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ON STRONGLY ALMOST HEREDITARY RINGS

By

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M. Harada defined an almost projective module in [8] and showed that semisimple rings, serial rings, QF-rings and H-rings are well-characterized by the property of an almost projective module in [8], [9]. Using an almost projective module he further considered the following generalized condition of a hereditary ring in [7]:

$(*)_r$ Every submodule of a finitely generated projective right R-module is almost projective.

In this paper we call an artinian ring R a *right strongly almost hereditary ring* (abbreviated *right SAH ring*) if R satisfies $(*)_r$. On the other hand, an artinian hereditary ring is characterized by the following equivalent conditions:

(a) Every submodule of a projective right R-module is also projective;

(b) every submodule of a projective left R-module is also projective;

(c) every factor module of an injective right R-module is also injective;

(d) every factor module of an injective left R-module is also injective.

In section 2 we consider the following generalized condition of (c):

$(*^\sharp)_r$ Every factor module of an injective right R-module is a direct sum of an injective module and finitely generated almost injective modules.

Similarly we define $(*^\sharp)_l$ for left R-modules. The first aim of this paper is to show that an artinian ring R is right SAH if and only if R satisfies $(*^\sharp)_l$. But we see that the equivalence between a right SAH ring and an artinian ring which satisfies $(*^\sharp)_r$ does not hold in general.

In [7] M. Harada further considered the following two stronger conditions than $(*)_r$:

$(**)_r$ The Jacobson radical of M is almost projective for any finitely generated almost projective right R-module M;

$(***)_r$ every submodule of a finitely generated almost projective right R-module is also almost projective.

And he showed that an artinian ring R satisfies $(**)_r$ iff it satisfies $(***)_r$. In section 3 we consider the following generalized conditions of (c):

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$(**^{\sharp})_r$ $M/Socle(M)$ is a direct sum of an injective module and finitely generated almost injective modules for any injective or finitely generated almost injective right R-module M;

$(***^{\sharp})_r$ every factor module of an injective or finitely generated almost injective right R-module is a direct sum of an injective module and finitely generated almost injective modules.

We also consider $(**^{\sharp})_l$ and $(***^{\sharp})_l$ for left R-modules. The second aim of this paper is to show that an artinian ring R satisfies $(**)_r$ if and only if R satisfies $(**^{\sharp})_l$ if and only if R satisfies $(***^{\sharp})_l$. But we see that the equivalence between the two conditions $(**)_r$ and $(**^{\sharp})_r$ does not hold in general.

1. Preliminaries

In this paper, we always assume that every ring is a basic artinian ring with identity and every module is unitary. Let R be a ring and let $P(R) = \{e_i\}_{i=1}^n$ be a complete set of pairwise orthogonal primitive idempotents in R. We denote the *Jacobson radical*, an *injective hull* and the *composition length* of a module M by $J(M)$, $E(M)$ and $|M|$, respectively. Especially, we put $J := J(R_R)$. For a module M we denote the *socle* of M by $S(M)$ and the *k-th socle* of M by $S_k(M)$ (i.e., $S_k(M)$ is a submodule of M defined by $S_k(M)/S_{k-1}(M) = S(M/S_{k-1}(M))$ inductively).

Let M and N be modules. M is called *N-projective* (resp. *N-injective*) if for any homomorphism $\phi : M \to L$ (resp. $\phi' : L \to M$) and any epimorphism $\pi : N \to L$ (resp. monomorphism $\iota : L \to N$) there exists a homomorphism $\tilde{\phi} : M \to N$ (resp. $\tilde{\phi}' : N \to M$) such that $\phi = \pi\tilde{\phi}$ (resp. $\phi' = \tilde{\phi}'\iota$). And M is called *almost N-projective* (resp. *almost N-injective*) if for any homomorphism $\phi : M \to L$ (resp. $\phi' : L \to M$) and any epimorphism $\pi : N \to L$ (resp. monomorphism $\iota : L \to N$) either there exists a homomorphism $\tilde{\phi} : M \to N$ (resp. $\tilde{\phi}' : N \to M$) such that $\phi = \pi\tilde{\phi}$ (resp. $\phi' = \tilde{\phi}'\iota$) or there exist a nonzero direct summand N' of N and a homomorphism $\theta : N' \to M$ (resp. $\theta' : M \to N'$) such that $\phi\theta = \pi i$ (resp. $\theta'\phi' = p\iota$), where i is an inclusion of N' in N (resp. p is a projection on N' of N). Further M is called *almost projective* (resp. *almost injective*) if M is always almost N-projective (resp. almost N-injective) for any finitely generated R-module N.

We call an artinian ring R a *right almost hereditary ring* if J is almost projective as a right R-module. By [8, Theorem 1] this definition is equivalent to the condition: $J(P)$ is almost projective for any finitely generated projective right R-module P.

A module is called *uniserial* if its lattice of submodules is a finite chain, i.e., any two submodules are comparable. An artinian ring R is called a *right serial ring* if every indecomposable projective right R-module is uniserial. And we call a ring R a *serial ring* if R is a right and left serial ring. Let $f_1, f_2, \dots, f_n$ be primitive idempotents in a serial ring R. Then a sequence $\{f_1R, f_2R, \dots, f_nR\}$ (resp. $\{Rf_1, Rf_2, \dots, Rf_n\}$) of indecomposable projective right (resp. left) R-modules is called a *Kupisch series* if $f_jJ/f_jJ^2 \cong f_{j+1}R/f_{j+1}J$ (resp. $Jf_j/J^2f_j \cong Rf_{j+1}/Jf_{j+1}$) holds for any $j = 1, \dots, n-1$. Further $\{f_1R, f_2R, \dots, f_nR\}$ (resp. $\{Rf_1, Rf_2, \dots, Rf_n\}$) is called a *cyclic Kupisch series* if it is a Kupisch series with $f_nJ/f_nJ^2 \cong f_1R/f_1J$ (resp. $Jf_n/J^2f_n \cong Rf_1/Jf_1$) holds. Let R be a serial ring with a Kupisch series $\{f_1R, f_2R, \dots, f_nR\}$. If $f_nJ = 0$ and $P(R) = \{f_1, \dots, f_n\}$, then R is called a serial ring *in the first category*. And if $\{f_1R, f_2R, \dots, f_nR\}$ is a cyclic Kupisch series and $P(R) = \{f_1, \dots, f_n\}$, then R is called a serial ring *in the second category*. Moreover a serial ring is called a *strongly serial ring* if it is a direct sum of indecomposable serial rings R with a Kupisch series $\{f_{1,1}R, f_{1,2}R, \dots, f_{1,\beta_1}R, f_{2,1}R, \dots, f_{m,\beta_m}R\}$ such that $|f_{i,\beta_i}R| = 2$ for any $i = 1, \dots, m-1$ and $|f_{m,\beta_m}R| = 1$ or 2, where $P(R) = \{f_{i,j}\}_{i=1,j=1}^{m\ \beta_i}$ and $f_{i,j}R$ is injective iff $j = 1$. Then, if $|f_{m,\beta_m}R| = 1$ (resp. $= 2$), then R is a serial ring in the first (resp. second) category. Further we can easily check the following characterization of a strongly serial ring.

Lemma 1. *Let R be an indecomposable strongly serial ring with a Kupisch series $\{f_{1,1}R, f_{1,2}R, \dots, f_{1,\beta_1}R, f_{2,1}R, \dots, f_{m,\beta_m}R\}$, where $P(R) = \{f_{i,j}\}_{i=1,j=1}^{m\ \beta_i}$ and $f_{i,j}R$ is injective iff $j = 1$. Then the following hold*:

(1) $S(f_{i,j}R) \cong f_{i+1,1}R/f_{i+1,1}J$ *for any* $i = 1, \dots, m-1$ *and* $j = 1, \dots, \beta_i$ *and* $S(f_{m,k}R) \cong f_{m,\beta_m}R/f_{m,\beta_m}J$ (*resp.* $\cong f_{1,1}R/f_{1,1}J$) *for any* $k = 1, \dots, \beta_m$ *if* $|f_{m,\beta_m}R| = 1$ (*resp.* $= 2$);

(2) $\{f_{1,1}R/f_{1,1}J^j\}_{i=1}^{\beta_1+1} \cup \{f_{i,1}R/f_{i,1}J^j\}_{i=2,j=2}^{m-1\ \beta_i+1} \cup \{f_{m,1}R/f_{m,1}J^j\}_{j=2}^{\beta_m}$ (*resp.* $\{f_{i,1}R/f_{i,1}J^j\}_{i=1,j=2}^{m\ \beta_i+1}$) *is a basic set of indecomposable injective right R-modules if* $|f_{m,\beta_m}R| = 1$ (*resp.* $= 2$);

(3) $\{Rf_{m,\beta_m}, Rf_{m,\beta_m-1}, \dots, Rf_{m,1}, Rf_{m-1,\beta_{m-1}}, \dots, Rf_{1,1}\}$ *is a Kupisch series* (*resp. a cyclic Kupisch series*) *of left R-modules with* $|Rf_{i,2}| = 2$ *for any* $i = 1, \dots, m$ *and* $|Rf_{1,1}| = 1$ (*resp.* $= \beta_m + 1$) *if* $|f_{m,\beta_m}R| = 1$ (*resp.* $= 2$);

(4) $S(Rf_{1,1}) \cong Rf_{1,1}/Jf_{1,1}$ (*resp.* $\cong Rf_{m,1}/Jf_{m,1}$) *if* $|f_{m,\beta_m}R| = 1$ (*resp.* $= 2$), $S(Rf_{i,1}) \cong Rf_{i-1,1}/Jf_{i-1,1}$ *for any* $i = 2, \dots, m$, *and* $S(Rf_{k,j}) \cong Rf_{k,1}/Jf_{k,1}$ *for any* $k = 1, \dots, m$ *and* $j = 2, \dots, \beta_k$;

(5) $\{Rf_{i,1}/J^jf_{i,1}\}_{i=2,j=2}^{m\ \beta_{i-1}+1} \cup \{Rf_{m,\beta_m}/J^jf_{m,\beta_m}\}_{j=1}^{\beta_m}$ (*resp.* $\{Rf_{1,1}/J^jf_{1,1}\}_{j=2}^{\beta_m+1} \cup \{Rf_{i,1}/J^jf_{i,1}\}_{i=2,j=2}^{m\ \beta_{i-1}+1}$) *is a basic set of indecomposable injective left R-modules if* $|f_{m,\beta_m}R| = 1$ (*resp.* $= 2$).

For a set S of R-modules, a subset S' of S is called a *basic set* of S if
(a) for any $M, M' \in S'$, $M \approx M'$ as R-modules iff $M = M'$ and
(b) for any $N \in S$, there exists $M \in S'$ such that $M \approx N$ as R-modules.

2. Strongly almost Hereditary Rings

The following is a structure theorem of a right SAH ring given by M. Harada.

Theorem A ([7, Theorem 3]). *A ring is right SAH if and only if it is a direct sum of the following rings:*
(i) *Hereditary rings;*
(ii) *strongly serial rings;*
(iii) *rings R with $P(R) = \{h_1, \dots, h_m, f_1^{(1)}, f_2^{(1)}, \dots, f_{n_1}^{(1)}, f_1^{(2)}, \dots, f_{n_2}^{(2)}, f_1^{(3)}, \dots, f_{n_k}^{(k)}\}$ such that, for each $l = 1, \dots, k$ we put $S_l := \sum_{j=1}^{n_l} f_j^{(l)}$ and $H := \sum_{s=1}^{m} h_s + \sum_{l=1}^{k} f_1^{(l)}$, the following three conditions hold for any $l = 1, \dots, k$:*
(x) *$S_l R S_l$ is a strongly serial ring in the first category with a Kupisch series $\{f_1^{(l)} R S_l, f_2^{(l)} R S_l, \dots, f_{n_l}^{(l)} R S_l\}$ of right $S_l R S_l$-modules,*
(y) *$S_l R(1 - S_l) = 0$, $(h_1 + \cdots + h_m) R f_1^{(l)} \neq 0$ and $(h_1 + \cdots + h_m) \cdot R(f_2^{(l)} + \cdots + f_{n_l}^{(l)}) = 0$, and*
(z) *HRH is a hereditary ring.*

We note that by [4, Lemma 3.1] a ring in Theorem A (iii) coincides with a ring in [4, Theorem B (iii)] if it satisfies that $\alpha_l = 1$ and $S_l R S_l$ is a strongly serial ring for any $l = 1, \dots, k$, where α_l and S_l are as in it.

Moreover, the condition (ii) in the above Theorem is not the same as [7, Theorem 3], i.e., when R is a serial ring in the second category, he wrote that "R is a serial ring in the second category with $J^2 = 0$". But this original condition is not suitable. We give an example. Let R be a serial ring in the second category with $P(R) = \{f_1, f_2, f_3, f_4\}$ such that $\{f_1 R, f_2 R, f_3 R, f_4 R\}$ is a Kupisch series and $|f_1 R| = 4$, $|f_2 R| = 3$, $|f_3 R| = 2$, $|f_4 R| = 2$. Then R is a strongly serial ring. So it is right SAH by the following proof. But $J^2 \neq 0$. In an unpublished lecture note written by M. Harada the condition is already corrected. Now we give a proof with respect to this part for reader's convenience.

Proof. Assume that R is an indecomposable right SAH serial ring in the second category. And we show that R is a strongly serial ring. Let

$\{f_1R, f_2R, \dots, f_nR\}$ be a Kupisch series with $P(R) = \{f_i\}_{i=1}^n$. We may assume that f_1R is injective and $|f_1R| \geq |f_iR|$ for any $i = 1, \dots, n$.

First suppose that $f_1Jf_1 \neq 0$. Then we claim that f_1Jf_1 is simple as a right f_1Rf_1-module. Since $f_1Jf_1 \neq 0$, $f_1J^n/f_1J^{n+1} \cong f_1R/f_1J$. Then a right R-module f_1J^n is almost projective (but not projective) because R is right SAH. So $f_1R/S_i(f_1R)$ is injective for any $i = 0, \dots, n-1$ by [8, Theorem 1] since the kernel of the projective cover: $f_1R \to f_1J^n$ is $S_n(f_1R)$. Hence

(†) $\{f_1R/S_i(f_1R)\}_{i=0}^{n-1}$ is a basic set of indecomposable injective right R-modules.

Assume that $f_1J^2f_2 \neq 0$. Then $f_1J^{n+1}/f_1J^{n+2} \cong f_2R/f_2J$. On the other hand, f_1J^{n+1} is almost projective (but not projective) because R is right SAH. Therefore f_2R must be injective by [8, Theorem 1]. This contradicts with (†). So $f_1J^2f_2 = 0$. Hence f_1Jf_1 is simple as a right f_1Rf_1-module. Therefore $S(f_iR) \cong f_1R/f_1J$ for any $i = 1, \dots, n$ and $|f_nR| = 2$ since f_jR is not injective for any $j = 2, \dots, n$ by (†). In consequence, R is a strongly serial ring.

Next suppose that $f_1Jf_1 = 0$. Then we note that $f_iJf_i = 0$ for any $i = 1, \dots, n$ since $|f_1R| \geq |f_iR|$. Let k be an integer with $S(f_1R) \cong f_kR/f_kJ$. Then we claim that $S(f_jR) \cong f_kR/f_kJ$ for any $j = 1, \dots, k-1$ and $|f_{k-1}R| = 2$. Assume that $S(f_{k-1}R) \not\cong f_kR/f_kJ$. Then there exists an integer $t \geq 2$ with $f_{k-1}J \cong f_kR/f_kJ^t$ since $f_{k-1}J/f_{k-1}J^2 \cong f_kR/f_kJ$. On the other hand, $S(f_1R)$ $(\cong f_kR/f_kJ)$ is almost projective because R is right SAH. But it is not projective since R is a serial ring in the second category. So f_kR/f_kJ^i is injective for any $i = 2, \dots, |f_kR|$ by [8, Theorem 1]. Therefore $f_{k-1}J$ $(\cong f_kR/f_kJ^t)$ is injective since $t \geq 2$. This contradicts with $f_{k-1}J \subset f_{k-1}R$. So $S(f_{k-1}R) \cong f_kR/f_kJ$. Hence $S(f_jR) \cong f_kR/f_kJ$ for any $j = 1, \dots, k-1$ and $|f_{k-1}R| = 2$ hold since $S(f_1R) \cong f_kR/f_kJ$ and $f_1Jf_1 = 0$. Moreover, let $S(f_kR) \cong f_lR/f_lJ$ for some l. Then we obtain that $S(f_jR) \cong f_lR/f_lJ$ for any $j = k, \dots, l-1$ and $|f_{l-1}R| = 2$ by the same argument as f_1R. Continue this argument, we see that R is a strongly serial ring.

Conversely, assume that R is a strongly serial ring in the second category. We can show that R is right SAH by the same way as the case that R is a strongly serial ring in the first category (see the proof of [7, Theorem 3]).

The purpose of this section is to show the following theorem.

Theorem 2. *A ring R is right SAH if and only if R satisfies* $(*^{\sharp})_l$.

To complete the proof, we give a lemma.

Lemma 3. *Let R be a ring in* [4, *Theorem B* (iii)] *and we use the same* *notations as in it. Put* $E_s := E(Rh_s/Jh_s)$ *and* $E_j^{(l)} := E(Rf_j^{(l)}/Jf_j^{(l)})$ *for any* $s = 1, \dots, m$, $l = 1, \dots, k$ *and* $j = 1, \dots, n_l$. *Then the following hold for each* s, l *and* j.

(1) $HRh_s = Rh_s$, $HRf_i^{(l)} = Rf_i^{(l)}$, $HE_s = E_s$ *and* $E({}_{HRH}Rh_s/Jh_s) = E_s$ *for any* $i = 1, \dots, \alpha_l$.

(2) $E_j^{(l)} \cong Rf_{j'}^{(l)}/J^u f_{j'}^{(l)}$ *for some positive integers* j' $(\geq \alpha_l + 1)$ *and* u *and they are uniserial left* R*-modules.*

(3) $S_l E_j^{(l)} = E_j^{(l)}$ *and* $E({}_{S_lRS_l}S_lRf_j^{(l)}/S_lJf_j^{(l)}) = E_j^{(l)}$.

(4) *If* $E_j^{(l)}/N$ *is an almost injective left* R*-module for some submodule* N *of* $E_j^{(l)}$, *then it is almost injective also as a left* S_lRS_l*-module.*

(5) *If* R *satisfies* $(*^\sharp)_l$, *then so does* S_lRS_l.

Proof. (1). $HRh_s = Rh_s$, $HRf_i^{(l)} = Rf_i^{(l)}$ and $HE_s = E_s$ by [4, Theorem 3.3 (a′), (b′)]. So E_s is considered as a left HRH-module. And further we can easily see that E_s is injective also as a left HRH-module by [4, Lemma 3.1 and Theorem 3.3 (a′), (b′)] using Baer's criterion and Azumaya's theorem (see, for instance, [1, 16.13. Proposition (2)]), i.e., $E({}_{HRH}Rh_s/Jh_s) = E_s$.

(2). By $(^{**})$ in the proof for "if" part of [4, Theorem 4.1].

(3). $S_lE_j^{(l)} = E_j^{(l)}$ by (2) and [4, Lemma 3.1 and Theorem B (iii)(b)]. So $E_j^{(l)}$ is considered as a left S_lRS_l-module. And further we can easily see that $E_j^{(l)}$ is $S_lRf_i^{(l)}$-injective for any $i = 1, \dots, n_l$ by [4, Theorem 3.3 (a′), (b′)]. Therefore $E_j^{(l)}$ is injective as a left S_lRS_l-module using Baer's criterion and Azumaya's theorem (see, for instance, [1, 16.13. Proposition (2)]), i.e., $E({}_{S_lRS_l}S_lRf_j^{(l)}/S_lJf_j^{(l)}) = E_j^{(l)}$.

(4). If $E_j^{(l)}/N$ is injective as a left R-module, it is also injective as a left S_lRS_l-module by (3). Assume that a (uniserial) left R-module $E_j^{(l)}/N$ is almost injective but not injective. Then there is a positive integer p such that $J^pE(E_j^{(l)}/N) = E_j^{(l)}/N$ and $J^iE(E_j^{(l)}/N)$ is projective for any $i = 0, \dots, p-1$ by [8, Theorem 1♯]. Let j'' be an integer with $J^{p-1}E(E_j^{(l)}/N) \cong Rf_{j''}^{(l)}$. We note that $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \dots, Rf_1^{(l)}\}$ is a Kupisch series of left R-modules by [4, Lemma 3.4 (1)]. So $J^iE(E_j^{(l)}/N) \cong Rf_{j''+p-1-i}^{(l)}$ for any $i = 0, \dots, p-1$. Further $j'' \geq \alpha_l + 1$ from (2) since $Jf_{j''}^{(l)} = J^pE(E_j^{(l)}/N) = E_j^{(l)}/N$. Therefore $J^iE(E_j^{(l)}/N)$ is projective also as a left S_lRS_l-module for any $i = 0, \dots, p-1$ by [4, Lemma 3.1 and Theorem B (iii)(b)] since $j'' + p - 1 - i \geq j'' \geq \alpha_l + 1$. Hence $E_j^{(l)}/N$ is almost injective also as a left S_lRS_l-module by (3) and [8, Theorem 1♯].

(5). By (3) and (4).

Proof of Theorem 2. (⇒). We may assume that R is an indecomposable ring in (i), (ii) or (iii) of Theorem A.

Suppose that R is a hereditary ring, then it is well known that $(*^\sharp)_l$ holds (see, for instance, [4, §1 Preliminaries]).

Suppose that R is a strongly serial ring with a Kupisch series $\{f_{1,1}R, f_{1,2}R, \dots, f_{1,\beta_1}R, f_{2,1}R, \dots, f_{m,\beta_m}R\}$, where $P(R) = \{f_{i,j}\}_{i=1,j=1}^{m\ \beta_i}$ and $f_{i,j}R$ is injective iff $j = 1$. Let E be an injective left R-module and let N be a proper submodule of E. First we consider that E is indecomposable. Then $E/N \cong Rf_{m,\beta_m}/J^v f_{m,\beta_m}$ or $\cong Rf_{u,1}/J^v f_{u,1}$ by Lemma 1 (5), where u and v are positive integers. If $v \geq 2$ or $E/N \cong Rf_{m,\beta_m}/Jf_{m,\beta_m}$, then E/N is injective again by Lemma 1 (5). Assume that $E/N \cong Rf_{u,1}/Jf_{u,1}$ for some $u \in \{1, \dots, m-1\}$. Then $E/N \cong S(Rf_{u+1,1})$ by Lemma 1 (4). And $E(E/N) \cong Rf_{u+1,1}$ with $J^{\beta_u}E(E/N) = E/N$ and $J^j E(E/N) \cong Rf_{u,\beta_u - j+1}$, i.e., it is projective, for any $j = 1, \dots, \beta_u - 1$ by Lemma 1 (3), (4), (5). Therefore E/N is almost injective by [8, Theorem $1^\sharp$]. If $E/N \cong Rf_{m,1}/Jf_{m,1}$ and $|f_{m,\beta_m}R| = 1$ (resp. $= 2$), then $E/N \cong S(Rf_{m,\beta_m})$ (resp. $\cong S(Rf_{1,1})$). And we can see that E/N is almost injective by the same way as the case that $E/N \cong Rf_{u,1}/Jf_{u,1}$ for some $u \in \{1, \dots, m-1\}$. In consequence, E/N is (injective or) almost injective, if E is indecomposable. Next we consider that E is not indecomposable. Since R is a serial ring, we can represent $N = \bigoplus_{i \in I} N_i$, where N_i is a nonzero uniserial submodule of N for any $i \in I$. There is a direct summand E' of E with $E = E' \oplus (\bigoplus_{i \in I} E(N_i))$. Then $E/N \cong E' \oplus (\bigoplus_{i \in I} E(N_i)/N_i)$. Therefore E/N is a direct sum of an injective module and finitely generated almost injective modules because a uniserial module $E(N_i)/N_i$ is (injective or) almost injective for any $i \in I$ by the case that E is indecomposable.

Suppose that R is a ring in Theorem A (iii). Let E be an injective left R-module and let N be a submodule of E. We may assume that $E = (\bigoplus_{s=1}^m E(Rh_s/Jh_s)^{u_s}) \oplus (\bigoplus_{l=1,j=1}^{k\ n_l} E(Rf_j^{(l)}/Jf_j^{(l)})^{v_j^l})$, where u_s and v_j^l are non-negative integers. Put $E_1 := \bigoplus_{s=1}^m E(Rh_s/Jh_s)^{u_s}$ and $E_2 := \bigoplus_{l=1,j=1}^{k\ n_l} E(Rf_j^{(l)}/Jf_j^{(l)})^{v_j^l}$. For each $i = 1, 2$, let $\pi_i : E \to E_i$ be the projection with respect to $E = E_1 \oplus E_2$ and put $N^i := \pi_i(N)$ and $N_i := N \cap E_i$. Then there is an isomorphism $\eta : N^1/N_1 \to N^2/N_2$ with $N = \{x + y_x \mid x \in N^1, y_x \in N^2$ with $y_x + N_2 = \eta(x + N_1)\} + N_1 + N_2$ (see, for instance, [6, p449] or [3, p54]). And we claim that there exists a homomorphism $\eta' : N^1/N_1 \to N^2$ such that $v_2 \eta' = \eta$, where $v_2 : N^2 \to N^2/N_2$ is the natural epimorphism. Let H and S_l as in Theorem A (iii). By Lemmas 3 (1), (3) $HN^1 = N^1$ and $(\sum_{l=1}^k S_l)N^2 = N^2$. So we can represent $N^1/N_1 \overset{\eta}{\cong} N^2/N_2 \cong \bigoplus_{l=1}^k (Rf_1^{(l)}/Jf_1^{(l)})^{w_l}$ by the definitions of H and S_l, where $w_1, \dots, w_k$ are non-negative integers. On the other hand, $(\sum_{l=1}^k f_1^{(l)})N^2 \subseteq (\sum_{l=1}^k f_1^{(l)})E_2 \subseteq S(E_2)$ by [4, Theorem 3.3 (a′)] since $(\sum_{l=1}^k S_l)E_2 = E_2$ from Lemma 3 (3). Hence there exists a homomorphism $\eta' : N^1/N_1 \to N^2$ such that $v_2\eta' = \eta$. Then we note that $N = \{x + y_x \mid x \in N^1, y_x \in N^2$ with $y_x + N_2 = \eta(x + N_1)\} + N_1 + N_2 = \{x + \eta'(x + N_1) \mid x \in N^1\} + N_2$. Let $v_1 : N^1 \to N^1/N_1$ be the natural epimorphism and put $\psi := \eta' v_1$. Then we obtain a homomorphism $\tilde{\psi} : E_1 \to E_2$ with $\tilde{\psi}|_{N^1} = \psi$. Put $E_1(\tilde{\psi}) := \{x + \tilde{\psi}(x) \mid x \in E_1\}$ and $N^1(\tilde{\psi}) := \{x + \tilde{\psi}(x) \mid x \in N^1\}$. Then $E = E_1(\tilde{\psi}) \oplus E_2$ and $N = N^1(\tilde{\psi}) \oplus N_2$ hold because $N = \{x + \eta'(x + N_1) \mid x \in N^1\} + N_2 = \{x + \tilde{\psi}(x) \mid x \in N^1\} + N_2$. Therefore $E/N \cong (E_1(\tilde{\psi})/N^1(\tilde{\psi})) \oplus E_2/N_2 \cong E_1/N^1 \oplus E_2/N_2$ since the restrictions of π_1 induce isomorphisms $E_1(\tilde{\psi}) \cong E_1$ and $N^1(\tilde{\psi}) \cong N^1$. Now E_1/N^1 is injective by Lemma 3 (1) and Theorem A (iii)(z). And E_2/N_2 is a direct sum of (uniserial) almost injective modules by Lemma 3 (3), Theorem A (iii)(x) and the case that R is a strongly serial ring. In consequence, E/N is a direct sum of an injective module and finitely generated almost injective modules.

($\Leftarrow$). We may assume that R is an indecomposable ring satisfying $(*^{\sharp})_l$. And we show that R is a ring in either (i), (ii) or (iii) of Theorem A.

R satisfies the condition $(\sharp)_l$. So we may assume that R is a ring in either (i), (ii) or (iii) of [4, Theorem B] by [4, Theorem 4.1].

Suppose that R is a serial ring in the first category. Let $P(R) = \{g_{i,j}\}_{i=1,j=1}^{m,\ \gamma_i}$ such that $\{Rg_{1,1}, Rg_{1,2}, \ldots, Rg_{1,\gamma_1}, Rg_{2,1}, \ldots, Rg_{m,\gamma_m}\}$ is a Kupisch series and $Rg_{i,j}$ is injective iff $j = 1$. If $m = 1$, then clearly R is a strongly serial ring. Assume that $m \geq 2$. For each $i = 2, \ldots, m$, $Rg_{i,1}/Jg_{i,1}$ is almost injective by $(*^{\sharp})_l$. But it is not injective since there is a monomorphism: $Rg_{i,1}/Jg_{i,1} \to Rg_{i-1,\gamma_{i-1}}/J^2 g_{i-1,\gamma_{i-1}}$. Put $p := |E(Rg_{i,1}/Jg_{i,1})|$. Then $J^{p-1}E(Rg_{i,1}/Jg_{i,1}) = Rg_{i,1}/Jg_{i,1}$ and $J^j E(Rg_{i,1}/Jg_{i,1})$ is projective for any $j = 0, \ldots, p-2$ by [8, Theorem 1$^{\sharp}$]. So, in particular, $|J^{p-2}E(Rg_{i,1}/Jg_{i,1})| = 2$ and $J^{p-2}E(Rg_{i,1}/Jg_{i,1}) \cong Rg_{i-1,\gamma_{i-1}}$ because $Jg_{i-1,\gamma_{i-1}}/J^2 g_{i-1,\gamma_{i-1}} \cong Rg_{i,1}/Jg_{i,1}$. Therefore $|Rg_{i-1,\gamma_{i-1}}| = 2$. Further $|Rg_{m,\gamma_m}| = 1$ since R is a serial ring in the first category. Hence R is a strongly serial ring.

Next suppose that R is a serial ring in the second category. By the same argument as the case that R is a serial ring in the first category with $m \geq 2$, we see that R is a strongly serial ring.

Last suppose that R is a ring in [4, Theorem B (iii)] and we use the same notations as in it. By Theorem A and [4, Lemma 3.1] we only show that $S_l R S_l$ is a strongly serial ring and $\alpha_l = 1$ for any $l = 1, \ldots, k$. A serial ring $S_l R S_l$ satisfies $(*^{\sharp})_l$ by Lemma 3 (5). So $S_l R S_l$ is a strongly serial ring by the above case. Next we show that $\alpha_l = 1$. $Rf_{\alpha_l}^{(l)}$ has a simple subfactor which is isomorphic to Rh_s/Jh_s for some $s \in \{1, \ldots, m\}$ by the definition of α_l. Therefore there exist a submodule N of $Rf_{\alpha_l}^{(l)}$ and a nonzero homomorphism $\phi : N \to Rh_s/Jh_s$. Put $E_s := E(Rh_s/Jh_s)$ and let $\tilde{\phi} : Rf_{\alpha_l}^{(l)} \to E_s$ be an extension homomorphism of ϕ. Then we claim that $\tilde{\phi}(f_{\alpha_l}^{(l)}) \in E_s - J(E_s)$. Let $\bigoplus_{i=1}^{p} Re_i$ be the projective cover of E_s, where $\{e_1, \dots, e_p\} \subseteq P(R)$. Then $h_s Re_i \neq 0$ for any $i = 1, \dots, p$. So $e_i \notin \{f_{\alpha_l+1}^{(l)}, \dots, f_{n_l}^{(l)}\}$ because $h_s R(f_{\alpha_l+1}^{(l)} + \cdots + f_{n_l}^{(l)}) = 0$ by the definition of α_l. On the other hand, if $g \in P(R)$ with $f_{\alpha_l}^{(l)} Jg \neq 0$, then $g \in \{f_{\alpha_l+1}^{(l)}, \dots, f_{n_l}^{(l)}\}$ by [4, Theorem 3.3 (a′)]. Hence $f_{\alpha_l}^{(l)} Je_i = 0$ for any $i = 1, \dots, p$, i.e., $f_{\alpha_l}^{(l)} J(E_s) = 0$. Therefore $\tilde{\phi}(f_{\alpha_l}^{(l)}) \in E_s - J(E_s)$. So we have a submodule X of E_s with $E_s/X \cong Rf_{\alpha_l}^{(l)}/Jf_{\alpha_l}^{(l)}$. Therefore $Rf_{\alpha_l}^{(l)}/Jf_{\alpha_l}^{(l)}$ is almost injective by $(*^\sharp)_l$. But $Rf_{\alpha_l}^{(l)}/Jf_{\alpha_l}^{(l)}$ is not injective by Lemma 3 (2). Hence, put $E_{\alpha_l}^{(l)} := E(Rf_{\alpha_l}^{(l)}/Jf_{\alpha_l}^{(l)})$ and $q := |E_{\alpha_l}^{(l)}|$, then $J^{q-1}E_{\alpha_l}^{(l)} \cong Rf_{\alpha_l}^{(l)}/Jf_{\alpha_l}^{(l)}$ and $J^i E_{\alpha_l}^{(l)}$ is projective for any $i = 0, \dots, q-2$ by [8, Theorem 1$^\sharp$]. So, in particular, $S(Rf_{\alpha_l+1}^{(l)}) \cong Rf_{\alpha_l}^{(l)}/Jf_{\alpha_l}^{(l)}$ since $\{Rf_{n_l}^{(l)}, Rf_{n_l-1}^{(l)}, \dots, Rf_1^{(l)}\}$ is a Kupisch series of left R-modules by [4, Lemma 3.4 (1)]. But $S(Rf_{\alpha_l+1}^{(l)}) \cong Rf_1^{(l)}/Jf_1^{(l)}$ by [4, Lemma 3.4 (2)]. Hence $\alpha_l = 1$.

A right SAH ring does not always satisfy $(*^\sharp)_r$ and a ring satisfying $(*^\sharp)_r$ is not always a right SAH ring. Now we give an example.

Example 4. Consider a factor ring

$$R := \begin{bmatrix} D & D & 0 & D & \bar{0} & \bar{0} \\ 0 & D & 0 & D & \bar{0} & \bar{0} \\ 0 & 0 & D & D & \bar{0} & \bar{0} \\ 0 & 0 & 0 & D & D & \bar{0} \\ 0 & 0 & 0 & 0 & D & D \\ 0 & 0 & 0 & 0 & 0 & D \end{bmatrix},$$

where D is a division ring. And we consider that R is a ring by the ordinary addition and the multiplication of matrices. Put $H := e_1 + e_2 + e_3 + e_4$ and $S_1 := e_4 + e_5 + e_6$, where e_i is the (i,i)-matrix unit for any i.

Then HRH is a hereditary ring and S_1RS_1 is a strongly serial ring in the first category. And R is a ring in Theorem A(iii), i.e., R is a right SAH ring.

But we claim that R does not satisfies $(*^\sharp)_r$. e_4R is an injective right R-module with $e_4R/S(e_4R) \cong e_4R/e_4J$. And $e_4R/S(e_4R)$ is not injective. Further $e_4R/S(e_4R)$ is not almost injective by [8, Corollary 1$^\sharp$] since $e_1R \oplus e_3R$ is a projective cover of $E(e_4R/e_4J)$.

By Theorem 2 R satisfies $(*^\sharp)_l$ but is not a left SAH ring.

3. Stronger Conditions than that of a SAH Ring

The following is a structure theorem of an artinian ring which satisfies $(**)_r$ and $(***)_r$ which are stronger conditions than that of a right SAH ring:

Theorem B ([7, Theorem 4]). *For a ring the following are equivalent*:
(a) *It satisfies* $(**)_r$;
(b) *it satisfies* $(***)_r$;
(c) *it is a direct sum of the following rings*:
  (i) *Hereditary rings which are not serial*;
  (ii) *serial rings with the radical square zero*;
  (iii) *rings* R *in Theorem* A (iii) *such that* HRH *is not a serial ring and* $J(S_lRS_l)^2 = 0$ *for any* $l = 1, \dots, k$, *where* H *and* S_l *are as in Theorem* A (iii).

The purpose of this section is to show the following theorem.

Theorem 5. *For a ring* R *the following are equivalent*:
(a) R *satisfies* $(**)_r$ $(\Leftrightarrow (***)_r)$;
(b) R *satisfies* $(**^\sharp)_l$;
(c) R *satisfies* $(***^\sharp)_l$.

To complete the proof, we give a lemma.

Lemma 6. *Let* R *be a ring in* [4, *Theorem* B (iii)] *and we use the same notations as in it*.
(1) *Suppose that* $\alpha_l = 1$. *And let* M *be an indecomposable left* R*-module with* $HM = M$. *Then the following hold.*
  (i) $Rf_1^{(l)}/Jf_1^{(l)}$ *is injective as a left* HRH*-module but not injective as a left* R*-module for any* l.
  (ii) *If* M *is injective or finitely generated almost injective as a left* R*-module*, *then* M *is injective or finitely generated almost injective also as a left* HRH*-module*.
  (iii) *If* M *is finitely generated almost injective but not injective as a left* HRH*-module*, *then* M *is finitely generated almost injective but not injective also as a left* R*-module*.
(2) *Suppose that* $\alpha_l = 1$. *If* R *satisfies* $(**^\sharp)_l$, *then* HRH *also satisfies* $(**^\sharp)_l$.
(3) *Let* M *be an indecomposable left* R*-module with* $S_lM = M$ *for some* l. *Then* M *is almost injective but not injective as a left* R*-module if and only if* M *is almost injective but not injective as a left* S_lRS_l*-module*.
(4) *If* R *satisfies* $(**^\sharp)_l$, *then* S_lRS_l *also satisfies* $(**^\sharp)_l$ *for any* $l = 1, \dots, k$.

Proof. Put $E_s := E(Rh_s/Jh_s)$ and $E_j^{(l)} := E(Rf_j^{(l)}/Jf_j^{(l)})$ for any $s = 1, \dots, m$, $l = 1, \dots, k$ and $j = 1, \dots, n_l$.

(1)(i). Since $\alpha_l = 1$, $H = \sum_{s=1}^m h_s + \sum_{l=1}^k f_1^{(l)}$. So we can easily see that $Rf_1^{(l)}/Jf_1^{(l)}$ is injective as a left HRH-module by [4, Lemma 2.3 and Theorem 3.3 (a′), (b′)] using Baer's criterion. And $Rf_1^{(l)}/Jf_1^{(l)}$ is not injective as a left R-module by Lemma 3 (2).

(ii). First assume that M is injective as a left R-module. Then $M \cong E_s$ for some s by (i) since $HM = M$. Therefore M is injective also as a left HRH-module by Lemma 3 (1).

Next assume that M is finitely generated almost injective but not injective as a left R-module. Then $S({}_RM)$ is simple by [8, Theorem $1^\sharp$]. And $S({}_RM) \cong Rf_1^{(l)}/Jf_1^{(l)}$ for some l or $\cong Rh_s/Jh_s$ for some s since $HM = M$. If $S({}_RM) \cong Rf_1^{(l)}/Jf_1^{(l)}$ for some l, then M is simple, i.e., $M \cong Rf_1^{(l)}/Jf_1^{(l)}$, by [4, Theorem 3.3 (a′), (b′)] since $\alpha_l = 1$. Therefore M is injective as a left HRH-module by (i). So we consider that $S({}_RM) \cong Rh_s/Jh_s$ for some s. Then there exists a positive integer p such that $M \cong J^pE_s$ and J^iE_s is projective as a left R-module for any $i = 0, \dots, p-1$ by [8, Theorem $1^\sharp$]. And $J^jE_s = J(HRH)^jE_s$ for any $j = 0, \dots, p$ and J^iE_s is projective also as a left HRH-module for any $i = 0, \dots, p-1$ by Lemma 3 (1). So M is almost injective but not injective as a left HRH-module by [8, Theorem $1^\sharp$].

(iii). $S({}_{HRH}M)$ is simple by [8, Theorem $1^\sharp$]. But $S({}_{HRH}M) \not\cong HRf_1^{(l)}/HJf_1^{(l)}$ $(= Rf_1^{(l)}/Jf_1^{(l)})$ for any l by (i) because M is not injective as a left HRH-module. So $S({}_{HRH}M) \cong HRh_s/HJh_s$ for some s since $HM = M$. Then there is a positive integer p such that $M \cong J(HRH)^pE_s$ and $J(HRH)^iE_s$ is projective as a left HRH-module for any $i = 0, \dots, p-1$ by [8, Theorem $1^\sharp$] and Lemma 3 (1). And $J(HRH)^jE_s = J^jE_s$ for any $j = 0, \dots, p$ and $J(HRH)^iE_s$ is projective also as a left R-module for any $i = 0, \dots, p-1$ by Lemma 3 (1). So M is almost injective but not injective as a left R-module by [8, Theorem $1^\sharp$].

(2). Let M be an injective or finitely generated almost injective left HRH-module. We may assume that M is indecomposable and not simple.

Assume that M is injective as a left HRH-module. Then $M \cong E_s$ for some s by [4, Theorem 3.3 (a′), (b′)] and Lemma 3 (1) since $\alpha_l = 1$ and M is not simple. Therefore M is injective also as a left R-module. So $M/S(M)$ is a direct sum of an injective left R-module and finitely generated almost injective left R-modules by $(**^\sharp)_l$. Hence $M/S(M)$ is a direct sum of an injective left HRH-module and finitely generated almost injective left HRH-modules by (1)(ii).

Next assume that M is finitely generated almost injective but not injective as a left HRH-module. Then M is almost injective as a left R-module by (1)(iii). So

$M/S(M)$ is a direct sum of an injective left R-module and finitely generated almost injective left R-modules by $(**^\sharp)_l$. Hence $M/S(M)$ is a direct sum of an injective left HRH-module and finitely generated almost injective left HRH-modules by (1)(ii).

(3). First we note that M is a uniserial left R- and S_lRS_l-module since $S_lM = M$, M is indecomposable and a ring S_lRS_l is serial.

Assume that M is almost injective but not injective as a left R-module. Then $S(M)$ is simple by [8, Theorem $1^\sharp$]. So $E(M) \cong E_j^{(l)}$ for some j since $S_lM = M$. And there exists a positive integer p such that $M \cong J^pE_j^{(l)}$ and $J^iE_j^{(l)}$ is projective as a left R-module for any $i = 0, \dots, p-1$ by [8, Theorem $1^\sharp$]. Now $S_lE_j^{(l)} = E_j^{(l)}$ by Lemma 3 (3). And $S_l \cdot S(Rf_t^{(l)}) \neq S(Rf_t^{(l)})$ for any $t \in \{1, \dots, \alpha_l\}$ by [4, Lemma 3.1 and Lemma 3.4 (1)]. So there is $j_i \in \{\alpha_l + 1, \dots, n_l\}$ with $J^iE_j^{(l)} \cong Rf_{j_i}^{(l)}$ for any $i = 0, \dots, p-1$. Therefore $J^iE_j^{(l)} \cong S_lRf_{j_i}^{(l)}$, i.e., $J^iE_j^{(l)}$ is projective also as a left S_lRS_l-module, by [4, Theorem B (iii)(b) and Lemma 3.1] since $j_i \geq \alpha_l + 1$. Hence M is almost injective but not injective as a left S_lRS_l-module by [8, Theorem $1^\sharp$] and Lemma 3 (3).

We can show the converse by the same way.

(4). By the same way as the proof of (2) we can show using (3) and Lemma 3 (3).

PROOF OF THEOREM 5. We may assume that R is an indecomposable ring.

(a) $\Rightarrow$ (c). We may assume that R is a ring in either (i), (ii) or (iii) in Theorem B (c).

Suppose that R is a hereditary ring which are not serial. Then Rg is not injective for any $g \in P(R)$ by [7, Corollary 3]. Therefore every finitely generated almost injective left R-module is injective by [8, Theorem $1^\sharp$]. So $(***^\sharp)_l$ holds since R is a hereditary ring.

Suppose that R is a serial ring with $J^2 = 0$. Let $\{Rf_1, Rf_2, \dots, Rf_n\}$ be a Kupisch series with $\{f_1, f_2, \dots, f_n\} = P(R)$. If R is a serial ring in the first (resp. second) category, then $\{Rf_j, Rf_1/Jf_1\}_{j=1}^{n-1}$ (resp. $\{Rf_j\}_{j=1}^n$) is a basic set of indecomposable injective left R-modules. So $\{Rf_j, Rf_n, Rf_j/Jf_j\}_{j=1}^{n-1}$ (resp. $\{Rf_j, Rf_j/Jf_j\}_{j=1}^n$) is a basic set of finitely generated almost injective left R-modules by [8, Theorem $1^\sharp$]. Therefore because R is a serial ring with $J^2 = 0$, every factor module of a finitely generated almost injective module is represented as $\bigoplus_{j=1}^{n-1}((Rf_j)^{u_j} \oplus (Rf_n)^{u_n} \oplus (Rf_j/Jf_j)^{v_j})$ (resp. $\bigoplus_{j=1}^{n}((Rf_j)^{u_j} \oplus (Rf_j/Jf_j)^{v_j})$), where u_j, u_n, v_j are non-negative integers. Hence $(***^\sharp)_l$ holds.

Last suppose that R is a ring in Theorem B (c)(iii). We use the same notations as in Theorem A (iii). It is obvious that S_lRS_l is a serial ring in the first category with a Kupisch series $\{S_lRf_{n_l}^{(l)}, S_lRf_{n_l-1}^{(l)}, \dots, S_lRf_1^{(l)}\}$ of left S_lRS_l-modules from Theorem A (iii)(x). So $S_lRf_j^{(l)}$ is injective as a left S_lRS_l-module for any l and $j = 2, \dots, n_l$ since $J(S_lRS_l)^2 = 0$. Therefore $Rf_j^{(l)}$ is an injective left R-module with $|Rf_j^{(l)}| = 2$ for any l and $j = 2, \dots, n_l$ by Lemma 3 (3). On the other hand, we claim that Rh_s and $Rf_1^{(l)}$ are not injective for any s and l. Assume that Rh_s (resp. $Rf_1^{(l)}$) is injective for some s (resp. l). Then Rh_s (resp. $Rf_1^{(l)}) \cong E_{s'}$ for some s' by Lemma 3 (1) and Theorem A (iii)(y). Therefore Rh_s (resp. $Rf_1^{(l)}$) is injective also as a left HRH-module by Lemma 3 (1), i.e., there exists an injective projective left HRH-module. So HRH is a serial ring by [7, Corollary 3] and Theorem A (iii)(z). But HRH is not serial by assumption, a contradiction. In consequence, we obtain that $\{Rf_j^{(l)}\}_{l=1,j=2}^{k\ n_l}$ is a basic set of indecomposable injective projective left R-modules. Therefore $\{Rf_j^{(l)}, Jf_j^{(l)}\ (\cong Rf_{j-1}^{(l)}/Jf_{j-1}^{(l)})\}_{l=1,j=2}^{k\ n_l}$ is a basic set of finitely generated indecomposable almost injective modules by [8, Theorem $1^\sharp$]. So $(***^\sharp)_l$ holds by the same reason as the case that R is a serial ring with $J^2 = 0$.

(c) $\Rightarrow$ (b). Clear.

(b) $\Rightarrow$ (a). Since R satisfies $(**^\sharp)_l$, it satisfies $(\sharp)_l$, i.e., R is a right almost hereditary ring by [4, Theorem 4.1]. So we may assume that R is a ring in either (i), (ii) or (iii) of [4, Theorem B]. And we show that it is a ring in either (i), (ii) or (iii) of Theorem B (c).

Suppose that R is a hereditary ring. Assume that Rg is not injective for any $g \in P(R)$, then R is not serial, i.e., R is a ring in Theorem B (c)(i). Assume that there is $f \in P(R)$ with Rf injective, then R is a serial ring by [7, Corollary 3].

Suppose that R is a serial ring. Assume that there exists $f \in P(R)$ with $|Rf| \geq 3$. Then further we may assume that Rf is injective. Jf is almost injective by [8, Theorem $1^\sharp$]. And $Jf/S(Rf)$ is also almost injective by $(**^\sharp)_l$. But $Jf/S(Rf)$ is not injective since there is an inclusion map: $Jf/S(Rf) \to Rf/S(Rf)$. Therefore there exist $e \in P(R)$ and a positive integer p such that $Re \cong E(Jf/S(Rf))$, $J^pe \cong Jf/S(Rf)$ and J^ie is projective for any $i = 0, \dots, p-1$ by [8, Theorem $1^\sharp$]. So, in particular, $J^{p-1}e$ is projective. But $J^{p-1}e \cong Rf/S(Rf)$, a contradiction.

Suppose that R is a ring in [4, Theorem B (iii)]. And let H and S_l as in it. Then S_lRS_l satisfies $(**^\sharp)_l$ for any $l = 1, \dots, k$ by Lemma 6 (4). Therefore $J(S_lRS_l)^2 = 0$ from the previous case that R is a serial ring. So $\alpha_l = 1$ since $E(Rf_1^{(l)}/Jf_1^{(l)}) \cong Rf_j^{(l)}/J^uf_j^{(l)}$ for some j $(\geq \alpha_l + 1)$ and u by Lemma 3 (2). Therefore HRH also satisfies $(**^\sharp)_l$ by Lemma 6 (2). Hence HRH is not serial or serial with $J(HRH)^2 = 0$ by the previous two cases. In consequence, R is a ring in Theorem B (c)(ii) or (iii).

References

[1] F. W. Anderson and K. R. Fuller: Rings and categories of modules (second edition), Graduate Texts in Math. 13, Springer-Verlag (1991).

[2] M. Auslander: On the dimension of modules and algebras (III), global dimensions, Nagoya Math. J. **9** (1995), 66–77.

[3] Y. Baba and M. Harada, On almost M-projectives and almost M-injectives, Tsukuba J. Math. **14** No. 1 (1990), 53–69.

[4] Y. Baba and H. Miki: Symmetry of almost hereditary rings, Math. J. of Okayama Univ. 42 (2000), 29–54.

[5] M. Harada: Hereditary semi-primary rings and tri-angular matrix rings, Nagoya Math. J. **27** (1966), 463–484.

[6] ——— and A. Tozaki: Almost M-projectives and Nakayama rings, J. Algebra **122** (1989), 447–474.

[7] ———: Almost hereditary rings, Osaka J. Math. **28** (1991), 793–809.

[8] ———: Almost projective modules, J. Algebra **159** (1993), 150–157.

[9] ———: Almost QF rings and almost $QF^\sharp$ rings, Osaka J. Math. **30** (1993), 887–892.

[10] I. Murase: On the structure of generalized uni-serial rings I, Sci. Paper College Gen. Ed. Univ. Tokyo **13** (1963), 1–21.

[11] J. J. Rotman: An introduction to homological algebra, Academic Press (1979).

[12] R. Wisbauer: Foundations of module and ring theory, Gordon and Breach Science Publishers (1991).

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