x = \sum_{n=1}^{\infty} \lambda_n u_n \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \lambda_n a_n.$$

Write

$$x = \sum_{k=1}^{\infty} \sum_{n \in N(k)} \lambda_n u_n \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} \sum_{n \in N(k)} \lambda_n a_n.$$

$$x_k = \sum_{n \in N(k)} \lambda_n u_n \quad \text{and} \quad y_k = \sum_{n \in N(k)} \lambda_n a_n. \tag{41}$$

We claim that $y_k = \theta$ for every $k \in N$. In fact, if it is not the case, let $K \in N$ denote the largest number such that $y_K \neq \theta$. (By (39) only finitely many y_k are non-zero.) From (34) (41) we get $y_K = (\sum_{n \in N(K)} \lambda_n) b_K$. Observe that $g(x) = y_1 + \cdots + y_K$. Since $g(x) = \theta$ we get $y_K \in V_1 + \cdots + V_{K-1}$. Since $y_K \neq \theta$ we have $\sum_{n \in N(K)} \lambda_n \neq 0$. Therefore

$$b_K = \left(\sum_{n \in N(k)} \lambda_n \right)^{-1} y_K \in V_1 + \cdots + V_{K-1}.$$

This contradicts the definition of b_K, and the claim is proved.

Observe that

$$\begin{aligned}\|\|y_k\|\| &= \inf\left\{\sum_{n\in N(k)} \|\|\lambda_n a_n\|\|_n : \sum_{n\in N(k)} \lambda_n a_n = y_k\right\}\\ &= \inf\left\{\sum_{n\in N(k)} \|\|\lambda_n b_k\|\|_n : \sum_{n\in N(k)} \lambda_n b_k = y_k\right\}\\ &\geq \inf\left\{\sum_{n\in N(k)} \|\|\lambda_n b_k\|\|_k : \sum_{n\in N(k)} \lambda_n b_k = y_k\right\}\\ &\qquad (\text{since } n \geq k, \|\|\cdot\|\|_n \geq 2\|\|\cdot\|\|_k, \text{ see (14)})\\ &\geq \inf\left\{\left\|\left\|\sum_{n\in N(k)} \lambda_n)b_k\right\|\right\|_k : \sum_{n\in N(k)} \lambda_n b_k = y_k\right\}.\end{aligned}$$

Since $y_k = \theta$ we get $\sum_{n\in N(k)} \lambda_n = 0$. Hence by (30)

$$x_k \in G_k \quad \text{for every } k \in N.$$

Consequently

$$x = \sum_{k=1}^{\infty} x_k \in \bigoplus_{k=1}^{\infty} G_k \subset G.$$

Hence $g^{-1}(\theta) \cap U \subset G$. Since U is dense in Z, it follows that $g^{-1}(\theta) \subset G$. The lemma is proved

Proof of Proposition 1. We have to prove that

$$\|\|g(x)\|\| = \|\|x + G\|\| \quad \text{for every } x \in Z,$$

which is quivalent to

$$\inf\{\|\|x - y\|\| : y \in G\} = \|\|g(x)\|\| \quad \text{for every } x \in Z.$$

It suffices to show that

$$\inf\{\|\|x - y\|\| : y \in G \cap U\} = \|\|g(x)\|\| \quad \text{for every } x \in U.$$

Observe that for any $x \in U$ and $y \in G \cap U$ we have

$$x = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n u_i^n \quad (\text{with only finitely many } x_i^n \text{ are non-zero}),$$

$$y = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} y_i^n u_i^n \quad (\text{with only finitely many } y_i^n \text{ are non-zero}).$$

Since $y \in G \cap U$,

$$g(x-y) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} (x_i^n - y_i^n) a_i^n = g(x).$$

Therefore

$$\begin{aligned} |||g(x)||| &\leq \sum_{n=1}^{\infty} \left| \sum_{i=1}^{\ell(n)} (x_i^n - y_i^n) a_i^n \right| \\ &= \sum_{n=1}^{\infty} \left| \sum_{i=1}^{\ell(n)} (x_i^n - y_i^n) u_i^n \right| = |||x-y|||. \end{aligned}$$

It follows that

$$|||g(x)||| \leq |||x-y||| \quad \text{for every } y \in G \cap U.$$

Consequently

$$|||g(x)||| \leq |||x + G \cap U||| \quad \text{for every } x \in U.$$

To prove that the above inequality must be an equality, we assume on the contrary that there exists $x \in U$ such that $|||g(x)||| < |||x + G \cap U|||$. By Lemma 5 there exist $x_i^n \in \boldsymbol{R}$, $i = 1, \dots, \ell(n)$, $n \in N$ (with only finitely many x_i^n are non-zero) such that $g(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n a_i^n$, and

$$\sum_{n=1}^{\infty} \left| \sum_{i=1}^{\ell(n)} x_i^n a_i^n \right|_n < |||x + G \cap U|||.$$

Denote $y = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n u_i^n \in U$. Then we have $y \in x + G \cap U$, and

$$|||y||| = \sum_{n=1}^{\infty} \left| \sum_{i=1}^{\ell(n)} x_i^n u_i^n \right|_n = \sum_{n=1}^{\infty} \left| \sum_{i=1}^{\ell(n)} x_i^n a_i^n \right|_n < |||x + G \cap U|||$$

a contradiction. Consequently Proposition 1 is proved.

§7. Proof of the Main Result

Let

$$Y = \left\{ x = \sum_{n=1}^{\infty} \lambda_n u_n \in Z : \sum_{n=1}^{\infty} |\lambda_n u_n|_n < \infty \right\}, \tag{42}$$

where $\{u_n : n \in N\}$ was defined by (35).

Observe that, see (35)

$$|\lambda_n u_n|_n = \left|\lambda_n[\ell(n)]^{-1}\sum_{i=1}^{\ell(n)} u_i^n\right|_n .$$

Since $|\cdot|_n$ is the (p_n, β_n)-norm on U_n, from Corollary 1 we get

$$(43) \qquad |\lambda_n u_n|_n = \begin{cases} |\lambda_n| & \text{if } |\lambda_n| \le \ell(n)\beta_n^{1/(1-p_n)} \\ [\ell(n)]^{1-p_n}\beta_n|\lambda_n|^{p_n} & \text{if } |\lambda_n| > \ell(n)\beta_n^{1/(1-p_n)}. \end{cases}$$

Proposition 2. *For certain choice of $\{p_n\}$ satisfying condition* (5), *Y is a locally convex linear subspace of Z.*

For the proof of Proposition 2, we need the following fact established in [11]. Let $\{p_n\}$ be a sequence of positive numbers satisfying condition (5). Let $\ell(\{p_n\})$ denote the space of all sequences $x = \{x_n\}$ such that

$$\|x\| = \sum_{n=1}^{\infty} |x_n|^{p_n} < \infty.$$

Lemma 7. [11] *There exists a sequence $\{p_n^0\}$ satisfying condition* (5) *such that for any sequence $\{p_n\}$ satisfying condition* (5) *with $p_n \ge p_n^0$ for $n \in N$, the resulting space $\ell(\{p_n\})$ is locally convex.*

In fact, it was proved in [11] that for any $\varepsilon > 0$ and for any $x^i = \{x_n^i\}$, $i = 1, \dots, m$, with

$$\|x^i\| = \sum_{n=1}^{\infty} |x_n^i|^{p_n} \le \varepsilon \quad \text{for } i = 1, \dots, m$$

and for any $\alpha_i \ge 0$, $i = 1, \dots, m$, with $\sum_{i=1}^m \alpha_i = 1$, we have

$$\left\|\sum_{i=1}^{m} \alpha_i x^i\right\| \le 3\varepsilon.$$

Let us observe that the proof given in [11] also shows that for any sequence $\{c_n\}$ of positive numbers and for any $x^i = \{x_n^i\}$, $i = 1, \dots, m$, with

$$(44) \qquad x^i = \sum_{n=1}^{\infty} x_n^i u_n; \quad \|x^i\| = \sum_{n=1}^{\infty} c_n |x_n^i|^{p_n} \le \varepsilon, \quad i = 1, \dots, m,$$

and for any $\alpha_i \geq 0$, $i = 1, \dots, m$, with $\sum_{i=1}^m \alpha_i = 1$, we have

$$\left\| \sum_{i=1}^{m} \alpha_i x^i \right\| \leq 3\varepsilon. \tag{45}$$

Now using the above observation we are able to complete the proof of Proposition 2. We shall prove that, under the above situation, the space Y will be a locally convex space. First observe that the F-norm on Y is given by (43).

Let $x^i \in Y$ with $|||x^i||| \leq \varepsilon$ for $i = 1, \dots, m$. Then we have

$$|||x^i||| = \sum_{n=1}^{\infty} 4^{n-1}[|\varphi_n(x_n^i)| + c_n|\psi_n(x_n^i)|^{p_n}] \leq \varepsilon, \tag{46}$$

where $c_n = \beta_n[\ell(n)]^{1-p_n}$, see (43), and

$$\varphi_n(x) = \begin{cases} x & \text{if } |x| \leq \ell(n)\beta_n^{1/(1-p_n)}; \\ 0 & \text{if } |x| > \ell(n)\beta_n^{1/(1-p_n)}, \end{cases}$$

and

$$\psi_n(x) = \begin{cases} 0 & \text{if } |x| \leq \ell(n)\beta_n^{1/(1-p_n)}; \\ x & \text{if } |x| > \ell(n)\beta_n^{1/(1-p_n)}. \end{cases}$$

It follows that

$$\sum_{n=1}^{\infty} 4^{n-1}|\varphi_n(x_n^i)| \leq \varepsilon \quad \text{and} \quad \sum_{n=1}^{\infty} 4^{n-1}c_n|\psi_n(x_n^i)|^{p_n} \leq \varepsilon \tag{47}$$

for every $i = 1, \dots, m$. Hence from (44) (45) we get

$$\sum_{n=1}^{\infty} 4^{n-1}c_n\left|\sum_{i=1}^{m} \alpha_i\psi_n(x_n^i)\right|^{p_n} \leq 3\varepsilon \tag{48}$$

for any $\alpha_i \geq 0$, $i = 1, \dots, m$ and $\sum_{i=1}^m \alpha_i = 1$. Since $\alpha_i \in [0, 1]$ for $i = 1, \dots, m$, from (47) we get

$$\sum_{n=1}^{\infty} 4^{n-1}\left|\sum_{i=1}^{m} \alpha_i\varphi_n(x_n^i)\right| \leq \varepsilon.$$

Hence from (48) we obtain

$$\left|\left|\left| \sum_{i=1}^{\infty} \alpha_i x^i \right|\right|\right| \leq \sum_{n=1}^{\infty} 4^{n-1}\left(\left|\sum_{i=1}^{m} \alpha_i\varphi_n(x_n^i)\right| + c_n\left|\sum_{i=1}^{m} \alpha_i\psi_n(x_n^i)\right|^{p_n}\right)$$

$$\leq \varepsilon + 3\varepsilon = 4\varepsilon.$$

Consequently Y is locally convex and Proposition 2 is proved.

Since G is a linear subspace of Y, see (35) (36) (37) (42), from Proposition 2 we get

COROLLARY 4. *Under the assumption of Proposition* 2, *G is a locally convex linear subspace of Z.*

PROOF OF THEOREM 1. By Lemma 6, $g^{-1}(\theta) = G$. By Corollary 4, G is a locally convex linear subspace of Z, by Michael's selection theorem, see for instance, [1], Proposition 7-1, p. 87, there exists a continuous map $h : X \to Z$ such that $h(x) \in g^{-1}(x)$ for every $x \in X$. By Theorem 2, Z is an AR. Consequently X is an AR and Theorem 1 is proved.

§8. Proof of Theorem 2

We use the following characterization of ANR-spaces to be found in [8]: Let $\{\mathcal{U}_n\}$ be a sequence of open covers of a metric space X. For a given cover $\mathcal{U}_n$, let

$$\mathrm{mesh}(\mathcal{U}_n) = \sup\{\mathrm{diam}\ U : U \in \mathcal{U}_n\}.$$

We say that $\{\mathcal{U}_n\}$ is a zero sequence if $\mathrm{mesh}(\mathcal{U}_n) \to 0$ as $n \to \infty$.

For a given cover $\mathcal{U}$ of X, let $\mathcal{N}(\mathcal{U})$ denote the nerve of $\mathcal{U}$. Let

$$\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n \quad \text{and} \quad \mathcal{K}(\mathcal{U}) = \bigcup_{n=1}^{\infty} \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})$$

and for $\sigma \in \mathcal{K}(\mathcal{U})$, write

$$n(\sigma) = \max\{n \in N : \sigma \in \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})\}.$$

The following characterization of ANR-spaces was established in [8], see also [9] [10].

THEOREM 3. *A metric space with no isolated points is an ANR if and only if there exists a zero sequence $\{\mathcal{U}_n\}$ of open covers of X and a map $g : \mathcal{K}(\mathcal{U}) \to X$ such that $g|\mathcal{U} \to X$ is a selection; i.e. $g(U) \in U$ for every $U \in \mathcal{U}$, and for any sequence of simplices $\{\sigma_k\}$ in $\mathcal{K}(\mathcal{U})$ with $n(\sigma_k) \to \infty$ and $g(\sigma_k^0) \to x_0 \in X$, we have $g(\sigma_k) \to x_0$, here σ_k^0 represents the vertices of σ_k.*

We are going to prove Theorem 2. Our aim is to verify the conditions of Theorem 3.

First we define two functions

(49) $$\alpha_n(x) = \begin{cases} x & \text{if } |x| \le \beta_n^{1/(1-p_n)}; \\ |x|^{p_n}\delta(x) & \text{if } |x| > \beta_n^{1/(1-p_n)}, \end{cases}$$

(50) $$\alpha_n^*(x) = \begin{cases} x & \text{if } |x| \le \beta_n^{1/(1-p_n)}; \\ |x|^{1/p_n}\delta(x) & \text{if } |x| > \beta_n^{1/(1-p_n)}, \end{cases}$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x \ge 0; \\ -1 & \text{if } x < 0. \end{cases}$$

Let $\{\mathscr{U}_k\}$ be a sequence of open covers of Z. Let $\mathscr{U} = \bigcup_{k=1}^{\infty} \mathscr{U}_k$ and $f_0 : \mathscr{U} \to Z$ be a selection.

We shall extend f_0 to a map $f : \mathscr{N}(\mathscr{U}) \to Z$ as follows: For any simplex $\sigma = \langle U_1, \dots, U_m \rangle \in \mathscr{K}(\mathscr{U})$, $U_j \in \mathscr{U}$ for $j = 1, \dots, m$. Since $f_0(U_j) \in Z$,

$$f_0(U_j) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_{j_i}^n u_i^n, \quad j = 1, \dots, m.$$

For any $x \in \sigma$,

$$x = \sum_{j=1}^{m} \lambda_j U_j, \quad \lambda_j \ge 0, \quad j = 1, \dots, m \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j = 1,$$

we define

(51) $$f(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} \alpha_n^* \left(\sum_{j=1}^{m} \lambda_j \alpha_n(x_{j_i}^n) \right) e_i^n,$$

where α_n and α_n^* were defined by (49) and (50) respectively.

Observe that for every $U \in \mathscr{U}$, we have

$$f(U) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} \alpha_n^* \alpha_n(x_i^n) u_i^n$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n e_i^n = f_0(U).$$

Therefore $f|_{\mathscr{U}} = f_0$.

Now assume that $\{\sigma_k\}$ be a sequence of simplices in $\mathscr{K}(\mathscr{U})$ with $n(\sigma_k) \to \infty$, such that $f(\sigma_k^0) \to x_0 \in Z$ as $k \to \infty$. We need to show that

$$f(\sigma_k) \to x_0 \quad \text{as } k \to \infty.$$

Since $x_0 \in Z$,

$$x_0 = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_i^n u_i^n.$$

Let $\sigma_k = \langle U_1^k, \dots, U_{m(k)}^k \rangle$. Then we have

$$f(U_j^k) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} x_{ji}^n(k) u_i^n, \quad \text{for } j = 1, \dots, m(k).$$

For every $x_k \in \sigma_k$,

$$x_k = \sum_{j=1}^{m(k)} \lambda_j(k) U_j^k, \quad \lambda_j(k) \geq 0, \quad j = 1, \dots, m(k) \quad \text{and} \quad \sum_{j=1}^{m(k)} \lambda_j(k) = 1,$$

we have

$$f(x_k) = \sum_{n=1}^{\infty} \sum_{i=1}^{\ell(n)} \alpha_n^* \left(\sum_{j=1}^{m(k)} \lambda_j(k) \alpha_n(x_{ji}^n(k)) \right) u_i^n. \tag{52}$$

We will show that given $\varepsilon > 0$ there exists $K \in N$ such that

$$|||f(x_k) - x_0||| < 6\varepsilon \quad \text{for any } x_k \in \sigma_k \text{ and } k > K. \tag{53}$$

Since $f_0(\sigma_k^0) \to x_0$,

$$\max\{|||f(U_j) - x_0|||, j = 1, \dots, m(k)\} \to 0 \quad \text{as } k \to \infty.$$

It follows that, see Corollary 3,

$$\max\left\{ \sum_{n=1}^{\infty} 4^{n-1} \left(\sum_{i \in I_n(k)} |x_{ji}^n(k) - x_i^n| + \sum_{i \in J_n(k)} \beta_n |x_{ji}^n(k) - x_i^n|^{p_n} \right) : \right. \tag{54}$$

$$\left. j = 1, \dots, m(k) \right\} \to 0 \quad \text{as } k \to \infty,$$

where

$$I_n(k) = \{ i : |x_{ji}^n(k) - x_i^n| \leq \beta_n^{1/(1-p_n)} \text{ for } j = 1, \dots, m(k) \}, \tag{55}$$

$$J_n(k) = \{ i : |x_{ji}^n(k) - x_i^n| > \beta_n^{1/(1-p_n)} \text{ for } j = 1, \dots, m(k) \}. \tag{56}$$

First we take $N_0 \in N$ such that

$$(57)\qquad \max\left\{ \sum_{n=1}^{\infty} 4^{n-1}\left(\sum_{i\in I_n(k)} |x_{ji}^n(k) - x_i^n| + \sum_{i\in J_n(k)} \beta_n |x_{ji}^n(k) - x_i^n|^{p_n}\right) \; j = 1,\ldots,m(k)\right\} < \varepsilon$$

for every $k > N_0$.

Observe that by Corollary 3

$$(58)\qquad |||x_0||| = \sum_{n=1}^{\infty} 4^{n-1}\left(\sum_{i\in I_n} |x_1^n| + \sum_{i\in J_n} \beta_n |x_i^n|^{p_n}\right) < \infty$$

where

$$(59)\qquad I_n = \{i : |x_i^n| \le \beta_n^{1/(1-p_n)}\} \quad \text{and} \quad J_n = \{i : |x_i^n| > \beta_n^{1/(1-p_n)}\}.$$

Take $N_1 \in N$ so that

$$(60)\qquad \sum_{n=N_1+1}^{\infty} 4^{n-1}\left(\sum_{i\in I_n} |x_i^n| + \sum_{i\in J_n} \beta_n |x_i^n|^{p_n}\right) < \varepsilon.$$

Let

$$(61)\qquad x_0(N_1) = \sum_{n=1}^{N_1}\sum_{i=1}^{\ell(n)} x_i^n u_k^n; \quad \text{and} \quad x_0(N_1,\infty) = \sum_{n=N+1}^{\infty}\sum_{i=1}^{\ell(n)} x_i^n u_i^n;$$

$$(62)\qquad x_j^k(N_1) = \sum_{n=1}^{N_1}\sum_{i=1}^{\ell(n)} x_{ji}^n(k) u_i^n; \quad \text{and} \quad x_j^k(N_1,\infty) = \sum_{n=N_1+1}^{\infty}\sum_{i=1}^{\ell(n)} x_i^n u_i^n.$$

Then we have

$$x_0(N_1) + x_0(N_1,\infty) = x_0,$$

and

$$x_j^k(N_1) + x_j^k(N_1,\infty) = f(U_j^k) \quad \text{for } j = 1,\ldots,m(k).$$

From (57) we get

$$|||x_j^k(N_1) - x_0(N_1)||| \le |||f(U_j^k) - x_0||| < \varepsilon$$

for every $j = 1, \dots, m(k)$ and $k > N_0$. Observe that

$$\begin{aligned}|||x_j^k(N_1, \infty) - x_0(N_1, \infty)||| &= |||f(U_j^k) - x_j^k(N_1) - x_0 + x_0(N_1)||| \\ &\le |||f(U_j^k) - x_0||| + |||x_j^k(N_1) - x_0(N_1)||| \\ &< \varepsilon + \varepsilon = 2\varepsilon,\end{aligned}$$

for every $j = 1, \dots, m(k)$ and $k > N_0$.

By (60), $|||x_0(N_1, \infty)||| < \varepsilon$. Therefore

$$\begin{aligned}|||x_j^k(N_1, \infty)||| &\le |||x_j^k(N_1, \infty) - x_0(N_1, \infty)||| + |||x_0(N_1, \infty)||| \\ &< 2\varepsilon + \varepsilon = 3\varepsilon\end{aligned} \tag{63}$$

for every $j = 1, \dots, m(k)$ and $k > N_0$.

We claim that

CLAIM 4. There exists an $N_2 \in N$ such that for every $j = 1, \dots, m(k)$ and $k > N_2$ we have

(i) $|x_i^n| \le \beta_n^{1(1-p_n)}$ if and only if $|x_{ji}^n(k)| \le \beta_n^{1/(1-p_n)}$;
(ii) $|x_i^n| > \beta_n^{1/(1-p_n)}$ if and only if $|x_{ji}^n(k)| > \beta_n^{1/(1-p_n)}$.

PROOF. From (58) we get

$$\lim_{n\to\infty} \sum_{i \in I_n} |x_i^n| = 0.$$

Therefore from (54) we get (i). Observe that (ii) also follows from (54) and the claim is proved.

Let

$$B_i^n(k) = \alpha_n^* \left(\sum_{j=1}^{m(k)} \lambda_j(k) \alpha_n(x_{ji}^n(k)) \right). \tag{64}$$

Then from Claim 4 we get, see (59)

$$I_n = \{i : |B_i^n(k)| \le \beta_n^{1/(1-p_n)}\} \quad \text{and} \quad J_n = \{i : |B_i^n(k)| > \beta_n^{1(1-p_n)}\} \tag{65}$$

for every $k > N_2$. Let

$$x_k(N_1) = \sum_{n=1}^{N_1} \sum_{i=1}^{\ell(n)} B_i^n(k) u_i^n; \quad x_k(N_1, \infty) = \sum_{n=N_1+1}^{\infty} \sum_{i=1}^{\ell(n)} B_i^n(k) u_i^n. \tag{66}$$

Then $x_k(N_1) + x_k(N_1, \infty) = f(x_k)$, see (52). We claim that

(67) $$|||x_k(N_1, \infty)||| < 3\varepsilon, \quad \text{for every } k > \max\{N_0, N_2\}.$$

In fact, from (62) (63) (65) we get

$$\begin{aligned}
|||x_k(N_1, \infty)||| &= \sum_{n=N_1+1}^{\infty} 4^{n-1}\left(\sum_{i\in I_n}|B_i^n(k)| + \sum_{i\in J_n}\beta_n|B_i^n(k)|^{p_n}\right)\\
&\le \sum_{n=N_1+1}^{\infty} 4^{n-1}\left(\sum_{i\in I_n}\sum_{j=1}^{m(k)}\lambda_j(k)|x_{ji}^n(k)| + \sum_{i\in J_n}\sum_{j=1}^{m(k)}\lambda_j(k)\beta_n|x_{ji}^n(k)|^{p_n}\right)\\
&= \sum_{j=1}^{m(k)}\lambda_j(k)\sum_{n=N_1+1}^{\infty} 4^{n-1}\left(\sum_{i\in I_n}|x_{ji}^n(k)| + \sum_{i\in J_n}\beta_n|x_{ji}^n(k)|^{p_n}\right)\\
&= \sum_{j=1}^{m(k)}\lambda_j(k)|||x_j^k(N_1, \infty)||| < \sum_{j=1}^{m(k)}\lambda_j(k)3\varepsilon = 3\varepsilon.
\end{aligned}$$

The claim is proved. We show

CLAIM 5. For each $n = 1, \dots, N_1$, there exists $K_n \in N$ such that, see (64)

$$|B_i^n(k) - x_i^n| < 4^{-n+1}[2^{-n}(\ell(n))^{-1}\varepsilon]^{1/p_n}$$

for every $i = 1, \dots, \ell(n)$ and $k > K_n$.

PROOF. Consider three cases:

CASE 1. $|x_i^n| < \beta_n^{1/(1-p_n)}$. Then from (54) there exists $K_1(n) \in N$ such that

$$|x_{ji}^n(k)| < \beta_n^{1/(1-p_n)} \quad \text{and} \quad |x_{ji}^n(k) - x_i^n| < 4^{-n+1}[2^{-n}(\ell(n))^{-1}\varepsilon]^{1/p_n}$$

for every $j = 1, \dots, m(k)$ and $k > K_1(n)$. Therefore from (49) we get

$$\alpha_n(x_{ji}^n(k)) = x_{ji}^n(k) \quad \text{for } j = 1, \dots, m(k) \text{ and } k > k_1(n);$$

and

$$\sum_{j=1}^{m(k)}\lambda(k)\alpha_n(x_{ji}^n(k)) < \beta_n^{1(1-p_n)} \quad \text{for } k > K_1(n).$$

Hence from (50) we have

$$\alpha_n^*\left(\sum_{j=1}^{m(k)}\lambda_j(k)\alpha_n(x_{ji}^n(k))\right) = \sum_{j=1}^{m(k)}\lambda_j(k)x_{ji}^n(k)$$

for every $k > K_1(n)$. Therefore

$$\begin{aligned} |B_i^n(k) - x_i^n| &= \left|\sum_{j=1}^{m(k)} \lambda_j(k) x_{ji}^n(k) - x_i^n\right| \\ &\le \sum_{j=1}^{m(k)} \lambda_j(k) |x_{ji}^n(k) - x_i^n| \\ &< \sum_{j=1}^{m(k)} \lambda_j(k) 4^{-n+1} [2^{-n}(\ell(n))^{-1}\beta_n \varepsilon]^{p_n} \\ &= 4^{-n+1}[2^{-n}(\ell(n))^{-1}\varepsilon]^{1/p_n} \end{aligned}$$

for every $K > K_1(n)$.

CASE 2. $|x_i^n| > \beta_n^{1/(1-p_n)}$. Them from (54) there exists $K_2(n) \in N$ such that

$$|x_{ji}^n(k)| > \beta_n^{1/(1-p_n)} \quad \text{for } j = 1, \ldots, m(k) \quad \text{and} \quad k > K_2(n).$$

Then we get

$$|\alpha_n(x_{ji}^n(k))| = |x_{ji}^n(k)|^{p_n} > \beta_n^{p_n/(1-p_n)}$$

for $j = 1, \ldots, m(k)$ and $k > K_2(n)$. Observe that x_i^n and $x_{ji}^n(k)$, $j = 1, \ldots, m(k)$, are of the same signs. Therefore

$$\left|\sum_{j=1}^{m(k)} \lambda_j(k) \alpha_n(x_{ji}^n(k))\right| > \beta_n^{p_n/(1-p_n)} \quad \text{for every } k > K_2(n).$$

Consequently

$$\left|\alpha_n^*\left(\sum_{j=1}^{m(k)} \lambda_j(k)\alpha_n(x_{ji}^n(k))\right)\right| > (\beta_n^{p_n/(1-p_n)})^{1/p_n} = \beta_n^{1/(1-p_n)} \quad \text{for every } k > K_2(n).$$

By the continuity of α_n and α_n^* there exists $\delta_i^n > 0$ such that

(67)
$$\begin{aligned} |B_i^n(k) - x_i^n| &= \left|\alpha_n^*\left(\sum_{j=1}^{m(k)} \lambda_j(k)\alpha_n(x_{ji}^n(k))\right) - x_i^n\right| \\ &< 4^{-n+1}[2^{-n}(\ell(n))^{-1}\varepsilon]^{1/p_n} \end{aligned}$$

whenever

$$\max\{|x_{ji}^n(k) - x_i^n|, j = 1, \ldots, m(k)\} < \delta_i^n.$$

Since, see (54)

$$\max\{|x_{ji}^n(k) - x_i^n| : j = 1, \dots, m(k)\} \to 0 \quad \text{as } k \to \infty,$$

there exists $K_3(n) \in N$ such that

$$\max\{|x_{ji}^n(k) - x_i^n| : j = 1, \dots, m(k)\} < \delta_i^n \quad \text{for any } k > K_3(n).$$

Consequently (67) holds true for $k > K_3(n)$.

CASE 3. $|x_i^n| = \beta_n^{1/(1-p_n)}$. We shall prove the claim for $x_i^n = \beta_n^{1/(1-p_n)}$. The case $x_i^n = -\beta_n^{1/(1-p_n)}$ is similar. From (54) we get

$$\begin{aligned}&\max\{|\alpha_n(x_{ji}^n(k)) - x_i^n| : j = 1, \dots, m(k)\} \\ &\quad = \max\{|\alpha_n(x_{ji}^n(k)) - \beta_n^{1/(1-p_n)}| : j = 1, \dots, m(k)\} \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

It follows that

$$\sum_{j=1}^{m(k)} \lambda_j(k)\alpha_n(x_{ji}^n(k)) \to \beta_n^{1/(1-p_n)} \quad \text{as } k \to \infty.$$

Therefore there exists $K_4(n) \in N$ such that

$$|B_i^n(k) - x_i^n| = \left| \alpha_n^* \left(\sum_{j=1}^{m(k)} \lambda_j(k)\alpha_n(x_{ji}^n(k)) \right) - \beta_n^{1/(1-p_n)} \right| < 4^{-n+1}[2^{-n}(\ell(n))^{-1}\varepsilon]^{1/p_n}$$

for every $k > K_4(n)$. Finally, letting

$$K_n = \max\{K_1(n), K_2(n), K_3(n), K_4(n)\}$$

we get

$$|B_i^n(k) - x_i^n| < 4^{-n+1}[2^{-n}(\ell(n))^{-1}\varepsilon]^{1/p_n}$$

for every $i = 1, \dots, \ell(n)$ and $k > K_n$. The claim is proved.

Now we are already in the position to complete the proof of Theorem 2. Let

$$K = \max\{N_0, N_2, K_1, \dots, K_{N_1}\}.$$

Then by Claim 5 we get

$$|B_i^n(k) - x_i^n| < 4^{-n+1}[2^{-n}(\ell(n))^{-1}\varepsilon]^{1/p_n} \quad \text{for every } k > K.$$

Let

$$I_n(k) = \{i : |B_i^n(k) - x_i^n| \le \beta_n^{1/(1-p_n)}\}; \quad J_n(k) = \{i : |B_i^n(k) - x_i^n| > \beta_n^{1/(1-p_n)}\}.$$

Then $\mathrm{card}(I_n(k)) \leq \ell(n)$ and $\mathrm{card}(J_n(k)) \leq \ell(n)$. Therefore, see (61) (66)

$$\begin{aligned}
|||x_k(N_1) - x_0(N_1)||| &= \sum_{n=1}^{N_1} 4^{n-1}\left(\sum_{i\in I_n} |B_i^n(k) - x_i^n| + \sum_{i\in J_n} \beta_n |B_i^n(k) - x_i^n|^{p_n}\right) \\
&\leq \sum_{n=1}^{N_1} 4^{n-1}(\mathrm{card}(I_n(k)) 4^{-n+1}[2^{-n}(\ell(n))^{-1}\varepsilon]^{1/P_n} \\
&\quad + \mathrm{card}(J_n(k))\beta_n 4^{-n+1} 2^{-n}(\ell(n))^{-1}\varepsilon) \\
&\leq \sum_{n=1}^{N_1} [\ell(n) 2^{-n}(\ell(n))^{-1}\varepsilon + \ell(n) 2^{-n}(\ell(n))^{-1}\varepsilon] \\
&= \sum_{n=1}^{N_1} 2^{-n}(2\varepsilon) < 2\varepsilon \sum_{n=1}^{\infty} 2^{-n} = 2\varepsilon,
\end{aligned}$$

for every $k > K$. Consequently from (63) (67) we get

$$\begin{aligned}
|||f(x_k) - x_0||| &= |||x_k(N_1) + x_k(N_1, \infty) - x_0(N_1) - x_0(N_1, \infty)||| \\
&\leq |||x_k(N_1) - x_0||| + |||x_k(N_1, \infty)||| + |||x_0(N_1, \infty)||| \\
&< 2\varepsilon + 3\varepsilon + \varepsilon = 6\varepsilon,
\end{aligned}$$

for every $k > K$ and $x_k \in \sigma_k$ and so (53) is proved.

Accordingly, $f(\sigma_k) \to x_0$ as $k \to \infty$. The proof of Theorem 2 is complete.

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