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ON THE EXISTENCE OF SOLUTIONS OF PRESCRIBING SCALAR CURVATURE PROBLEM

By

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Abstract. We consider the scalar curvature problem with nonzero Dirichlet boundary data and with prescribed scalar function changing sign as a nonlinear eigenvalue problem. We use the super-sub solution method and our early result [MW] to obtain a solution of the problem and give a lower bound on the eigenvalue.

1. Introduction

In the study of the prescribed scalar curvature problem, one may treat the corresponding boundary problem. Such a problem were studied by H. Brezis and L. Nirenberg, L. Caffarelli and J. Spruck (see [BN] and [CS]). We can extend some of their results. Here we continue our early study [MW] into the prescribed scalar curvature problem with non-zero Dirichlet data and the scalar curvature function changing sign. There are relative a few results in this direction (see [Ni]). This is because we can not use the standard variational method to study the difficulty arising from the negative part of the scalar curvature. Hence we try to understand this problem in other ways. A little thinking gives us the idea using the implicit function theorem (see [BN] or [M2]), and we can get a general existence theorem for general scalar curvature functions. However, one wants to know more. The new point here is that we do get more by using our early result (see [MW]), which will be stated as follows.

Let Ω be a bounded smooth domain in R^n with $n \geq 3$ and recall the result proved in [MW].

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PROPOSITION 1. *Let $Q(x)$ be a bounded differentiable function on Ω satisfying the following conditions*:

$$0 \leq Q(x) \leq b < \infty, \tag{1}$$

and there exist some point $0 \in \Omega$ and a integer $m > (n-2)/2$ such that

$$Q(0) = b, \ d^{(j)}Q(0) = 0, \quad for \ j = 1, \dots, m-1. \tag{2}$$

Given a function $\phi \in H^{1/2}(\partial\Omega), \phi \neq 0, \phi(x) \geq 0$ on $\partial\Omega$. Let h be the harmonic extension of ϕ and let d be a constant such that $\int_\Omega Q|h|^{2n/(n-2)} < d^{2n/(n-2)}$. Then there is a minimizer to the minimization problem

$$Inf\left\{\int_\Omega |du|^2; u \in H^1(\Omega), u = \phi \ on \ \partial\Omega, \int_\Omega Q|u|^{2n/(n-2)} = d^{2n/(n-2)}\right\}.$$

Therefore there are a positive number λ and a positive smooth function u satisfying

$$-\Delta u = \lambda Q(x) u^p \quad on \ \Omega \tag*{(3),}$$

and

$$u = \phi \quad on \ \partial\Omega \tag*{(4).}$$

where $p+1 = 2n/(n-2)$ is the Sobolev exponent.

Based on this result we will study the scalar curvature problem with the scalar curvature function changing sign. Our result in this paper is

THEOREM 1. *Suppose $K \in C^\infty(\Omega)$ changes its sign in Ω. Assume the positive part K_+ of K satisfies (1) and (2) above. Then there is a positive constant $\lambda_1 \geq (2b^{2/p+1}/nS(p+1)^2)(n/(n+2))^{(n+2)/n}(\int_\Omega K_+ h^{p+1})^{-2/n}$ (see the precise formula (9) in next section) such that for every $0 \leq \lambda \leq \lambda_1$ there is at least one positive solution to (3) and (4) with Q replaced by K, where S is the best Sobolev constant of Ω.*

Recall that $K_+ = \sup\{K, 0\}$ is the positive part and $K_- = K - K_+$ is the negative part of K. We remark that this theorem gives a good explicit lower bound estimate on λ_1. This is the reason we use our Proposition 1.

To prove this theorem, we first recall the following well-known super-sub solution method (see [Ni]).

PROPOSITION 2. *Assume the function $f \in C^3(\Omega \times R)$ and we consider the semilinear elliptic equation*

$$\Delta u + f(x,u) = 0. \tag{5}$$

If there are two functions $u_1, v_1 \in C^\infty(\Omega)$ *satisfy*

$$\Delta u_1 + f(x,u_1) \geq 0,$$

and

$$\Delta v_1 + f(x,v_1) \leq 0,$$

and $u_1 \leq v_1$, *then there is a* C^2 *solution* u *of* (5) *satisfying* $u_1 \leq u \leq v_1$.

Now the idea to prove our Theorem 1 is to use Proposition 2. We use Proposition 1 for K_+ to get a pair $(\lambda_*, v_1) \in R_+ \times C^\infty(\Omega)$. Then we use such a pair as the super-solution.

One the other hand, one can use the direct method (see [A] or our paper [M1]) to get a positive solution u_1 of (3) and (4) for the negative part K_- (note $K = K_+ + K_-$) and any $0 \leq \lambda \leq \lambda_*$. We use this u_1 as a sub-solution.

Before closing this introduction we pose a problem of studying a similar problem on a compact riemmannian manifold with boundary.

2. The Proof of the Result

We use the notation in the Introduction. Using Proposition 1 to the positive part K_+ we get the pair (v_1, λ_*). By the maximum principle (see 3.71 in [A]) we see that $v_1 > h$ on Ω. Now, because we have, for every $0 \leq \lambda \leq \lambda_*$ and for every $x \in \Omega$,

$$-\Delta v_1(x) = \lambda_* K_+(x) v_1(x)^{(n+2)/(n-2)} \geq \lambda K(x) v_1(x)^{(n+2)/(n-2)},$$

v_1 is a super-solution of (3) with this λ.

One the other hand, we define, for the negative part K_- and every $\lambda > 0$, the functional

$$I_\lambda(u) = \int_\Omega |\nabla u|^2 - \frac{\lambda}{p+1} K_- |u|^{p+1}$$

on the set

$$\mathrm{M} := \{u \in H^1(\Omega), u = \phi \text{ on } \partial\Omega\}.$$

It is easy to know that I_λ is a convex functional on M. By using the direct method (see [A] or [M1]) we find a positive smooth function u_λ satisfying (3) and (4) for λ. By the maximum principle [A] again we find that $u_\lambda < h$ on Ω. It is easy to see that u_λ is a sub-solution of (3) with $0 \leq \lambda \leq \lambda_*$.

Now for every $0 \le \lambda \le \lambda_*$, we can use v_1 as super-solution and u_λ as sub-solution as in Proposition 2 to get a positive smooth solution u of (3) and (4).

Define λ_1 be the maximum number λ such that there is a positive smooth solution to (3) and (4) for λ. Then $\lambda_1 \ge \lambda_*$.

In the following, we estimate $\lambda := \lambda_*$. For short we write $v = v_1$. Multiplying (3) by $v - h$ and integrating by part on Ω we obtain

$$\begin{aligned} \int(|\nabla v|^2 - \nabla v \nabla h) &= \lambda \int K_+(v-h)v^p = \lambda\left(d^{p+1} - \int K_+ h v^p\right) \\ &\le \lambda\left(d^{p+1} - \int K_+ h^{p+1}\right). \end{aligned} \tag{6}$$

Here we used the fact $h < v$. Because h is a harmonic function, we have

$$\int \nabla v \nabla h = \int |\nabla h|^2$$

and

$$\int(|\nabla v|^2 - \nabla v \nabla h) = \int |\nabla(v-h)|^2.$$

Using the Sobolev inequality, we get

$$\begin{aligned} \int |\nabla(v-h)|^2 &\ge S\left(\int (v-h)^{p+1}\right)^{2/p+1} \\ &\ge \frac{S}{b^{2/p+1}}\left(\int K_+(v-h)^{p+1}\right)^{2/p+1}, \end{aligned} \tag{7}$$

where S is the best Sobolev constant of Ω. Using the mean value theorem, we get

$$v^{p+1} - h^{p+1} \le (p+1)v^p(v-h)$$

and

$$\begin{aligned} \int K_+(v^{p+1} - h^{p+1}) &\le (p+1)\int K_+ v^p(v-h) \\ &\le (p+1)\left(\int K_+ v^{p+1}\right)^{p/p+1}\left(\int K_+(v-h)^{p+1}\right)^{1/p+1} \\ &= (p+1)d^p\left(\int K_+(v-h)^{p+1}\right)^{1/p+1}. \end{aligned} \tag{8}$$

Combining (6) and (7) with (8), we have

$$\left(d^{p+1} - \int K_+ h^{p+1}\right)^2 \le (p+1)^2 d^{2p} (S/b^{2/p+1}) \lambda \left(d^{p+1} - \int K_+ h^{p+1}\right).$$

Hence

$$\lambda \ge \frac{b^{2/p+1}}{S(p+1)^2} \left(\frac{d^{p+1} - \int K_+ h^{p+1}}{d^{2p}}\right). \tag{9}$$

Choose d such that $d^{p+1} = 2p/(p+1) \int_\Omega K_+ h^{p+1}$ and we get the formula wanted in our Theorem 1.

References

[A] Th. Aubin, Nonlinear Analysis on Manifolds, Monge-Ampere Equations. Springer, Berlin, 1982.

[BN] H. Brezis, L. Nirenberg, A minimization problem with critical exponents and nonzero Dirichlet data, in Symmetry in Nature, Vol. 1, SNSP 129–40, (1989).

[CS] L. Caffarelli, J. Spruck, Variational problems with critical growth and positive Dirichlet data, Indiana Univ. Math. J. Vol. 39, 1–18 (1990).

[M1] Li Ma, Conformal deformations on a non-compact Riemannian manifold. Math. Ann. **295** (1993) 75–80.

[M2] Li Ma, The Yamabe problem with Dirichlet data. C.R. Academic Sci. Paris, T 320, Serie I, 709–712 (1995).

[MW] Li Ma, Hui Wang, A minimization problem arising from prescribing scalar curvature functions, Math. Z. 222, 1–6 (1996).

[Ni] W. M. Ni, On the elliptic equation $\Delta u + K(x) u^{(n+2)/(n-2)} = 0$, its generalization and application in geometry, Indiana Univ. Math. J. **31** (1982) 493–529.

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