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EXPLICIT STRUCTURES OF THREE-DIMENSIONAL HYPERSURFACE PURELY ELLIPTIC SINGULARITIES OF TYPE (0,1)

By

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Introduction

In this paper, we give an explicit description of a certain class of singularities of algebraic varieties of dimension greater than or equal to two using toric geometry. Singularities appearing in an algebraic variety which is a closed subset in an affine space C^n for some positive integer n defined by a regular function on C^n is called hypersurface singularities, which we will investigate in the following sections. Especially, our subject is investigating so-called hypersurface *purely elliptic singularities*.

Watanabe [18] introduced the notion of purely elliptic singularities. In two-dimensional case, the notion of purely elliptic singularities is equivalent to that of cusps and simple elliptic singularities. Cusps are characterized as normal two-dimensional singularities the exceptional sets of whose minimal resolutions are circles of rational curves and appears, for example, in Hilbert modular surfaces, while simple elliptic singularities are characterized as two-dimensional normal singularities the exceptional sets of whose minimal resolutions consist of non-singular elliptic curves. These two-dimensional purely elliptic singularities are much investigated by many researchers.

We already know due to Ishii, Watanabe and other researchers that in three-dimensional Gorenstein purely elliptic singularities, some analogies of two-dimensional cases hold. For example, Ishii-Watanabe [9] defined a simple $K3$ singularity to be a normal Gorenstein isolated singularity of which the exceptional set of Q-factorial terminal modification consists of a normal $K3$ surface, of course which is an analogy of simple elliptic singularities in two-dimensional cases. And simple $K3$ singularities are three-dimensional purely elliptic singularities.

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Furthermore, Ishii [6] classified n-dimensional purely elliptic singularities into n classes, from type $(0,0)$ to type $(0,n-1)$, by means of the mixed Hodge structure of the cohomology of the exceptional set of each singularity. For example, in two-dimensional cases, purely elliptic singularities are classified into two classes, one of which is said to be of type $(0,0)$ and the other of type $(0,1)$. The former corresponds to the class of cusps and the latter corresponds to the class of simple elliptic singularities. In three-dimensional cases, Ishii [6] unveiled structures of the *essential divisors* of good resolutions of purely elliptic singularities. If a singularity is Gorenstein and of type $(0,2)$, it is a simple $K3$ singularity and its essential divisor consists of a $K3$ surface. If a singularity is Gorenstein and of type $(0,0)$, it is a singularity whose essential divisor consists of rational surfaces and, roughly speaking, forms a sphere. And if a singularity is Gorenstein and of type $(0,1)$, the essential divisor forms a chain of surfaces and the intersection of any pair of surfaces adjacent to each other is an elliptic curve. The last class is the one having no analogue in two-dimensional cases. Here, we note that elliptic curves and $K3$ surfaces are so-called Calabi-Yau varieties of dimension one and of dimension two, respectively.

Hypersurface singularities are Gorenstein singularities. Watanabe [18] found conditions whether a singularity is a purely elliptic singularity or not by means of a character of a diagram, called the Newton diagram, associated with its defining equation. We start to investigate hypersurface purely elliptic singularities with this criterion. Here we note that we restrict ourselves to *nondegenerate* hypersurface isolated singularities, see § 1.2 for the definition of nondegeneracy, because Watanabe's criterion or toric method does not work well without the nondegenerate condition.

Let us give the outline of this paper.

§ 1 is devoted to the review of resolution of singularities of nondegenerate hypersurface isolated singularities using toric geometry, and to the review of the definition and some properties of (hypersurface) purely elliptic singularities. Especially, we recall Watanabe's criterion. This indicates a special face of the Newton boundary of the defining polynomial of a purely elliptic singularity, which we call the *fundamental face*, and the sum of the terms of the defining polynomial on it, which we call the *fundamental part* of the defining polynomial. In the following sections, we will see roles of the fundamental parts in the structures of the essential divisors and their characters as algebraic varieties.

In § 2, we try to describe the essential divisor of a resolution of an r-dimensional nondegenerate hypersurface purely elliptic singularity by means of a stratified diagram in the $(r+1)$-dimensional Euclidean space $\boldsymbol{R}^{r+1}$ associated with the singularity, which we call the *dual essential diagram* of the singularity, and of simplicial complexes, which gives simplicial subdivision of the dual essential diagram. This attempt succeeds in case the dimension of the fundamental face is greater than or equal to two.

§3 is devoted to study the varieties associated with the fundamental parts in case the dimensions of the fundamental faces are greater than or equal to two. Of course, these varieties have direct relations with the structure of the essential divisor.

Hypersurface simple elliptic singularities are deeply related to elliptic curves in two-dimensional weighted projective spaces as well as hypersurface simple $K3$ singularities are related to $K3$ surfaces in three-dimensional weighted projective spaces. Hypersurfaces which are Calabi-Yau varieties in weighted projective spaces, more generally in toric varieties, are spotlighted by many physicists and mathematicians as candidates of examples of the mirror symmetry phenomena after Batyrev's article [1]. Before those, in the study of fundamental faces of hypersurface simple $K3$ singularities, Yonemura [19] listed up all possible weights of which weighted projective spaces contain normal $K3$ surfaces as anticanonical divisors. Here, we note that Fletcher [2] independently obtain the same result by a different approach.

Then do the fundamental parts, or fundamental faces of hypersurface purely elliptic singularities have some relations to Calabi-Yau varieties? For this question, we give partial answers: (1) smooth models of the variety associated with the fundamental part has the geometric genus one in general; (2) under a special condition, it is birational to a Calabi-Yau variety.

In §4, we apply the results obtained in the former parts of this paper to nondegenerate three-dimensional purely elliptic singularities of type $(0,1)$, which are the simplest non-semi-quasi-homogeneous cases and have relations to Calabi-Yau varieties, in fact, elliptic curves. Here we note that Fujisawa [3] generalized the notion of the weight system of a semi-quasi-homogeneous singularity to the cases of non-semi-quasi-homogeneous purely elliptic singularities and, from this point of view, gave a classification of three-dimensional hypersurface purely elliptic singularities of type $(0,1)$.

Although, as we mentioned above, many of the results contained in this paper are already known in more general and abstract contexts, we believe in the significance of reviewing from the point of view of the geometry of toric hypersurfaces because hypersurface cases are good examples for general theories and there are many theories on toric hypersurfaces available to further investigations of hypersurface singularities.

Finally, we notes that this paper consists of results in the doctor's thesis of the author submitted at the university of Tsukuba [10].

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1. Preliminaries

1.1. Notations and Terminologies in Toric Geometry

For guidance on toric geometry, we refer the reader to Oda [14] and Fulton [4].

Let N be a free module over the ring of the rational integers $\boldsymbol{Z}$ of finite rank n and $M = \mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ be the dual $\boldsymbol{Z}$-module. Denote by $N_{\boldsymbol{R}}$ (resp. $M_{\boldsymbol{R}}$) the scalar extension of N (resp. M) by the field of the real numbers $\boldsymbol{R}$. Let $\langle *, * \rangle :$ $M \times N \to \boldsymbol{Z}$ be the canonical bilinear form. We use the same notation $\langle *, * \rangle$ for the natural scalar extension of the bilinear form on $M_{\boldsymbol{R}} \times N_{\boldsymbol{R}}$.

A strongly convex rational polyhedral cone, or cone for short, σ in $M_{\boldsymbol{R}}$, denote by $\boldsymbol{C}[\check{\sigma} \cap M]$ the semigroup algebra $\bigoplus_{\mathbf{m} \in \check{\sigma} \cap M} \boldsymbol{C} \cdot \chi^{\mathbf{m}}$. For a fan Σ in N, we denote by V_Σ the toric variety associated with Σ.

A cone σ is said to be *nonsingular* if it is generated by a part of a basis of N. If a cone $\sigma \in \Sigma$ is nonsingular, then the corresponding affine toric variety $\mathrm{Spec}\, \boldsymbol{C}[\check{\sigma} \cap M]$ is a nonsingular variety. Moreover, if every cone in a fan Σ is nonsingular, we say that Σ is *nonsingular*. In this case, the corresponding toric variety V_Σ is a nonsingular variety.

We say that a fan Σ is *complete* if $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma = N_{\boldsymbol{R}}$. When Σ is complete, the corresponding toric varieties V_Σ is a complete variety.

A toric variety V_Σ contains an algebraic torus $T_N := \mathrm{Spec}\, \boldsymbol{C}[M]$ as an open dense subset. This T_N acts on V_Σ which is compatible with the multiplication of T_N as a group variety. We have a natural one-to-one correspondence between the orbits of V_Σ by the action of T_N and the cones in Σ. Denote by $\mathrm{orb}(\sigma)$ the orbit corresponding to a cone σ in Σ. The closure $\overline{\mathrm{orb}(\sigma)}$, denoted by $V(\sigma)$, of $\mathrm{orb}(\sigma)$ in V_Σ is a T_N-invariant closed subset of V_Σ. In particular, if a cone $\rho \in \Sigma$ is of dimension one, the corresponding (T_N-)invariant closed subset $V(\rho)$ is a Weil divisor and is denoted by D_ρ. For a cone σ (resp. a fan Σ), $\sigma(1)$ (resp. $\Sigma(1)$) denotes the set of one-dimensional faces of σ (resp. one-dimensional cones of Σ). For each $\rho \in \sigma(1)$ (or $\Sigma(1)$), $\mathbf{n}(\rho)$ denotes the primitive integral generator of ρ.

Finally, we note that $\boldsymbol{Z}_{\geq 0}$, $\boldsymbol{Q}_{\geq 0}$, $\boldsymbol{R}_{\geq 0}$ denotes the set of non-negative integers, the set of non-negative rational numbers, the set of non-negative real numbers, respectively.

1.2. Resolution of Singularities by Means of Toric Geometry

Let

$$f = f(z_0, z_1, \dots, z_r) = \sum_{m_0, m_1, \dots, m_r \in \boldsymbol{Z}_{\geq 0}} a_{m_0, m_1, \dots, m_r} \cdot z_0^{m_0} z_1^{m_1} \cdots z_r^{m_r}$$

be a polynomial over the complex number field $\boldsymbol{C}$ in variables $z_0, z_1, \dots, z_r$ such that $f(\mathbf{0}) = 0$ and that the hypersurface $X = \mathrm{V}(f) \subset \boldsymbol{C}^{r+1}$ defined by f has an isolated singular point at the origin $O = \mathbf{0} = (0, 0, \dots, 0)$ of the $(r+1)$-dimensional affine space $\boldsymbol{C}^{r+1}$.

Let $N \cong \boldsymbol{Z}^{r+1}$. Put $\mathbf{e}_i := (0, \dots, 0, 1, 0, \dots, 0)$ (the entries equal to 0 other than the i-th entry which equals to 1) for $i = 0, 1, \dots, r$ under the identification of N with $\boldsymbol{Z}^{r+1}$. Then $\{\mathbf{e}_i\}_{i=0}^r$ forms a basis of N. Let Σ be a fan consisting of the faces of the cone $\sum_{i=0}^r \boldsymbol{R}_{\geq 0}\mathbf{e}_i$. Then the corresponding toric variety V_Σ is the $(r+1)$-dimensional affine space $\boldsymbol{C}^{r+1}$ and the affine coordinate ring is

$$\boldsymbol{C}\left[\left(\sum_{i=0}^r \boldsymbol{R}_{\geq 0} \cdot \mathbf{e}_i\right)^\vee \cap M\right] = \bigoplus_{\mathbf{m} \in M \cap (\boldsymbol{R}_{\geq 0})^{r+1}} \boldsymbol{C} \cdot \chi^{\mathbf{m}} \cong \boldsymbol{C}[z_0, z_1, \dots, z_r],$$

where $M = \mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$. For every $\mathbf{m} = (m_0, m_1, \dots, m_r) \in (\boldsymbol{Z}_{\geq 0})^{r+1}$, denote $z_0^{m_0} z_1^{m_1} \cdots z_r^{m_r}$ (resp. $a_{m_0, m_1, \dots, m_r}$) by $\mathbf{z}^{\mathbf{m}}$ (resp. $a_{\mathbf{m}}$). Then the isomorphism just above is given by

$$\bigoplus_{\mathbf{m} \in M \cap (\boldsymbol{R}_{\geq 0})^{r+1}} \boldsymbol{C} \cdot \chi^{\mathbf{m}} \xrightarrow{\sim} \boldsymbol{C}[z_0, z_1, \dots, z_r], \quad a_{\mathbf{m}} \cdot \chi^{\mathbf{m}} \mapsto a_{\mathbf{m}} \cdot \mathbf{z}^{\mathbf{m}}.$$

We define $\Gamma_+(f)$ to be the convex hull of the union of the subset $\mathbf{m} + (\boldsymbol{R}_{\geq 0})^{r+1}$ of $\boldsymbol{R}^{r+1}$ for $\mathbf{m}$ such that $a_{\mathbf{m}} \neq 0$ and call it the *Newton diagram* of f. The *Newton boundary* $\Gamma(f)$ of f is the union of the compact faces of

$\Gamma_+(f)$. We associate a polynomial $f_\gamma(\mathbf{z}) = \sum_{\mathbf{m}\in\gamma\cap M} a_{\mathbf{m}} \cdot \mathbf{z}^{\mathbf{m}}$ with each face γ of $\Gamma(f)$. We say that f is *nondegenerate on* γ if $\partial f_\gamma/\partial z_0 = \cdots = \partial f_\gamma/\partial z_r = 0$ has no solution in $(\boldsymbol{C}^*)^{r+1}$. We say that f is *nondegenerate* if f_γ is nondegenerate on any face γ of $\Gamma(f)$. We also say that a hypersurface singularity is *nondegenerate* if its defining polynomial is nondegenerate. In the following of this paper, we always assume that the defining polynomials of singularities are nondegenerate.

Since $M_{\boldsymbol{R}}$ and $N_{\boldsymbol{R}}$ are dual to each other as vector spaces over $\boldsymbol{R}$, an element $\mathbf{n} = (n_0, n_1, \dots, n_r) \in N_{\boldsymbol{R}}$ gives rise to a family of hyperplanes in $M_{\boldsymbol{R}} \cong \boldsymbol{R}^{r+1}$ with the common ratio $(n_0, n_1, \dots, n_r)$. For $\mathbf{n} \in N \cap (\boldsymbol{R}_{\geq 0})^{r+1}$, define

$$l(\mathbf{n}) := \min\{\langle \mathbf{m}, \mathbf{n}\rangle \mid \mathbf{m} \in \Gamma_+(f)\}.$$

For a face γ of $\Gamma_+(f)$, define

$$\gamma^* := \{\mathbf{n} \in (\boldsymbol{R}_{\geq 0})^{r+1} \subset N_{\boldsymbol{R}} \mid \langle \mathbf{n}, \mathbf{m}\rangle = l(\mathbf{n}) \text{ for any } \mathbf{m} \in \gamma\}.$$

Then γ^* is a cone in $N_{\boldsymbol{R}}$. The set of cones

$$\Sigma(f) := \{\gamma^* \mid \gamma \prec \Gamma_+(f)\}$$

forms a fan, which we call the *dual fan* of $\Gamma_+(f)$. Note that $\Sigma(f)$ is a subdivision of Σ, that is, $|\Sigma(f)| = |\Sigma|$ and for every cone $\sigma' \in \Sigma(f)$, there exists a cone $\sigma \in \Sigma$ such that $\sigma' \subset \sigma$.

Take a nonsingular subdivision $\hat{\Sigma}(f)$ of $\Sigma(f)$, which is a finite subdivision of $\Sigma(f)$ consisting of nonsingular cones. There exists at least one nonsingular subdivision $\hat{\Sigma}(f)$ of $\Sigma(f)$—see Kempf, et al [11]. Then since $\Sigma(f)$ is a subdivision of Σ, $\hat{\Sigma}(f)$ is also a subdivision of Σ. Hence we have a map of fans

$$\varphi : (N, \hat{\Sigma}(f)) \to (N, \Sigma(f)) \to (N, \Sigma).$$

Let $\Pi : V_{\hat{\Sigma}(f)} \to V_\Sigma \cong \boldsymbol{C}^{r+1}$ be the equivariant morphism associated with φ. This is a proper, birational morphism since $\hat{\Sigma}(f)$ is a finite subdivision of Σ, and sometimes called the *equivariant blow-up associated with the subdivision* $\hat{\Sigma}(f)$ *of* Σ. The following is a well-known fact:

PROPOSITION 1.1 (Varchenko [16]). *If (X, O) is nondegenerate, the restriction $\pi : \tilde{X} \to X$ of Π on $\tilde{X}$ is a proper, birational morphism and a good resolution of the singularity (X, O).*

The fiber $E := \pi^{-1}(O) = \tilde{X} \cap \Pi^{-1}(O)$ of the origin $O \in X$ is called the *exceptional set* of $\tilde{X}$.

1.3. The Laurent Polynomial Associated with the Pair (f_γ, γ)

In § 1.2, we defined a polynomial $f_\gamma = \sum_{\mathbf{m} \in (\boldsymbol{Z}_{\geq 0})^{r+1}} a_\mathbf{m} \cdot \mathbf{z}^\mathbf{m}$ for each face γ of the Newton boundary $\Gamma(f)$ to define the notion of nondegeneracy of the defining polynomial f of hypersurface isolated singularity $(X, x) = (\mathbf{V}(f), \mathbf{0})$.

To describe the exceptional sets, it is useful to define a slightly different pair (f_γ^L, M_γ) from the pair (f_γ, γ) as follows:

Definition 1.2. For a face γ of $\Gamma(f)$, define M_γ to be the free $\boldsymbol{Z}$-module generated by the vectors $\gamma \cap M - \mathbf{m}_0$ and define the Laurent polynomial f_γ^L, that is an element of $\boldsymbol{C}[M_\gamma] = \bigoplus_{\mathbf{m} \in M_\gamma} \boldsymbol{C} \cdot \chi^\mathbf{m}$, to be

$$\sum_{\mathbf{m} \in \gamma \cap M} a_\mathbf{m} \cdot \chi^{\mathbf{m} - \mathbf{m}_0},$$

where $\mathbf{m}_0$ is an element of $\gamma \cap M$.

Next, we recall a way to construct a complete hypersurface in a complete toric variety from a Laurent polynomial canonically.

Let M be a lattice of rank n and let $M_\boldsymbol{R}$ be its scalar extension $M \otimes_\boldsymbol{Z} \boldsymbol{R}$ by the real number field $\boldsymbol{R}$. Let Δ be an n-dimensional integral polyhedron in $M_\boldsymbol{R}$. We associate a complete fan $\Sigma(\Delta)$ in $N := \mathrm{Hom}_\boldsymbol{Z}(M, \boldsymbol{Z})$ and a toric variety $\boldsymbol{P}_\Delta := V_{\Sigma(\Delta)}$ with Δ as follows:

For every l-dimensional face $\Theta \subset \Delta$, we define the convex n-dimensional cone $\check{\sigma}(\Theta) \subset M_\boldsymbol{R}$ consisting all vectors $\lambda(\mathbf{p} - \mathbf{p}')$, where $\lambda \in \boldsymbol{R}_{\geq 0}$, $\mathbf{p} \in \Delta$, $\mathbf{p}' \in \Theta$. Let $\sigma(\Theta) \subset N_\boldsymbol{R} = \mathrm{Hom}_\boldsymbol{Z}(M, \boldsymbol{Z}) \otimes_\boldsymbol{Z} \boldsymbol{R}$ be the $(n - l)$-dimensional cone dual to $\check{\sigma}(\Theta)$. Then, the set $\Sigma(\Delta)$ of all cones $\sigma(\Theta)$, where Θ runs over all faces of Δ, forms a complete fan. We represent $\boldsymbol{P}_\Delta$ the toric variety $V_{\Sigma(\Delta)}$ associated with $\Sigma(\Delta)$ (see Batyrev [1], Proposition 2.1.1).

Let $f = \sum_{\mathbf{m} \in M} c_\mathbf{m} \cdot \chi^\mathbf{m}$ be a Laurent polynomial, i.e., an element of $\boldsymbol{C}[M] = \bigoplus_{\mathbf{m} \in M} \boldsymbol{C} \cdot \chi^\mathbf{m}$. This f defines a hypersurface $\mathbf{V}(f)$ in $T_N = \mathrm{Spec}\, \boldsymbol{C}[M]$, denoted by $Z(f, M)$.

On the other hand, if the Newton polyhedron $\Delta = \Delta(f)$ of f, which is the convex hull of the set of points $\mathbf{m}$ with $c_\mathbf{m} \neq 0$ in $M_\boldsymbol{R}$, is of dimension n, we have a complete toric variety $\boldsymbol{P}_\Delta$ which contains T_N as an open dense subset. Here, denote by $\bar{Z}(f, M)$ the closure of $Z(f, M)$ in $\boldsymbol{P}_\Delta$.

As a result of the above discussion, to the pair (f_γ^L, M_γ), we can attach a toric hypersurface $Z(f_\gamma^L, M_\gamma)$ in $\mathrm{Spec}\, \boldsymbol{C}[M_\gamma]$ and a complete toric hypersurface $\bar{Z}(f_\gamma^L, M_\gamma)$ in $\boldsymbol{P}_{\gamma - \mathbf{m}_0}$, where the Newton polyhedron $\Delta(f_\gamma^L)$ is just $\gamma - \mathbf{m}_0$ in M_γ.

1.4. Purely Elliptic Singularities and the Essential Divisors

Here we recall the definition of purely elliptic singularities and that of the essential divisors of good resolutions of them.

DEFINITION 1.3 (Watanabe [17]). Let (X, x) be a normal isolated Gorenstein singularity of dimension $r \geq 2$. Let $\pi : \tilde{X} \to X$ be a good resolution of (X, x). Denote the reduced exceptional divisor $\pi^{-1}(x)_{\mathrm{red}}$ by E. Then, we define

$$\begin{aligned}\delta_m(X, x) &:= \dim_C \Gamma(X \backslash \{x\}, \mathcal{O}(mK))/L^{2/m}(X \backslash \{x\}) \\ &= \dim_C \Gamma(\tilde{X} \backslash E, \mathcal{O}(mK))/\Gamma(\tilde{X}, \mathcal{O}(mK + (m-1)E))\end{aligned}$$

for each $m \in N$.

DEFINITION 1.4 (Watanabe [18]). A normal isolated Gorenstein singularity is called a *purely elliptic singularity* if $\delta_m(X, x) = 1$ for all $n \in N$.

DEFINITION 1.5 (Ishii [6]). For a good resolution $\pi : \tilde{X} \to X$ of a normal isolated Gorenstein singularity (X, x), we can write

$$K_{\tilde{X}} = \pi^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_j,$$

where $m_i \geq 0$ for $i \in I$, $m_j > 0$ for $j \in J$ and E_i $(i \in I)$, E_j $(j \in J)$ are irreducible components of $E = \pi^{-1}(x)_{\mathrm{red}}$.

The divisor $E_J := \sum_{j \in J} m_j E_j$ is called the *essential divisor*.

PROPOSITION 1.6 (Ishii [6]). *Under the assumption of Definition* 1.5, (X, x) *is a purely elliptic singularity if and only if* E_J *is a reduced divisor.*

1.5. Hypersurface Purely Elliptic Singularities; Watanabe's Criterion

Let $(X, O) = (\mathrm{V}(f), \mathbf{0})$ be an isolated singularity of dimension $r \geq 2$ defined by a nondegenerate polynomial $f = \sum_{\mathbf{m} \in (\mathbf{Z}_{\geq 0})^{r+1}} a_{\mathbf{m}} \cdot \mathbf{z}^{\mathbf{m}} \in C[z_0, z_1, \dots, z_r]$.

Let $\Gamma_+(f)$ be the Newton diagram of f, let $\Gamma(f)$ be the Newton boundary, let $\Gamma_+^*(f)$ be the dual decomposition of the positive quadrant $(\boldsymbol{R}_{\geq 0})^{r+1}$ and let $\Sigma(f)$ be the dual fan. Take a nonsingular subdivision $\hat{\Sigma}(f)$ of $\Sigma(f)$. Let $\Pi : V_{\hat{\Sigma}(f)} \to V_\Sigma \cong C^{r+1}$ be the equivariant blow-up associated with $\hat{\Sigma}(f)$, let $\tilde{X}$ be the proper transform of X with respect to Π and let $E = \Pi^{-1}(O)_{\mathrm{red}}$.

PROPOSITION 1.7 (Ishii [7]). *Under the assumption above,*

$$K_{\tilde{X}} = \pi^*(K_X) + \sum_{\hat{\rho} \in \hat{\Sigma}(f)(1) \setminus \Sigma(1)} (\langle \mathbf{1}, \mathbf{n}(\hat{\rho}) \rangle - 1 - l(\mathbf{n}(\hat{\rho})))D_{\hat{\rho}}|_{\tilde{X}},$$

where $\mathbf{1} = (1, 1, \dots, 1) \in M$, $l(\mathbf{n}(\hat{\rho})) = \min\{\langle \mathbf{m}, \mathbf{n}(\hat{\rho}) \rangle \mid \mathbf{m} \in \Gamma_+(f)\}$ *and* $D_{\hat{\rho}}$ *is the invariant divisor on* $V_{\hat{\Sigma}(f)}$ *associated with* $\hat{\rho}$.

We have the following criterion whether $(X, O) = (\mathbf{V}(f), \mathbf{0})$ is a purely elliptic singularity.

Proposition 1.8 (Watanabe [18]). *Under the condition that* $(X, O) = (\mathbf{V}(f), \mathbf{0})$ *is nondegenerate,* (X, O) *is a purely elliptic singularity if and only if* $\mathbf{1} \in \Gamma(f)$.

By this proposition, there exists a unique face of $\Gamma(f)$ containing $\mathbf{1}$ in its relative interior if $(X, O) = (\mathbf{V}(f), \mathbf{0})$ is a purely elliptic singularity.

Definition 1.9. Let $\Gamma(f)$ to be the Newton boundary of a hypersurface purely elliptic singularity $(X, O) = (\mathbf{V}(f), \mathbf{0})$. We call the face of $\Gamma(f)$ containing $\mathbf{1} \in M$ in its relative interior the *fundamental face* of $\Gamma(f)$ and express it as $\gamma_1(f)$, or simply γ_1. And then, we call the polynomial $f_{\gamma_1(f)} = \sum_{\mathbf{m} \in \gamma_1(f) \cap M} a_{\mathbf{m}} \cdot \mathbf{z}^{\mathbf{m}}$ the *fundamental part* of the defining polynomial f.

2. Description of the Essential Divisors of Hypersurface Purely Elliptic Singularities

2.1. The Essential Cone

At the beginning of this section, we introduce a notion due to Ishii which is useful to study hypersurface singularities.

Let $(X, O) = (\mathbf{V}(f), \mathbf{0})$ be a hypersurface isolated singularity defined by a nondegenerate polynomial f and $\hat{\Sigma}(f)$ be a nonsingular subdivision for the dual fan $\Sigma(f)$.

Definition 2.1 (Ishii [7]). For a polynomial $f \in \boldsymbol{C}[z_0, z_1, \dots, z_r]$, define

$$C_1(f) := \{\mathbf{n} \in (\boldsymbol{R}_{\geq 0})^{r+1} \subset N_{\boldsymbol{R}} \mid l(\mathbf{n}) - \langle \mathbf{1}, \mathbf{n} \rangle \geq 0\}.$$

$C_1(f)$ is called the *essential cone*.

Remark 2.2. We note some properties of the essential cone $C_1(f)$. If the Newton diagram $\Gamma_+(f)$ contains $\mathbf{1} \in M$ in its interior, $C_1(f) = \{O\}$. If $\Gamma_+(f)$

does not contain $\mathbf{1}$ in its interior, that is, the boundary of $\Gamma_+(f)$ or $\Gamma_+(f)$ itself does not contain $\mathbf{1}$, then $C_1(f)$ is the cone spanned by the one-dimensional cones $\gamma_1^*, \gamma_2^*, \dots, \gamma_s^*$, which are dual to r-dimensional faces of $\Gamma_+(z_0z_1\cdots z_r+f)$ containing $\mathbf{1}$. See [7], Remark 2.3.

The relation between the essential cone $C_1(f)$ and the essential divisor E_J of a toric resolution of a hypersurface isolated singularity is given by the following proposition, which follows directly Proposition 1.7:

PROPOSITION 2.3. *Let E_J be the essential divisor of the resolution of the singularity (X,O) given by a nonsingular subdivision $\hat{\Sigma}(f)$. Then*

$$E_J = \sum_{\hat{\rho}\in\hat{\Sigma}(f)(1)\setminus\Sigma(1),\ \subset C_1(f)} D_{\hat{\rho}}|_{\tilde{X}}$$

holds, where the sum runs over all the one-dimensional cones of $\hat{\Sigma}(f)$ contained by $C_1(f)$, but not contained by Σ.

When (X,O) is a hypersurface purely elliptic singularity, that is, $\Gamma_+(f)$ contains $\mathbf{1}$ in its boundary, the essential cone $C_1(f)$ has a nice relationship with the diagram $C_1^*(f) := \bigcup_{\mathbf{n}\in C_1(f)} \gamma(\mathbf{n})$, where $\gamma(\mathbf{n}) = \{\mathbf{m}\in\Gamma_+(f) \mid \langle\mathbf{m},\mathbf{n}\rangle = l(\mathbf{n})\}$. Including non-purely-elliptic cases, the sum of the monomials whose indices lie in $C_1^*(f)$ with the same coefficients as in the defining polynomial f only affect the algebraic-geometric structure of the support of the essential divisor. In case (X,O) is a purely elliptic singularity, $C_1(f)$ and $C_1^*(f)$ enjoy "duality". We state it here more precisely:

LEMMA 2.4. *If (X,O) is a nondegenerate purely elliptic singularity, $C_1(f)$ is a cone in the dual fan $\Sigma(f)$.*

PROOF. If (X,O) is a nondegenerate purely elliptic singularity, $\mathbf{1}\in M$ is on the boundary of $\Gamma_+(f)$ by Watanabe's criterion: Proposition 1.8. Then the essential cone $C_1(f)$ is just $\{\mathbf{n}\in(\boldsymbol{R}_{\geq 0})^{r+1}\subset N_{\boldsymbol{R}} \mid l(\mathbf{n}) = \langle\mathbf{1},\mathbf{n}\rangle\}$. Define

$$\gamma(C_1(f)) := \bigcap_{\mathbf{n}\in C_1(f)} \gamma(\mathbf{n}) = \bigcap_{\mathbf{n}\in C_1(f)} \{\mathbf{m}\in\Gamma_+(f) \mid \langle\mathbf{m},\mathbf{n}\rangle = l(\mathbf{n})\}.$$

Then $\gamma(C_1(f))$ is a face of $\Gamma_+(f)$ and $C_1(f)\subset\gamma(C_1(f))^*$ holds. Since $l(\mathbf{n}) = \langle\mathbf{1},\mathbf{n}\rangle$ for each $\mathbf{n}\in C_1(f)$ and $\mathbf{1}\in\Gamma_+(f)$, $\gamma(C_1(f))$ contains $\mathbf{1}$. Because $\gamma_1(f)$ is the minimum face of $\Gamma_+(f)$ containing $\mathbf{1}\in M$, $\gamma_1(f)\subset\gamma(C_1(f))$. Therefore, $C_1(f)\subset\gamma_1(f)^*$.

Next, note that, by definition,

$$\gamma_1(f)^* = \{\mathbf{n} \in (\mathbf{R}_{\geq 0})^{r+1} \subset N_{\mathbf{R}} \mid \langle \mathbf{m}, \mathbf{n} \rangle = l(\mathbf{n}) \text{ for any } \mathbf{m} \in \gamma_1(f)\}.$$

Since $\mathbf{1} \in \gamma_1(f)$, $\langle \mathbf{1}, \mathbf{n} \rangle = l(\mathbf{n})$ holds for any $\mathbf{n} \in \gamma_1(f)^*$. Hence $\gamma_1(f)^* \subset C_1(f)$.

Thus, $C_1(f) = \gamma_1(f)^*$. □

The next corollary follows the lemma just above and the relationship between the dual fan and the Newton diagram:

COROLLARY 2.5. *There is a natural one-to-one, order reversing, dual correspondence between the non-zero faces of $C_1(f)$ and the faces of $C_1^*(f)$ containing of $\mathbf{1}$ as follows: A face σ of $C_1(f)$ corresponds to the face $\gamma(\sigma) := \bigcap_{\mathbf{m} \in \sigma} \gamma(\mathbf{n})$. Conversely, a face γ of $C_1^*(f)$ containing $\mathbf{1}$ corresponds to the face $\sigma(\gamma) := \gamma^*$, where* $\dim \sigma + \dim \gamma(\sigma) = \dim \gamma + \dim \sigma(\gamma) = r + 1$ *hold.*

In particular, the essential cone $C_1(f)$ itself corresponds to the fundamental face $\gamma_1(f)$.

Consequently, for a nondegenerate hypersurface purely elliptic singularity (X, O), every nonsingular subdivision $\hat{\Sigma}(f)$ of the dual fan $\Sigma(f)$ gives a nonsingular subdivision of $C_1(f)$:

DEFINITION 2.6. Let $(X, O) = (\mathbf{V}(f), \mathbf{0})$ be a nondegenerate purely elliptic singularity. For a nonsingular subdivision $\hat{\Sigma}(f)$ of the dual fan $\Sigma(f)$, define the fan

$$\hat{C}_1(f) := \{\hat{\sigma} \in \hat{\Sigma}(f) \mid \hat{\sigma} \subset C_1(f)\}.$$

In the following part of this section, we will see that the "duality" between $C_1(f)$ and $C_1^*(f)$ simplifies algebraic-geometric description of the essential divisors of purely elliptic singularities (Theorem 2.10). But, we will restrict ourselves to nondegenerate hypersurface purely elliptic singularities of type $(0, i)$ $(i \geq 1)$. We need nondegeneracy of singularities because toric method does not work well without this assumption as we saw before. We also need the assumption that the dimension of the fundamental face $\gamma_1(f)$ is greater than or equal to two to keep the direct relation of the one-dimensional cones of $\hat{C}_1(f)$ and the irreducible components of E_J (Proposition 2.7, Theorem 2.14).

2.2. The Stratification on the Essential Divisor and the Dual Essential Diagram

Each toric variety is stratified by its orbits by the action of algebraic torus, so that every closed subset of a toric variety is also stratified. Let $(X, O) = (\mathbf{V}(f), \mathbf{0})$

be a hypersurface isolated singularity and $\hat{\Sigma}(f)$ be a nonsingular subdivision of $\Sigma(f)$. Then, we can define a natural stratification on the exceptional set E or the essential divisor E_J using the stratification of the toric variety $V_{\hat{\Sigma}(f)}$.

To every cone $\hat{\sigma}$ in $\hat{\Sigma}(f)$, we can attach an orbit $\mathrm{orb}(\hat{\sigma})$ of $V_{\hat{\Sigma}(f)}$, which is the orbit of the smallest dimension in the affine open subset $\operatorname{Spec} \boldsymbol{C}[\hat{\sigma}^{\vee} \cap M]$. Define $\overset{\circ}{E}(\hat{\sigma}) := E \cap \mathrm{orb}(\hat{\sigma})$. Recall that $V(\hat{\sigma})$ denotes the closure of $\mathrm{orb}(\hat{\sigma})$ in $V_{\hat{\Sigma}(f)}$ and put $E(\hat{\sigma}) := E \cap V(\hat{\sigma})$. Note that $E(\hat{\sigma})$ is the closure of $\overset{\circ}{E}(\hat{\sigma})$ in E. A stratum $\overset{\circ}{E}(\hat{\sigma})$ may be the empty set or may not be connected in general.

Let (X, O) be a nondegenerate hypersurface purely elliptic singularity. The disjoint union of orbits for all the cones $\hat{\sigma}$ in $\hat{C}_1(f)$ covers almost all E_J. Indeed, define

$$E_J' := \coprod_{\hat{\sigma} \in \hat{C}_1(f) \setminus \{O\}} \overset{\circ}{E}(\hat{\sigma}),$$

where O is the origin of N. Then, the closure of E_J' is just E_J and we have

$$E_J = \bigcup_{\hat{\sigma} \in \hat{C}_1(f) \setminus \{O\}} E(\hat{\sigma}).$$

Therefore, it is sufficient for us to investigate the strata and their closures for the cones in $\hat{C}_1(f)$. For a cone $\hat{\sigma}$ in $\hat{C}_1(f)$, we will use the symbol $\overset{\circ}{E}_J(\hat{\sigma})$ (resp. $E_J(\hat{\sigma})$) for $\overset{\circ}{E}(\hat{\sigma})$ (resp. $E(\hat{\sigma})$).

For a cone $\hat{\sigma} \in \hat{\Sigma}(f)$, we also define $\gamma(\hat{\sigma}) := \bigcap_{\mathbf{n} \in \hat{\sigma}} \gamma(\mathbf{n})$, which is a face of $\Gamma_+(f)$. Let $\hat{\sigma}$ be a cone in $\hat{C}_1(f)$. Then the face $\gamma(\hat{\sigma})$ contains the fundamental face $\gamma_1(f)$. Hence, $\dim \gamma(\hat{\sigma}) \geq \dim \gamma_1(f)$ holds by Corollary 2.5. Then by Oka [15], Lemma 4.7, we have the following in case $\dim \gamma_1(f) \geq 2$:

PROPOSITION 2.7. *If (X, O) is nondegenerate and the dimension of the fundamental face $\gamma_1(f)$ is greater than or equals to two, then for any cone $\hat{\sigma}$ in $\hat{C}_1(f)$, the closure $E_J(\hat{\sigma})$ of the stratum $\overset{\circ}{E}_J(\hat{\sigma})$ is an irreducible nonsingular variety of complex dimension $r - \dim \hat{\sigma}$. Especially, it is non-empty.*

In particular, for a one-dimensional cone $\hat{\rho}$ in $\hat{C}_1(f)$, the corresponding divisor $D_{\hat{\rho}}|_{\tilde{X}}$ is irreducible.

Now, we introduce a diagram to describe the essential divisor E_J of a resolution of a hypersurface purely elliptic singularity $(X, O) = (\mathbf{V}(f), \mathbf{0})$ defined by a nondegenerate polynomial f.

DEFINITION 2.8. Let $C_1(f)$ be the essential cone. For $\mathbf{1} \in M$, we call the intersection of $C_1(f)$ and the hyperplane $H_1 := \{\mathbf{n} \in N_{\boldsymbol{R}} \mid \langle \mathbf{1}, \mathbf{n} \rangle = 1\}$ in $N_{\boldsymbol{R}}$ the *dual essential diagram* of (X, O), which is denoted by $B_1(f)$.

REMARK 2.9. Here, we note the reason why we call $B_1(f)$ the "dual" essential diagram. If necessary, we may call $C_1^*(f)$ the "essential diagram" because the sum of the terms of f whose indices are in the subset $C_1^*(f)$ of the Newton diagram $\Gamma_+(f)$ determines the algebraic-geometric structure of the support of the essential divisor E_J. In case (X, O) is a purely elliptic singularity, as we mentioned before, the essential cone $C_1(f)$ can be regarded as the "dual" of $C_1^*(f)$. Although $B_1(f)$ has the same information as $C_1(f)$, $B_1(f)$ is a little more convenient for us to visualize the essential divisor as we will see below.

For a cone σ in $N_{\boldsymbol{R}}$, define $\delta(\sigma)$ to be the intersection of σ and H_1 and $\mathring{\delta}(\sigma)$ to be the relative interior of $\delta(\sigma)$.

The dual essential diagram $B_1(f)$ has a natural stratification:

$$B_1(f) = \coprod_{\sigma \in C_1(f) \setminus \{O\}} \mathring{\delta}(\sigma),$$

which we call the *primitive stratification* of $B_1(f)$.

A nonsingular subdivision $\hat{\Sigma}(f)$ of $\Sigma(f)$ gives the dual essential diagram another stratification:

$$B_1(f) = \coprod_{\hat{\sigma} \in \hat{C}_1(f) \setminus \{O\}} \mathring{\delta}(\hat{\sigma}).$$

This stratification can be considered as a "subdivision" of the primitive stratification of $B_1(f)$ in the following meaning: For each $\hat{\sigma} \in \hat{C}_1(f)$, there is a cone $\sigma \in C_1(f)$ such that $\mathring{\delta}(\hat{\sigma}) \subset \mathring{\delta}(\sigma)$. If two strata of the stratification associated with a nonsingular subdivision are contained by the same stratum of the primitive stratification of $B_1(f)$, we say that these two strata are *primitively equivalent.*

Obviously, each stratum of a stratification of $B_1(f)$ as above is non-empty and connected.

THEOREM 2.10. (i) *There is a one-to-one correspondence between the strata of* $B_1(f) = \coprod_{\hat{\sigma} \in \hat{C}_1(f)} \mathring{\delta}(\hat{\sigma})$ *and the strata of* $E_J' = \coprod_{\hat{\sigma} \in \hat{C}_1(f)} \mathring{E}_J(\hat{\sigma})$ *given by*

$$\mathring{\delta}(\hat{\sigma}) \leftrightarrow \mathring{E}_J(\hat{\sigma})$$

for every cone $\hat{\sigma}$ *in* $\hat{C}_1(f) \setminus \{O\}$.

If $\dim \gamma_1(f) \geq 2$, *then the stratum* $\mathring{E}_J(\hat{\sigma})$ *is non-empty and connected in the Zariski topology for every* $\hat{\sigma} \in \hat{C}_1(f)$ *and*

$$\dim_{\boldsymbol{R}} \mathring{\delta}(\hat{\sigma}) + \dim_{\boldsymbol{C}} \mathring{E}_J(\hat{\sigma}) = r - 1$$

holds.

(ii) *The stratum* $\overset{\circ}{\delta}(\hat{\sigma})$ *corresponds to the stratum isomorphic to*

$$Z(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})}) \times_C (C^*)^n.$$

(iii) *The closure* $\delta(\hat{\sigma})$ of $\overset{\circ}{\delta}(\hat{\sigma})$ *corresponds to the closure* $E_J(\hat{\sigma})$ *of* $\overset{\circ}{E}_J(\hat{\sigma})$ *birational to*

$$\bar{Z}(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})}) \times_C \boldsymbol{P}^n_C.$$

In (ii), (iii), $n = r - (\dim \hat{\sigma} + \dim \gamma(\hat{\sigma})) + 1$.

(iv) *Strata of* E'_J *are isomorphic to each other and the closure of them are birationally equivalent if the corresponding strata of* $B_1(f)$ *are primitively equivalent.*

PROOF. (i) The operator $\overset{\circ}{\delta}$ gives the one-to-one correspondence between the non-zero cones in $\hat{C}_1(f)$ and the strata of $B_1(f)$. And the operator $\overset{\circ}{E}_J(*)$ gives the one-to-one correspondence between the nonzero cones in $\hat{C}_1(f)$ and the stratum of E'_J. These two operations give the correspondence in the theorem. When $\dim \gamma_1(f) \geq 2$, the non-emptiness and connectedness of $\overset{\circ}{E}_J(\hat{\sigma})$ follow Proposition 2.7. By the same proposition, we have $\dim_C \overset{\circ}{E}_J(\hat{\sigma}) = r - \dim_R \hat{\sigma}$. Since $\dim_R \overset{\circ}{\delta}(\hat{\sigma}) = \dim_R \hat{\sigma} - 1$, we obtain $\dim_R \overset{\circ}{\delta}(\hat{\sigma}) + \dim_C \overset{\circ}{E}_J(\hat{\sigma}) = r - 1$.

(ii) Let $\hat{\sigma} \in \hat{C}_1(f)$ be a cone and $\hat{v} \in \hat{\Sigma}(f)$ be an $(r+1)$-dimensional cone such that $\hat{\sigma}$ is a face of $\hat{v}$. Then, $U_{\hat{v}} := \operatorname{Spec} \boldsymbol{C}[(\hat{v})^\vee \cap M]$ contains $\operatorname{orb}(\hat{\sigma})$.

On $U_{\hat{v}}$, the proper, birational morphism $\Pi : U_{\hat{v}} \to \boldsymbol{C}^{r+1}$ is defined by the homomorphism $\boldsymbol{C}[(\boldsymbol{R}_{\geq 0})^{r+1} \cap M] \to \boldsymbol{C}[(\hat{v})^\vee \cap M]$, $\chi^{\mathbf{m}} \mapsto \chi^{\mathbf{m}}$. The proper transform $\tilde{X}$ of X is defined by the element $\hat{f} := \sum_{\mathbf{m} \in (\boldsymbol{R}_{\geq 0})^{r+1} \cap M} a_{\mathbf{m}} \cdot \chi^{\mathbf{m} - \mathbf{m}_\dagger} \in \boldsymbol{C}[(\hat{v})^\vee \cap M]$, where $\mathbf{m}_\dagger$ is the unique vertex of $\Gamma_+(f)$ such that $\{\mathbf{m}_\dagger\} = \bigcap_{\hat{\rho} \in \hat{v}(1)} \{\mathbf{m} \in M_{\boldsymbol{R}} \mid \langle \mathbf{m}, \mathbf{n}(\hat{\rho}) \rangle = l(\mathbf{n}(\hat{\rho}))\}$. Here we note that $\mathbf{m}_\dagger \in \gamma(\hat{\sigma})$.

In this proof, we denote the intersection $V(\hat{\sigma}) \cap U_{\hat{v}}$ simply by $V(\hat{\sigma})$. Then $\mathfrak{p}(\hat{\sigma}) := \bigoplus_{\mathbf{m} \in (\hat{v})^\vee \cap M,\, \notin (\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M} \boldsymbol{C} \cdot \chi^{\mathbf{m}}$ forms an ideal defining $V(\hat{\sigma})$ on $U_{\hat{v}}$, where $(\hat{\sigma})^\perp = \{\mathbf{m} \in M_{\boldsymbol{R}} \mid \langle \mathbf{m}, \mathbf{n} \rangle = 0$ *for any* $\mathbf{n} \in \hat{\sigma}\}$. On the other hand, $\boldsymbol{C}[(\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M] := \bigoplus_{\mathbf{m} \in (\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M} \boldsymbol{C} \cdot \chi^{\mathbf{m}}$ forms a group-algebra of $(\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M$. Then $\boldsymbol{C}[(\hat{v})^\vee \cap M]$ is the direct sum of $\mathfrak{p}(\hat{\sigma})$ and $\boldsymbol{C}[(\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M]$. The natural projection $\boldsymbol{C}[(\hat{v})^\vee \cap M] \to \boldsymbol{C}[(\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M]$, which is defined to be identity on $(\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M$ and zero on the complement, is a homomorphism of group-algebras and gives rise to an isomorphism $\boldsymbol{C}[(\hat{v})^\vee \cap M]/\mathfrak{p}(\hat{\sigma}) \cong \boldsymbol{C}[(\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M]$. Hence we can identify $\boldsymbol{C}[(\hat{v})^\vee \cap (\hat{\sigma})^\perp \cap M]$ with the affine coordinate ring of $V(\hat{\sigma})$ and the morphism associated with the above projection with the closed immersion $V(\hat{\sigma}) \hookrightarrow U_{\hat{v}}$. Thus we can omit the terms of $\hat{f} := \sum_{\mathbf{m} \in (\boldsymbol{R}_{\geq 0})^{r+1} \cap M} a_{\mathbf{m}} \cdot$

$\chi^{\mathbf{m}-\mathbf{m}_\dagger}$ whose indices $\mathbf{m}-\mathbf{m}_\dagger$ are not in $(\hat{\nu})^\vee \cap (\hat{\sigma})^\perp \cap M$ when we consider the intersection $\tilde{X}$ and $V(\hat{\sigma})$.

Take $\mathbf{m} \in \Gamma_+(f) \cap M$ such that $\langle \mathbf{m}-\mathbf{m}_\dagger, \mathbf{n}\rangle = 0$ for any $\mathbf{n} \in \hat{\sigma}$. Then $\langle \mathbf{m}, \mathbf{n}\rangle = \langle \mathbf{m}_\dagger, \mathbf{n}\rangle = l(\mathbf{n})$ since $\mathbf{m}_\dagger \in \gamma(\hat{\sigma})$, so that $\mathbf{m} \in \gamma(\hat{\sigma})$. In particular, $\mathbf{m}-\mathbf{m}_\dagger$ is contained by the sublattice $M_{\gamma(\hat{\sigma})} = \boldsymbol{R}(\gamma(\hat{\sigma}) - \mathbf{m}_\dagger) \cap M$ of $(\hat{\sigma})^\perp \cap M$. Therefore, we conclude that $E_J(\hat{\sigma}) = \tilde{X} \cap V(\hat{\sigma})$ is defined by $\hat{f}_{\gamma(\hat{\sigma})} := \sum_{\mathbf{m} \in \gamma(\hat{\sigma}) \cap M} a_{\mathbf{m}} \cdot \chi^{\mathbf{m}-\mathbf{m}_\dagger} \in \boldsymbol{C}[(\hat{\nu})^\vee \cap (\hat{\sigma})^\perp \cap M]$ on $V(\hat{\sigma})$, so that $\mathring{E}_J(\hat{\sigma})$ is defined by $f^L_{\gamma(\hat{\sigma})} = \sum_{\mathbf{m} \in \gamma(\hat{\sigma}) \cap M} a_{\mathbf{m}} \cdot \chi^{\mathbf{m}-\mathbf{m}_\dagger}$ considered as an element of $\boldsymbol{C}[(\hat{\sigma})^\perp \cap M]$. Indeed, the affine coordinate ring of the open subset $\mathrm{orb}(\hat{\sigma})$ of $V(\hat{\sigma})$ is just $\boldsymbol{C}[(\hat{\sigma})^\perp \cap M]$.

Now note that $(\hat{\sigma})^\perp \cap M$ can be expressed as a direct sum $M_{\gamma(\hat{\sigma})} \oplus M'$. Let n be the rank of M'. Then we have $\mathrm{rk}((\hat{\sigma})^\perp \cap M) = \mathrm{rk}\, M_{\gamma(\hat{\sigma})} + n$, and hence $n = r - (\dim \hat{\sigma} + \dim \gamma(\hat{\sigma})) + 1$. Thus we obtain $\mathring{E}_J(\hat{\sigma}) = Z(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})}) \times_{\boldsymbol{C}} \mathrm{Spec}\, \boldsymbol{C}[M'] \cong Z(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})}) \times_{\boldsymbol{C}} (\boldsymbol{C}^*)^n$.

At the end of the proof of (ii), we note that $Z(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})})$ is stable if we change $\mathbf{m}_\dagger$ into any element of $\gamma(\hat{\sigma}) \cap M$.

(iii) $E_J(\hat{\sigma})$ and $\bar{Z}(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})}) \times_{\boldsymbol{C}} \boldsymbol{P}^n_{\boldsymbol{C}}$ contain $\mathring{E}_J(\hat{\sigma})$ and $Z(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})}) \times_{\boldsymbol{C}} (\boldsymbol{C}^*)^n$ as open dense subsets, respectively. By (ii), $\mathring{E}_J(\hat{\sigma})$ is isomorphic to $Z(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})}) \times_{\boldsymbol{C}} (\boldsymbol{C}^*)^n$, so that $E_J(\hat{\sigma})$ and $\bar{Z}(f^L_{\gamma(\hat{\sigma})}, M_{\gamma(\hat{\sigma})}) \times_{\boldsymbol{C}} \boldsymbol{P}^n_{\boldsymbol{C}}$ are birational to each other.

(iv) The claim of (iv) follows directly (ii), (iii) of the theorem and Corollary 2.5. □

REMARK 2.11. In case (X, O) is a nondegenerate purely elliptic singularity of type $(0,0)$, some strata of E'_J are the empty set or consist of several connected components (see [15]).

COROLLARY 2.12. *In particular, if a stratum* $\mathring{\delta}(\hat{\sigma})$ *is in the interior of the dual essential diagram* $B_1(f)$, *it corresponds to the stratum* $\mathring{E}_J(\hat{\sigma})$ *isomorphic to*

$$Z(f^L_{\gamma_1(f)}, M_{\gamma_1(f)}) \times_{\boldsymbol{C}} (\boldsymbol{C}^*)^{\dim C_1(f) - \dim \hat{\sigma}}$$

and the closure $\delta(\hat{\sigma})$ *corresponds to the closure* $E_J(\hat{\sigma})$ *birational to*

$$\bar{Z}(f^L_{\gamma_1(f)}, M_{\gamma_1(f)}) \times_{\boldsymbol{C}} \boldsymbol{P}_{\boldsymbol{C}}^{\dim C_1(f) - \dim \hat{\sigma}}.$$

In the above, to define M_{γ_1} *and* $f^L_{\gamma_1}$ *as in Definition* 1.2, *we can always use* $\mathbf{1}$ *as* $\mathbf{m}_0$.

2.3. The Dual Complex of the Essential Divisor

Let Y be a nonsingular algebraic variety and let $E = \sum_{i=1}^r E_i$ be a simple normal crossing divisor on Y.

For E, we define the *dual complex* Γ_E of E as follows (cf. Ishii [8], Definition 7.4.7):

(0) We associate a vertex $\bullet$ for each irreducible component E_i;

(1) If a pair of irreducible components E_i, E_j intersect, then we associate a line segment between the vertices corresponding to E_i, E_j;

(2) If three irreducible components E_i, E_j, E_k intersect, we associate a triangle (two-dimensional simplex) with the vertices corresponding to E_i, E_j, E_k;

...

(i − 1) If i irreducible components intersect, then we associate an $(i-1)$-simplex with the vertices corresponding to $E_{v_1}, E_{v_2}, \dots, E_{v_i}$;

...

The dual complex of a simple normal crossing divisor is a simplicial complex.

Since the essential divisor E_J of a good resolution of the singularity (X, O) is a simple normal crossing divisor, we can associate the dual complex Γ_{E_J} with E_J.

Next, for the nonsingular subdivision $\hat{C}_1(f)$ of the essential cone $C_1(f)$ induced by the nonsingular subdivision $\hat{\Sigma}(f)$ of $\Sigma(f)$, we define a simplicial complex:

DEFINITION 2.13. We define $K_1(\hat{\Sigma}(f))$ to be the set of all $\delta(\hat{\sigma})$, where $\hat{\sigma}$ runs over all cones in $\hat{C}_1(f)\backslash\{O\}$.

Indeed, $K_1(\hat{\Sigma}(f))$ is a simplicial complex whose support is just the dual essential diagram $B_1(f)$ since the set of all cones in $\hat{C}_1(f)$ forms a nonsingular fan.

THEOREM 2.14. *If the dimension of the fundamental face γ_1 is greater than or equals to two, there exists a natural isomorphism of simplicial complexes between $K_1(\hat{\Sigma}(f))$ and Γ_{E_J}.*

PROOF. By Proposition 2.7, if $\dim \gamma_1(f) \geq 2$, then there exists a one-to-one correspondence between the set of all one-dimensional cones $\hat{\rho}$ in $\hat{C}_1(f)$ and the set of all irreducible components of the essential divisor E_J as follows:

$$\hat{\rho} \leftrightarrow D_{\hat{\rho}}|_{\tilde{X}}.$$

Hence we can define a bijection from the set of 0-dimensional simplexes of $K_1(\hat{\Sigma}(f))$ to the set of 0-dimensional simplexes of Γ_{E_J} by sending $\delta(\hat{\rho})$ to $D_{\hat{\rho}}|_{\tilde{X}}$.

To verify the theorem, we have to show that the intersection of irreducible components

$$(D_{\hat{\rho}_0}|_{\tilde{X}}) \cap (D_{\hat{\rho}_1}|_{\tilde{X}}) \cap \cdots \cap (D_{\hat{\rho}_n}|_{\tilde{X}}),$$

where $\hat{\rho}_0, \hat{\rho}_1, \ldots, \hat{\rho}_n$ are one-dimensional cones contained by $C_1(f)$, is not the empty set if and only if $\hat{\rho}_0 + \hat{\rho}_1 + \cdots + \hat{\rho}_n$ is a cone in $\hat{\Sigma}(f)$.

This follows the facts that $D_{\hat{\rho}_0} \cap D_{\hat{\rho}_1} \cap \cdots \cap D_{\hat{\rho}_n}$ is not an empty set if and only if $\hat{\rho}_0 + \hat{\rho}_1 + \cdots + \hat{\rho}_n$ is a cone in $\hat{\Sigma}(f)$ and that $E(\hat{\rho}_0 + \hat{\rho}_1 + \cdots + \hat{\rho}_n) = V(\hat{\rho}_0 + \hat{\rho}_1 + \cdots + \hat{\rho}_n) \cap \tilde{X}$ is an irreducible variety by Proposition 2.7 if $\hat{\rho}_0 + \hat{\rho}_1 + \cdots + \hat{\rho}_n$ is a cone in $\hat{C}_1(f)$ when $\dim \gamma_1(f) \geq 2$. □

COROLLARY 2.15. *If the dimension of the fundamental face γ_1 is greater than or equals to two, the dimension of the dual complex Γ_{E_J} equals to $r - \dim \gamma_1(f)$, and hence Γ_{E_J} is isomorphic to a triangulation of $(r - \dim \gamma_1)$-dimensional ball.*

3. Complete Toric Hypersurfaces Associated with the Fundamental Parts

For a purely elliptic singularity $(X, O) = (\mathbf{V}(f), \mathbf{0})$ defined by a nondegenerate polynomial f, we can associate a pair $(f_{\gamma_1}^L, M_{\gamma_1})$ consisting a Laurent polynomial $f_{\gamma_1}^L$ and a lattice M_{γ_1} with the fundamental part f_{γ_1} of f and construct canonically a complete toric hypersurface $\bar{Z}(f_{\gamma_1}^L, M_{\gamma_1})$ as in § 1.3, which we call the *complete toric hypersurface associated with the fundamental part* f_{γ_1} in this section.

In the following of this section, we will investigate the complete toric hypersurfaces associated with the fundamental parts of the defining polynomials for hypersurface purely elliptic singularities whose fundamental faces have the dimension greater than or equal to two.

In order to study toric hypersurfaces, we often investigate the Newton polyhedra of the defining Laurent polynomials. Similarly, studying the fundamental faces of the Newton diagrams is useful to understand the complete toric hypersurfaces associated with the fundamental parts.

We will use the scalar extensions by the field of rational numbers $\boldsymbol{Q}$ instead of $\boldsymbol{R}$ as we use before for the $\boldsymbol{Z}$-modules M and N etc in order to mind that our operations to vectors etc are closed in $\boldsymbol{Q}$ every time. Nevertheless, there are no differences technically.

3.1. Quasi-$\boldsymbol{Q}$-Reflexive Polyhedra and Hyperplanes Passing Through Them

The following definitions are weaker variations of the definition of *reflexive polyhedra* due to Batyrev for n-dimensional convex polyhedra in $M_{\boldsymbol{Q}}$, which may have non-integral points as its vertices.

DEFINITION 3.1 (cf. Batyrev [1], Definition 4.1.5). Let M be a free $\boldsymbol{Z}$-module of rank n and N be its dual $\boldsymbol{Z}$-module. Let Δ be an n-dimensional convex polyhedron in $M_{\boldsymbol{Q}}$ containing the zero $\mathbf{0} \in M$ in its interior. Then the pair (Δ, M) is said to be *quasi-$\boldsymbol{Q}$-reflexive* if every affine hyperplane generated by an $(n-1)$-dimensional face of Δ is of the form for an integral element $\mathbf{l} \in N$: $\{\mathbf{x} \in M_{\boldsymbol{Q}} \mid \langle \mathbf{x}, \mathbf{l} \rangle = -1\}$.

In the above, if we can take a primitive integral element $\mathbf{l} \in N$ for every affine hyperplane generated by an $(n-1)$-dimensional face of Δ, the pair (Δ, M) is said to be *$\boldsymbol{Q}$-reflexive*.

If (Δ, M) is a quasi-$\boldsymbol{Q}$-reflexive pair (resp. $\boldsymbol{Q}$-reflexive pair), we call Δ a *quasi-$\boldsymbol{Q}$-reflexive polyhedron* (resp. *$\boldsymbol{Q}$-reflexive polyhedron*)

REMARK 3.2. Note that a $\boldsymbol{Q}$-reflexive pair (Δ, M) is, of course, a quasi-$\boldsymbol{Q}$-reflexive pair and that a quasi-$\boldsymbol{Q}$-reflexive pair (Δ, M) is $\boldsymbol{Q}$-reflexive if and only if there exists an integral point on every affine hyperplane generated by an $(n-1)$-dimensional face of Δ.

REMARK 3.3. A $\boldsymbol{Q}$-reflexive pair (Δ, M) is a reflexive pair if and only if Δ is integral.

Recall that for a subset K in $M_{\boldsymbol{Q}}$, the *polar dual* K^* of K is defined by

$$K^* := \{\mathbf{y} \in N_{\boldsymbol{Q}} \mid \langle \mathbf{x}, \mathbf{y} \rangle \geq -1 \textit{ for all } \mathbf{x} \in K\}.$$

If (Δ, M) is a reflexive pair, then its polar dual (Δ^*, N) is also a reflexive pair. But for a $\boldsymbol{Q}$-reflexive polyhedron which is not integral, its polar dual is not $\boldsymbol{Q}$-reflexive.

We will show the following for later use:

LEMMA 3.4. *Let (Δ, M) be an n-dimensional quasi-$\boldsymbol{Q}$-reflexive pair. Then Δ contains no integral point in its interior except for the zero $\mathbf{0} \in M$.*

PROOF. Suppose that there exists an integral point in the interior of Δ, say $\mathbf{x}_0$, which is not the zero of $M_{\boldsymbol{Q}}$. Then there exists at least one $(n-1)$-dimensional face δ of Δ through which the one-dimensional cone generated by $\mathbf{x}_0$ passes. Let H be the hyperplane of $M_{\boldsymbol{Q}}$ spanned by δ. Then there exists an integral vector $\mathbf{l} \in N$ such that

$$H = \{\mathbf{x} \in M_{\boldsymbol{Q}} \mid \langle \mathbf{x}, \mathbf{l} \rangle = 1\}.$$

Here, note that since $\mathbf{x}_0$ is in the interior of Δ, there exists a positive rational number α greater than 1 such that the point $\alpha\mathbf{x}_0$ is on H. Hence, we have $\langle\alpha\mathbf{x}_0,\mathbf{l}\rangle = 1$, so that $\langle\mathbf{x}_0,\mathbf{l}\rangle \neq 0$ and $\alpha = 1/\langle\mathbf{x}_0,\mathbf{l}\rangle$ holds. Since both $\mathbf{x}_0$ and $\mathbf{l}$ are integral vectors, $\langle\mathbf{x}_0,\mathbf{l}\rangle$ is an integer. If $\langle\mathbf{x}_0,\mathbf{l}\rangle \geq 1$, then $\alpha \leq 1$: a contradiction. If $\langle\mathbf{x}_0,\mathbf{l}\rangle = 0$, then $\langle\alpha\mathbf{x}_0,\mathbf{l}\rangle = 0$: again a contradiction.

Thus Δ must have no integral point in its interior except for the zero. □

Given a $\boldsymbol{Q}$-reflexive pair (Δ, M), we can make new $\boldsymbol{Q}$-reflexive pairs of codimension 1:

Lemma 3.5. *Let M, $M_{\boldsymbol{Q}}$ be as before, respectively. Let H be a rational affine hyperplane passing through the zero $\mathbf{0} \in M_{\boldsymbol{Q}}$. If Δ is a quasi-$\boldsymbol{Q}$-reflexive polyhedron, then for $\Delta_H := \Delta \cap H$ and $M_H := M \cap H$, (Δ_H, M_H) is a quasi-$\boldsymbol{Q}$-reflexive pair.*

Proof. We can regard H as a $\boldsymbol{Q}$-vector subspace which contains $M \cap H$ as a lattice. Since Δ is a convex polyhedron containing the zero $\mathbf{0} \in M_{\boldsymbol{Q}}$ in its interior and H passes through the zero, Δ_H is an $(n-1)$-dimensional convex polyhedron in H containing the zero in its relative interior.

We have to show that the affine hyperplane generated by any $(n-2)$-dimensional face of Δ_H is of the form $\{\mathbf{x} \in (M_H)_{\boldsymbol{Q}} \mid \langle\mathbf{x},\mathbf{l}'\rangle = -1\}$ for some $\mathbf{l}' \in N_H := \mathrm{Hom}_{\boldsymbol{Z}}(M_H, \boldsymbol{Z})$. Now note that any $(n-2)$-dimensional face of Δ_H is the intersection of some $(n-1)$-dimensional face of Δ and H. Hence let $\delta_H = \delta \cap H$ be an $(n-2)$-dimensional face of Δ_H which is the intersection of an $(n-1)$-dimensional face δ of Δ and H. Since Δ is quasi-$\boldsymbol{Q}$-reflexive, there exists an integral element $\mathbf{l} \in N$ such that $\delta = \{\mathbf{x} \in \Delta \mid \langle\mathbf{x},\mathbf{l}\rangle = -1\}$. Let $\mathbf{l}'$ be the image of $\mathbf{l}$ by the homomorphism $\iota^* : N \to N_H := \mathrm{Hom}(M_H, \boldsymbol{Z})$, which is the dual map of the inclusion $\iota : M_H \to M$. Then, obviously, $\delta_H = \{\mathbf{x} \in \Delta \cap H \mid \langle\mathbf{x},\mathbf{l}'\rangle = -1\}$. Thus we are done. □

Corollary 3.6. *In the previous proposition, if Δ_H is integral, then (Δ_H, M_H) is a reflexive pair.*

3.2. The $\boldsymbol{Q}$-Reflexive Pair Associated with the Fundamental Face

Let $(X, O) = (\mathbf{V}(f), \mathbf{0})$ be an r-dimensional purely elliptic singularity defined by a nondegenerate polynomial $f = \sum_{\mathbf{m} \in (\boldsymbol{Z}_{\geq 0})^{r+1}} a_{\mathbf{m}} \cdot \mathbf{z}^{\mathbf{m}}$. Recall that $\gamma_1 = \gamma_1(f)$ denotes the unique face of $\Gamma(f)$ containing $\mathbf{1}$ in its relative interior. In the following, we assume that $\dim \gamma_1 \geq 2$.

Now we assume that $\dim_{\mathbf{Q}} \gamma_1 = r - k$. Then there exist exactly $k+1$ compact faces $\delta_1^{(0)}, \delta_1^{(1)}, \dots, \delta_1^{(k)}$ of $\Gamma(f)$ such that $\dim \delta_1^{(i)} = r$ for any i and

$$\gamma_1 = \delta_1^{(0)} \cap \delta_1^{(1)} \cap \cdots \cap \delta_1^{(k)}.$$

Let $H^{(0)}, H^{(1)}, \dots, H^{(k)}$ be the hyperplanes of $M_{\mathbf{Q}}$ spanned by $\delta_1^{(0)}, \delta_1^{(1)}, \dots, \delta_1^{(k)}$, respectively. Then, we have

$$\gamma_1 \subset (\mathbf{Q}_{\geq 0})^{r+1} \cap \left(\bigcap_{i=0}^{k} H^{(i)} \right).$$

Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ be the primitive integral generators of the dual cones $(\delta_1^{(0)})^*$, $(\delta_1^{(1)})^*, \dots, (\delta_1^{(k)})^*$, respectively. Then we have $H^{(i)} = \{\mathbf{m} \in M_{\mathbf{Q}} \mid \langle \mathbf{m}, \mathbf{v}_i \rangle = \langle \mathbf{1}, \mathbf{v}_i \rangle\}$ for $i = 0, 1, \dots, k$. Here we put

$$\Delta^{(i)} := (\mathbf{Q}_{\geq 0})^{r+1} \cap \left(\bigcap_{j=0}^{i} H^{(j)} \right) - \mathbf{1}.$$

On the other hand, we define

$$N^{(i)} := N / \left(\sum_{j=0}^{i} \mathbf{Q}\mathbf{v}_j \cap N \right).$$

Then we have a natural homomorphism $u^{(i)} : N^{(i)} \to N^{(i+1)}$ for $j = 0, 1, \dots, k-1$. Let $M_{\mathbf{Q}}^{(i)} := \mathbf{Q}\Delta^{(i)} - \mathbf{1}$ and $M^{(i)} := M_{\mathbf{Q}}^{(i)} \cap M$. Then $M_{\mathbf{Q}}^{(i)}$ and $N_{\mathbf{Q}}^{(i)}$, further, $M^{(i)}$ and $N^{(i)}$ are dual to each other.

Moreover, define

$$L^{(i)} := \bigcap_{j=0}^{i} H^{(j)} - \mathbf{1}$$

for $j = 0, 1, \dots, k$. Then $L^{(i)}$ is a hyperplane in $M_{\mathbf{Q}}^{(i-1)}$, where $M_{\mathbf{Q}}^{(-1)} = M_{\mathbf{Q}}$. Note that we have

$$\Delta^{(i+1)} = \Delta^{(i)} \cap L^{(i+1)}$$

for $i = 0, 1, \dots, k-1$.

Consequently, we obtain sequences of lattices:

$$N^{(0)} \to N^{(1)} \to \cdots \to N^{(k)}, \quad M^{(k)} \hookrightarrow M^{(k-1)} \hookrightarrow \cdots \hookrightarrow M^{(0)}$$

and a sequence of convex polyhedra:

$$(\Delta^{(0)}, M^{(0)}) \supset (\Delta^{(1)}, M^{(1)}) \supset \cdots \supset (\Delta^{(k)}, M^{(k)}) \supset (\gamma_1 - \mathbf{1}, M_{\gamma_1}).$$

Here, we prepare a lemma to progress further: In general, let M be a free $\boldsymbol{Z}$-module of rank $r+1$ and N be its dual $\boldsymbol{Z}$-module.

LEMMA 3.7. *For an integral positive vector* $\mathbf{w} \in N$, *define* $M(\mathbf{w}) := H(\mathbf{w}) \cap M - \mathbf{1}$ *and* $\Delta(\mathbf{w}) := (\boldsymbol{Q}_{\geq 0})^{r+1} \cap H(\mathbf{w}) - \mathbf{1}$, *where*

$$H(\mathbf{w}) := \{\mathbf{m} \in M_{\boldsymbol{Q}} \mid \langle \mathbf{m}, \mathbf{w} \rangle = \langle \mathbf{1}, \mathbf{w} \rangle\}.$$

Then $(\Delta(\mathbf{w}), M(\mathbf{w}))$ *is a* $\boldsymbol{Q}$*-reflexive pair.*

PROOF. Let $\mathbf{w} = (w_0, w_1, \dots, w_r)$ be a positive integral vector in N and let $d := \sum_{i=0}^{r} w_i$. We assume that $\mathbf{w}$ is primitive, i.e., $\gcd(w_0, w_1, \dots, w_r) = 1$.

For this $\mathbf{w}$, we define a lattice $M'(\mathbf{w})$ by

$$M'(\mathbf{w}) := \left\{ (\alpha_1, \alpha_2, \dots, \alpha_r) \in \boldsymbol{Z}^r \mid \sum_{i=1}^{r} w_i(\alpha_i + 1) \equiv d \bmod w_0 \right\}$$

and denote by $\Delta'(\mathbf{w})$ the convex hull of the set of points in $M'(\mathbf{w})_{\boldsymbol{Q}}$:

$$\{\mathbf{p}_0 := (-1, -1, \dots, -1), \mathbf{p}_1 := (-1 + d/w_1, -1, \dots, -1), \dots,$$
$$\mathbf{p}_r := (-1, -1, \dots, -1 + d/w_r)\}.$$

Then $(\Delta'(\mathbf{w}), M'(\mathbf{w}))$ is a $\boldsymbol{Q}$-reflexive simplex and the corresponding toric variety $\boldsymbol{P}_{\Delta'(\mathbf{w})}$ is a weighted projective space of weights $\mathbf{w}$, namely $\boldsymbol{P}(\mathbf{w}) = \boldsymbol{P}(w_0, w_1, \dots, w_r)$.

We define a homomorphism $\iota_{\Delta'} : M'(\mathbf{w}) \to M \cong \boldsymbol{Z}^{r+1}$ and its scalar extension $\iota_{\Delta'} : M'(\mathbf{w})_{\boldsymbol{Q}} \to M_{\boldsymbol{Q}} \cong \boldsymbol{Q}^{r+1}$, where

$$\iota_{\Delta'}(\alpha) = (\langle \alpha, \mathbf{l}_0 \rangle, \langle \alpha, \mathbf{l}_1 \rangle, \dots, \langle \alpha, \mathbf{l}_r \rangle),$$

$\mathbf{l}_0 = (-w_1/w_0, -w_2/w_0, \dots, -w_r/w_0)$ and $\mathbf{l}_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$ for $i = 1, 2, \dots, r$. Then $\iota_{\Delta'}$ is injective and the image of it is contained in the r-dimensional $\boldsymbol{Q}$-vector subspace defined by the equation $\sum_{i=0}^{r} w_i m_i = 0$, which we can identify with $M(\mathbf{w})_{\boldsymbol{Q}}$, in particular, the image of $M'(\mathbf{w})$ by $\iota_{\Delta'}$ is contained in the sublattice $M(\mathbf{w}) = \{\mathbf{m} \in M \mid \sum_{i=0}^{r} w_i m_i = 0\}$. Moreover, we can easily show that $\iota_{\Delta'}$ is surjective. Therefore, $\iota_{\Delta'}$ is an isomorphism of lattices.

The image of $\mathbf{p}_i$ by $\iota_{\Delta'}$ is $(-1, -1, \dots, -1, \langle \mathbf{p}_i, \mathbf{l}_i \rangle, -1, \dots, -1) = (-1, -1, \dots, -1, -1 + d/w_i, -1, \dots, -1)$ for each i. The fact that the convex hull of the set of the points $\{\iota_{\Delta'}(\mathbf{p}_i)\}$ is just $\Delta(\mathbf{w})$ completes the proof. □

By Lemma 3.5 and the lemma just above, we know that $(\Delta^{(i)}, M^{(i)})$ $(i=0,1,\ldots,k)$ are quasi-$\boldsymbol{Q}$-reflexive pairs. In fact, the last one $(\Delta^{(k)}, M^{(k)})$ is a $\boldsymbol{Q}$-reflexive pair. Although we need quasi-$\boldsymbol{Q}$-reflexivity for $(\Delta^{(k)}, M^{(k)})$, but not $\boldsymbol{Q}$-reflexivity in the following discussion, we state this fact as a proposition:

PROPOSITION 3.8. *$(\Delta^{(k)}, M^{(k)})$ is a $\boldsymbol{Q}$-reflexive pair.*

PROOF. As we saw just before, $(\Delta^{(k)}, M^{(k)})$ is a quasi-$\boldsymbol{Q}$-reflexive pair. Then, as we stated in Remark 3.2, we have to show that there exists an integral point on the affine hyperplane generated by any face of codimension-one of $\Delta^{(k)}$.

Now let δ be a face of codimension-one of $\Delta^{(k)}$. Then there is an integral element $\mathbf{l} \in N^{(k)}$ such that $\delta = \Delta^{(k)} \cap \{\mathbf{x} \in (M^{(k)})_{\boldsymbol{Q}} \mid \langle \mathbf{x}, \mathbf{l} \rangle = 1\}$ and $\Delta^{(k)}$ is contained by the half-space $\{\mathbf{x} \in (M^{(k)})_{\boldsymbol{Q}} \mid \langle \mathbf{x}, \mathbf{l} \rangle \leq 1\}$. If there is no integral point on the hyperplane $\{\mathbf{x} \in (M^{(k)})_{\boldsymbol{Q}} \mid \langle \mathbf{x}, \mathbf{l} \rangle = 1\}$, the integral convex polyhedron $\gamma_1 - \mathbf{1}$ must be contained by the half-space $\{\mathbf{x} \in (M^{(k)})_{\boldsymbol{Q}} \mid \langle \mathbf{x}, \mathbf{l} \rangle \leq 0\}$ since $\gamma_1 - \mathbf{1} \subset \Delta^{(k)}$, which contradicts the fact that $\gamma_1 - \mathbf{1}$ contains the zero $\mathbf{0} \in M^{(k)}$ in its interior, for $\dim \Delta^{(k)} = \gamma_1 - \mathbf{1} = \dim M^{(k)}$. □

By Lemma 3.4, we obtain the following proposition:

PROPOSITION 3.9. *The fundamental face γ_1 of a hypersurface purely elliptic singularity contains no integral point in its relative interior except for $\mathbf{1}$.*

THEOREM 3.10. *Let $\bar{Z} = \bar{Z}(f_{\gamma_1}^L, M_{\gamma_1})$ be the complete hypersurface in $\boldsymbol{P}_{\gamma_1 - 1}$ associated with the fundamental part of f. Then the geometric genus of a nonsingular model of $\bar{Z}$ equals to one.*

PROOF. To begin with, we mention the result of Khovanskiǐ [13] [12].

LEMMA 3.11 (Khovanskiǐ). *Let f be a nondegenerate Laurent polynomial, Δ be the Newton polyhedron of f and let Y be the hypersurface in $(\boldsymbol{C}^*)^{\dim \Delta}$ defined by f. Then there exists a complete nonsingular toric variety in which the closure $\tilde{Y}$ of Y is a compact nonsingular variety transverse to all the orbits of the toric variety. And the geometric genus $p(\tilde{Y}) = h^{\dim \tilde{Y}, 0}$ is given by the formula:*

$$p(\tilde{Y}) = l^*(\Delta),$$

where $l^*(\Delta)$ is the number of the integral points in the interior of Δ.

In Lemma 3.11, take $f_{\gamma_1}^L = \sum_{\mathbf{m} \in \gamma_1 \cap M} a_{\mathbf{m}} \cdot \chi^{\mathbf{m}-\mathbf{1}}$ as f and $\gamma_1 - \mathbf{1}$ as Δ, then we have a nonsingular compactification $\tilde{Z}$ with the geometric genus $p(\tilde{Z}) = l^*(\Delta)$. By Proposition 3.9, we have $l^*(\Delta) = 1$. This completes the proof. □

3.3. Special Cases

At the end of this section, we add some comments on special cases where $\gamma_1(f) = \Delta^{(k)}$ hold in the above discussion. The next proposition follows Corollary 3.6:

PROPOSITION 3.12. *If $\gamma_1(f) = \Delta^{(k)}$, then the pair $(\gamma_1 - \mathbf{1}, M_{\gamma_1})$ is a reflexive pair.*

REMARK 3.13. In general, the fundamental face $\gamma_1(f)$ does not satisfies the assumption in the above proposition. Indeed, for the polynomial

$$f = z_0^3 + z_1^3 z_2 + z_1^3 z_3 + z_2^5 + z_3^5,$$

γ_1 is contained by $(\boldsymbol{Q}_{\geq 0})^4 \cap H$, where $H = \{\mathbf{m} = (m_0, m_1, m_2, m_3) \in \boldsymbol{Q}^4 \mid \langle \mathbf{m}, (5,4,3,3) \rangle = 15\}$, but $\gamma_1 \neq (\boldsymbol{Q}_{\geq 0})^4 \cap H$.

A complex normal irreducible n-dimensional projective variety Y with only Gorenstein canonical singularities is called a *Calabi-Yau variety* if it has trivial canonical bundle and $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < n$. Due to Batyrev [1], Theorem 4.1.9, a Δ-*regular* toric hypersurface $\overline{Z}(f, M)$ is birational to a Calabi-Yau variety if $(\Delta(f), M)$ is a reflexive pair. See [1], Definition 3.1.1 for the definition of Δ-regular hypersurfaces.

The nondegeneracy of the defining polynomial of a purely elliptic singularity guarantees the $(\gamma_1 - \mathbf{1})$-regularity of $f_{\gamma_1}^L$. Therefore, by Corollary 2.12, we obtain the following statement:

COROLLARY 3.14. *If $\gamma_1 = \Delta^{(k)}$, then the closure $E_J(\hat{\sigma})$ of the stratum $\mathring{E}_J(\hat{\sigma})$ of the essential divisor corresponding to the stratum $\mathring{\delta}(\hat{\sigma})$ of the dual essential diagram $B_1(f)$ of dimension* $\dim B_1(f)$ *is birational to a Calabi-Yau variety of dimension* $(r - \dim B_1(f) - 1)$.

Moreover, the closure $E_J(\hat{\sigma})$ of the stratum $\mathring{E}_J(\hat{\sigma})$ is birational to a ruled variety over this Calabi-Yau variety if $\mathring{\delta}(\hat{\sigma})$ is contained in the interior of the dual essential diagram $B_1(f)$.

4. Three-Dimensional Purely Elliptic Singularities of Type $(0,1)$

4.1. The Type of a Purely Elliptic Singularity

Ishii [6] classified the r-dimensional purely elliptic singularities using the mixed Hodge structures of the $(r-1)$-th cohomology groups of the structure sheaves of the essential divisors of good resolutions of them as below:

Let (X,x) be a purely elliptic singularity of dimension $r \geq 2$ and let $\pi : \tilde{X} \to X$ be a good resolution of (X,x) with E_J the essential divisor. Then we have

PROPOSITION 4.1 (Ishii [6]).

$$C \cong H^{r-1}(E_J, \mathcal{O}_J) \cong \mathrm{Gr}_F^0 H^{r-1}(E_J) = \bigoplus_{i=0}^{r-1} H_{r-1}^{0,i}(E_J),$$

where $H_m^{i,j}(*)$ *is the* (i,j)*-component of* $\mathrm{Gr}_{i+j}^W H^m(*)$.

By the proposition just above, for a unique i $(0 \leq i \leq r-1)$,

$$H^{r-1}(E_J, \mathcal{O}_J) = H_{r-1}^{0,i}(E_J) \cong C.$$

DEFINITION 4.2 (Ishii). A purely elliptic singularity (X,x) is of *type* $(0,i)$ if $H^{r-1}(E_J, \mathcal{O}_{E_J})$ consists of the $(0,i)$-Hodge component.

Watanabe [18] gave the relation between the type of a hypersurface purely elliptic singularity and the dimension of the fundamental face of the Newton boundary of the defining equation:

PROPOSITION 4.3 (Watanabe). *Let* $(X,O) = (\mathbf{V}(f), \mathbf{0})$ *be an r-dimensional purely elliptic singularity defined by a nondegenerate polynomial* f. *Then* (X,O) *is of type* $(0, \dim \gamma_1 - 1)$ *if* $\dim \gamma_1 \geq 2$ *and of type* $(0,0)$ *if* $\dim \gamma_1 = 1$ *or* 0.

4.2. The Dual Essential Diagram of a Three-Dimensional Hypersurface Purely Elliptic Singularity of Type $(0,1)$ and the Stratification on It

Three-dimensional hypersurface purely elliptic singularities of type $(0,1)$ have two-dimensional fundamental faces and two-dimensional essential cones by Proposition 4.3. These have the simplest structures in hypersurface purely elliptic sin-

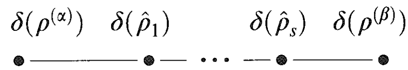

Figure 1

gularities with the fundamental faces of dimension greater than or equal to two and the non-trivial dual essential diagrams.

Let $f \in C[z_0, z_1, z_2, z_3]$ be a nondegenerate polynomial defining a purely elliptic singularity of type $(0, 1)$ at the origin $O \in C^4$. Then as we mentioned above, both the dimension of the fundamental face γ_1 and that of the essential cone $C_1(f)$ are two. Therefore, the dual essential diagram $B_1(f)$ is a line segment.

Take a nonsingular subdivision $\hat{\Sigma}(f)$ of the dual fan $\Sigma(f)$. In fact, we have only to take a nonsingular subdivision $\hat{C}_1(f)$ of the essential cone $C_1(f)$ to see the essential divisor. Then the essential divisor E_J of the induced resolution of singularities $\pi : (\tilde{X}, E) \to (X, O)$ is just

$$\sum_{\hat{\rho} \in \hat{C}_1(f)(1)} D_{\hat{\rho}}|_{\tilde{X}}.$$

Let $\gamma_1^{(\alpha)}$ and $\gamma_1^{(\beta)}$ are three-dimensional faces of $\Gamma_+(f)$ such that

$$\gamma_1(f) = \gamma_1^{(\alpha)} \cap \gamma_1^{(\beta)}.$$

and let $\rho^{(\alpha)}$ and $\rho^{(\beta)}$ be one-dimensional cones dual to the faces $\gamma_1^{(\alpha)}$ and $\gamma_1^{(\beta)}$, respectively. Moreover, let $\rho^{(\alpha)} = \hat{\rho}_0, \hat{\rho}_1, \dots, \hat{\rho}_s,\ \rho^{(\beta)} = \hat{\rho}_{s+1}$ be one-dimensional cones in $\hat{\Sigma}(f)$ in the essential cone $C_1(f)$, where in this order, one-dimensional cones appear in $C_1(f)$. Figure 1 shows the dual essential diagram of (X, O) and the stratification associated with $\hat{\Sigma}(f)$.

By Theorem 2.10, we know that any one-dimensional stratum of $B_1(f)$ corresponds to a stratum of E_J isomorphic to

$$Z(f_{\gamma_1}^L, M_{\gamma_1}) = \mathrm{V}(f_{\gamma_1}^L) \subset (C^*)^2.$$

The stratum $\delta(\hat{\rho}_i)$ corresponds to the stratum $\mathring{E}_J(\hat{\rho}_i)$ isomorphic to

$$Z(f_{\gamma_1}^L, M_{\gamma_1}) \times_C C^*$$

for $i = 1, 2, \dots, s$. And the stratum $\delta(\rho^{(\alpha)})$ (resp. $\delta(\rho^{(\beta)})$) corresponds to the stratum $\mathring{E}_J(\rho^{(\alpha)})$ (resp. $\mathring{E}_J(\rho^{(\beta)})$) isomorphic to

$$Z(f_{\gamma_1^{(\alpha)}}^L, M_{\gamma_1^{(\alpha)}}) \quad (\text{resp. } Z(f_{\gamma_1^{(\beta)}}^L, M_{\gamma_1^{(\beta)}})).$$

4.3. The Dual Complex of the Essential Divisor

By Theorem 2.14, we can easily read the structure of the dual complex Γ_{E_J} of E_J from Figure 1.

On the other hand, as we saw before, Theorem 2.10 and Corollary 2.12 give the structure of the closure of each stratum of the essential divisor up to birational equivalence. In particular, the closure of the stratum of E_J corresponding to a one-dimensional stratum of $B_1(f)$ is a nonsingular algebraic curve *isomorphic* to the toric hypersurface $\bar{Z}(f_{\gamma_1}^L, M_{\gamma_1})$ associated with the fundamental face γ_1, which is also a nonsingular algebraic curve, for two nonsingular curves which are birational to each other are isomorphic to each other.

Moreover, by Theorem 3.10, the geometric genus of the toric hypersurface associated with the fundamental face of a purely elliptic singularity is one, so that $\bar{Z}(f_{\gamma_1}^L, M_{\gamma_1})$ is an elliptic curve.

Summing up, we obtain the following final statements:

THEOREM 4.4 (cf. Ishii [6], Theorem 4.6). *The dual complex Γ_{E_J} of the essential divisor E_J of the singularity $(X, O) = (\mathbf{V}(f), \mathbf{0})$ is of dimension one.*

$D_{\rho^{(\alpha)}}|_{\tilde{X}} = E_J(\rho^{(\alpha)})$ (resp. $D_{\rho^{(\beta)}}|_{\tilde{X}} = E_J(\rho^{(\beta)})$) *is birational to*

$$\bar{Z}(f_{\gamma_1^{(\alpha)}}^L, M_{\gamma_1^{(\alpha)}}) \quad (\text{resp. } \bar{Z}(f_{\gamma_1^{(\beta)}}^L, M_{\gamma_1^{(\beta)}}))$$

and for $i = 1, 2, \ldots, s$, $D_{\hat{\rho}_i}|_{\tilde{X}} = E_J(\hat{\rho}_i)$ is birational to the elliptic ruled surface:

$$\bar{Z}(f_{\gamma_1}^L, M_{\gamma_1}) \times_C \boldsymbol{P}_C^1.$$

The intersection of every two irreducible components adjacent to each other $(D_{\hat{\rho}_j}|_{\tilde{X}}) \cap (D_{\hat{\rho}_{j+1}}|_{\tilde{X}}) = E_J(\hat{\rho}_j + \hat{\rho}_{j+1})$ is isomorphic to a nonsingular elliptic curve

$$\bar{Z}(f_{\gamma_1}^L, M_{\gamma_1}).$$

References

[1] V. V. Batyrev. Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. J. Algebraic Geom., **3**:493–535, 1994.

[2] A. R. Fletcher. Plurigenera of 3-folds and weighted hypersurfaces. PhD thesis, the University of Warwick, 1988.

[3] A. Fujisawa. On defining polynomials of hypersurface purely elliptic singularities. preprint.

[4] W. Fulton. Introduction to Toric Varieties. Princeton University Press, Princeton, NJ, 1993.

[5] R. Hartshorne. Algebraic Geometry. Number 52 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1977.

[6] S. Ishii. On isolated Gorenstein singularities. Math. Ann., **270**:541–554, 1985.

[7] S. Ishii. The canonical modifications by weighted blowups. J. Algebraic Geom., **5**:783–799, 1996.

[8] S. Ishii. Introduction to singularities (in Japanese). Springer-Verlag Tokyo, Tokyo, 1997.

[9] S. Ishii and K. Watanabe. A geometric characterization of a simple $K3$ singularity. Tôhoku Math. J., **44**:19–24, 1992.

[10] N. Kanesaka. Explicit structures of hypersurface purely elliptic singularities of type $(0, 1)$. PhD thesis, the University of Tsukuba, 2000.

[11] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. Toroidal Embeddings. Number 339 in Lecture Note in Math. Springer, 1973.

[12] A. G. Khovanskiĭ. Newton polyhedra and the genus of complete intersection. Functional. Anal. i Prilozhen, **11**:56–64, 1977. English transl. in Functional Anal. Appl. **11** (1977).

[13] A. G. Khovanskiĭ. Newton polyhedra and toral varieties. Functional. Anal. i Prilozhen, **11**:51–61, 1977. English transl. in Functional Anal. Appl. **11** (1977).

[14] T. Oda. Convex Bodies and Algebraic Geometry. Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge · Band 15. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.

[15] M. Oka. On the resolution of hypersurface singularities. In T. Suwa and P. Wagreich, editors, Complex Analytic Singularities, number 8 in Advanced Studies in Pure Math., pages 405–436. Kinokuniya, Tokyo and North-Holland, Amsterdam, New York, Oxford, 1986.

[16] A. N. Varchenko. Zeta-functions of monodromy and Newton's diagram. Inventiones Math., **37**:253–262, 1976.

[17] K. Watanabe. On plurigenera of normal isolated singularities I. Math. Ann., **250**:65–94, 1980.

[18] K. Watanabe. On plurigenera of normal isolated singularities II. In T. Suwa and P. Wagreich, editors, Complex Analytic Singularities, number 8 in Advanced Studies in Pure Math., pages 671–685. Kinokuniya, Tokyo and North-Holland, Amsterdam, New York, Oxford, 1986.

[19] T. Yonemura. Hypersurface simple $K3$ singularities. Tôhoku Math. J., **42**:351–380, 1990.

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