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ON SHIMURA LIFTING OF MODULAR FORMS

By

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0. Let $k, N \in \boldsymbol{N}$, $4|N$. Let $M_{k+1/2}(N, \chi_0)$ denote the space of modular forms for $\Gamma_0(N)$ of weight $k+1/2$ with a character $\chi_0 \pmod N$. Lifting maps of cusp forms in $M_{k+1/2}(N, \chi_0)$ to modular forms of integral weight was first studied by Shimura [7] and later by Niwa [4]. The domain of the map is extended to $M_{k+1/2}(N, \chi_0)$ by van Asch [1] in case that χ_0 is real and $N = 4p$ for p prime, and by Pei [5] in case that χ_0 is real and $N/4$ is square-free. In the present paper we consider the lifting map without any condition on N and χ_0, and extend the domain of the map to $M_{k+1/2}(N, \chi_0)$ for $k \geq 2$.

To show the assertion, we take some specific modular forms in $M_{k+1/2}(N, \chi_0)$ which together with cusp forms, span $M_{k+1/2}(N, \chi_0)$. Further we construct their liftings explicitly. It proves our main result. It may be expected to have further application to study of special values of L-series of Hecke eigen cusp forms, as in Zagier [9], Kohnen-Zagier [3] where the lifting of some particular modular forms plays an important role.

1. We denote by $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{C}$, the set of natural numbers, the ring of integers and the complex number field respectively. For a prime $p \in \boldsymbol{N}, v_p$ denotes the p-adic valuation. For $N \in \boldsymbol{N}, (\boldsymbol{Z}/N)^*$ denotes the group of Dirichlet characters $(\mathrm{mod}\, N)$. When $N = 1$, the group is consisting of a constant 1. The identity element of $(\boldsymbol{Z}/N)^*$ is denoted by 1_N. A group consisting of invertible elements in $\boldsymbol{Z}/N$ is denoted by $(\boldsymbol{Z}/N)^\times$. If $\chi \in (\boldsymbol{Z}/N)^*$ and $e \in \boldsymbol{N}$, the $\chi^{(e)}$ denotes a character $(\mathrm{mod}\, eN)$ obtained by $\chi^{(e)}(d) = \chi(d)$ $((d, e) = 1)$, 0 $((d, e) \neq 1)$. In case that all prime factors of e appear as factors of N, then $\chi^{(e)}$ is equal to χ. For $a \in \boldsymbol{Z}$ and for an odd $b \in \boldsymbol{N}$, (a/b) denotes the Jacobi-Legendre symbol where it is 0 if $(a, b) \neq 1$. If D is a discriminant of a quadratic field, then χ_D denote the Kronecker-Jacobi-Legendre symbol. We put $\chi_D = 1$ for $D = 1$.

Let $\mathfrak{H}$ denote the upper-half plane $\{z \in \boldsymbol{C} \mid \mathrm{Im}\, z > 0\}$. The group $SL_2(\boldsymbol{Z})$ acts on $\mathfrak{H}$ by the usual modular transformation sending $z \in \mathfrak{H}$ to $Mz = (az+b)/$

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$(cz+d), M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $N \in \boldsymbol{N}$ which is not necessarily divisible by 4, let $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid c \equiv 0 \pmod{N} \right\}$, $\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid a \equiv d \equiv 1, c \equiv 0 \pmod{N} \right\}$. Let $k \in \boldsymbol{N}$ and $\chi_0 \in (\boldsymbol{Z}/N)^*$. A holomorphic function f on $\mathfrak{H}$ is called *a modular form for* $\Gamma_1(N)$ of weight k (resp. *a modular form for* $\Gamma_0(N)$ of weight k with character χ_0) if it satisfies that (i) $(f|_M)(z) = f(z)$ for $M \in \Gamma_1(N)$ (resp. $(f|_{M,\chi_0})(z) = f(z)$ for $M \in \Gamma_0(N)$), and that (ii) f is holomorphic also at cusps, where $(f|_M)(z) = (cz+d)^{-k} f(Mz)$ (resp. $(f|_{M,\chi_0})(z) = \chi_0(d)^{-1}(cz+d)^{-k} f(Mz)$) for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We denote by $M_k(\Gamma_1(N))$ (resp. $M_k(N,\chi_0)$), the space of such modular forms, and by $S_k(\Gamma_1(N))$ (resp. $S_k(N,\chi_0)$), the space of such cusp forms. The orthogonal complement of $S_k(\Gamma_1(N))$ (resp. $S_k(N,\chi_0)$) in $M_k(\Gamma_1(N))$ (resp. $M_k(N,\chi_0)$) with respect to the Petersson product, is called a space of Eisenstein series, and it is denoted by $E_k(\Gamma_1(N))$ (resp. $E_k(N,\chi_0)$). We have

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi_0} M_k(N,\chi_0), \quad E_k(\Gamma_1(N)) = \bigoplus_{\chi_0} E_k(N,\chi_0)$$

where χ_0 runs over $(\boldsymbol{Z}/N)^*$.

The number $\nu_0(N)$ (resp. $\nu_1(N)$) of inequivalent cusps of $\Gamma_0(N)$ (resp. $\Gamma_1(N)$) is equal to $\sum_{N_1N_2=N} \varphi((N_1,N_2))$ (resp. 2 ($N=2$), 3 ($N=4$), $1/2\sum_{N_1N_2=N} \varphi(N_1) \cdot \varphi(N_2)$ ($N \geqq 3, \neq 4$)) where φ denotes the Euler function and N_i's are positive divisors of N. A complete system of representatives of inequivalent cusps of $\Gamma_0(N)$ is given by $\{i/N_2 | 0 < N_2, N_2|N$, i's being representatives of $(\boldsymbol{Z}/(N_1,N_2))^\times\}$. That of $\Gamma_1(N)$ is given as follows. Let $i \in (\boldsymbol{Z}/N_2)^\times, j \in (\boldsymbol{Z}/N_1)^\times$. Then for suitable $a, b \in \boldsymbol{Z}$, a fraction $(i+aN_2)/((j+bN_1)N_2)$ $((j+bN_1,N_2)=1)$ is reduced, one of which we denote by $[i,j,N_2;N]$. Then a complete system is $\{[i,j,N_2;N] | N_1N_2 = N, (i,j) \in (\boldsymbol{Z}/N_2)^\times \times (\boldsymbol{Z}/N_1)^\times/\{\pm 1\}\}$. By abuse of language, the rational numbers sometimes mean cusps on the corresponding modular curve.

LEMMA 1. (1) *Via the canonical map of the modular curve of* $\Gamma_1(N)$ *onto that of* $\Gamma_0(N)$, *a cusp* $[i,j,N_2;N]$ *is mapped to a cusp* ij^{-1}/N_2 *where* j^{-1} *is denoting an inverse of* $j \pmod{N_1}$.

(2) *Let* $N = N_1N_2$ *and let* $N_2|M, M|N$. *If* ρ *denotes the natural surjective map of the modular curve of* $\Gamma_1(N)$ *onto that of* $\Gamma_1(M)$, *then* $\rho^{-1}([i,j,N_2;M]) = \{[i',j',eN_2;N] \mid e|(N/M), v_p((e,N_1)) = 0$ *for any* p *with* $v_p(N_2) < v_p(M), i' \equiv i \pmod{eN_2}, j' \equiv j \pmod{N_1/e}\}$.

(3) *Let* $e|N, N_2|(N/e)$. *Let* ρ_e *be the surjective map of the modular curve of* $\Gamma_1(N)$ *onto that of* $\Gamma_1(N/e)$ *associated with* $z \to ez$. *Then* $\rho_e^{-1}([i, j, N_2; N/e]) = \{[i', j', e'N_2; N] \mid e'|e, (e/e', N_2) = 1, i'e \equiv i \,(\mathrm{mod}\, e'N_2), j' \equiv j \,(\mathrm{mod}\, N/e'N_2)\}$.

The lemma is easy to show, and so we skip the proof.

Now we consider Eisenstein series for $\Gamma_1(N)$. The result is an imitation of Hecke [2], Sect. 1 and 2. So we omit the detail. Let $k \in \boldsymbol{N}$. Let $N = N_1N_2$ ($N_1, N_2 \in \boldsymbol{N}$), and $a_1, a_2 \in \boldsymbol{Z}$. Then we put

$$E_k(z, a_1, a_2/N_2, N_1) := \sum_{\substack{m_1 \equiv a_1 \,(\mathrm{mod}\, N_1) \\ m_2 \equiv a_2/N_2 \,(\mathrm{mod}\, 1)}}{}' (m_1z + m_2)^{-k}|m_1z + m_2|^{-s}\,|_{s=0},$$

where the summation is over the pairs of numbers (m_1, m_2) such that m_1 runs through all the integers satisfying the congruence and m_2 runs through all the rational numbers such that $m_2 - a_2/N_2 \in \boldsymbol{Z}$, and where the notation $\sum'$ indicates as usual, that $(m_1, m_2) = (0, 0)$ is to be omitted. This is an element of $E_k(\Gamma_1(N))$ if $k \neq 2$. The space $E_k(\Gamma_1(N))$ is zero if k is odd and $N = 1, 2$. We exclude these cases from out argument. We have a Fourier expansion

$$E_k(z, a_1, a_2/N_2, N_1) = N^k \sum_{m \equiv a_2 \,(\mathrm{mod}\, N)}{}' m^{-k}|m|^{-s}\,|_{s=0} \quad (a_1 \in N_1\boldsymbol{Z})$$

$$+ \frac{(-2\sqrt{-1}\pi)^k}{(k-1)!} \sum_{n=1}^{\infty} \sum_{\substack{n/m \equiv a_1 \,(\mathrm{mod}\, N_1) \\ m \in \boldsymbol{Z}}} \mathrm{sgn}(m) e(a_2m/N_2) m^{k-1} e(nz),$$

where there is an additional term $-\sqrt{-1}\pi \sum'_{m \equiv a_1 \,(\mathrm{mod}\, N_1)} \mathrm{sgn}(m)|m|^{-s}\,|_{s=0}$ (resp. $\pi/(N_1 \mathrm{Im}(z))$) if $k = 1$ (resp. $k = 2$) and where the first term appears if the condition $a_1 \in N_1\boldsymbol{Z}$ is satisfied. When $k = 2$, the differences of Eisenstein series are contained in $E_2(\Gamma_1(N))$. If e is a common divisor of a_1, N_1, then $E_k(z, a_1, a_2/N_2, N_1) = E_k(ez, a_1/e, a_2/N_2, N_1/e)$.

Suppose that $(a_1, N_1) = (a_2, N_2) = 1$. Put

$$E_k^*(z, a_1, a_2/N_2, N_1) := \sum_{\substack{m_1 \equiv a_1 \,(\mathrm{mod}\, N_1) \\ m_2 \equiv a_2/N_2 \,(\mathrm{mod}\, 1) \\ (m_1, N_2m_2) = 1}}{}' (m_1z + m_2)^{-k}|m_1z + m_2|^{-s}\,|_{s=0}.$$

It is easily shown to be a modular form of weight k for $\Gamma_1(N)$ if $k \neq 2$. If $k = 2$, then it satisfies the same transformation law as a modular form of weight 2, but it is not holomorphic. When $k \geq 2$, it vanishes at all the cusps of $\Gamma_1(N)$ but $[a_2, a_1, N_2; N]$. In particular $\{E_k^*(z, a_1, a_2/N_2, N_1) | N_1N_2 = N, (a_1, N_1) = (a_2, N_2) = 1\}$ spans $E_k(\Gamma_1(N))$ for $k \geq 3$. The Eisenstein series $E_k(z, a_1, a_2/N_2, N_1)$ (resp.

$E_k^*(z, a_1, a_2/N_2, N_1))$ is written as a linear combination of $E_k^*(z, ta_1, ta_2/N_2, N_1)$ (resp. $E_k(z, ta_1, ta_2/N_2, N_1)$) with $t \in (\boldsymbol{Z}/N)^\times$. Let us denote by X_{k,N_1,N_2}, the set $\{E_k(z, a_1, a_2/N_2, N_1) | (a_1, a_2) \in (\boldsymbol{Z}/N_1)^\times \times (\boldsymbol{Z}/N_2)^\times / \{\pm 1\}\}$, and by $X_{k,N}$, the union $\bigcup_{N_1 N_2 = N} X_{k,N_1,N_2}$.

PROPOSITION 1. *Let $N, k \in \boldsymbol{N}$. (1) Let $k \geq 2$. Then Eisenstein series in X_{k,N_1,N_2} separate the cusps $[a_2, a_1, N_2; N], (a_1, a_2) \in (\boldsymbol{Z}/N_1)^\times \times (\boldsymbol{Z}/N_2)^\times / \{\pm 1\}$ and vanish at all other cusps. Eisenstein series in $X_{k,N}$ separate all the cusps of $\Gamma_1(N)$.*

(2) Let $k \geq 3$. The space $E_k(\Gamma_1(N))$ is spanned by $X_{k,N}$. The dimension is equal to 0 if k is odd and $N = 1, 2$ and to $v_1(N)$ if otherwise.

(3) Let $k = 2$. Then linear combinations of elements in $X_{2,N}$ which is holomorphic, span $E_2(\Gamma_1(N))$ whose dimension is $v_1(N) - 1$.

(4) Let $k = 1$ and $N \geq 3$. Then $X_{1,N}$ spans $E_k(\Gamma_1(N))$ whose dimension is $v_1(N)/2$.

2. We introduce Eisenstein series which are suitable for study of Shimura lifting in our method. Let $N = N_1 N_2$ as above. For $a_1, a_2 \in \boldsymbol{Z}$, let

$$
\begin{aligned}
& G_k(z, a_1, a_2, N_1, N_2) \\
& := \frac{2(k-1)!}{(2\sqrt{-1}\pi)^k N_2} \sideset{}{'}\sum_{\substack{m_1 \equiv -a_1 \pmod{N_1} \\ m_2 \in (1/N_2)\boldsymbol{Z}}} e(a_2 m_2)(m_1 z + m_2)^{-k} |m_1 z + m_2|^{-s} |_{s=0} \\
& = \frac{2(k-1)!}{(2\sqrt{-1}\pi)^k N_2} \sum_{a : \boldsymbol{Z}/N_2} e\left(\frac{a a_2}{N_2}\right) E_k(z, -a_1, a/N_2, N_1) \\
& = \sideset{}{'}\sum_{m \equiv a_2 \pmod{N_2}} \operatorname{sgn}(m) |m|^{k-1-s} |_{s=0} \quad (a_1 \in N_1 \boldsymbol{Z} \text{ and } 2 \nmid k) \\
& \quad + \sideset{}{'}\sum_{m \equiv a_2 \pmod{N_2}} |m|^{k-1-s} |_{s=0} \quad (a_1 \in N_1 \boldsymbol{Z} \text{ and } 2 | k) \\
& \quad + \frac{1}{2\pi N_1 y} \quad (k = 2 \text{ and } N_2 = 1) \\
& \quad + \sum_{m \equiv a_1 \pmod{N_1}} (\operatorname{sgn} m) |m|^{-s} |_{s=0} \quad (k = 1 \text{ and } N_2 = 1) \\
& \quad + 2 \sum_{n=1}^{\infty} \sum_{\substack{m \equiv a_2 \pmod{N_2} \\ n/m \equiv a_1 \pmod{N_1} \\ m \in \boldsymbol{Z}}} \operatorname{sgn}(m) m^{k-1} e(nz),
\end{aligned}
$$

where the first and forth terms should be understood to be special values of partial zeta functions at non-positive integers. Obviously $G_k(z,a_1,a_2,N_1,N_2) = G_k(z,a_1',a_2',N_1,N_2)$ for $a_1 \equiv a_1' \pmod{N_1}, a_2 \equiv a_2' \pmod{N_2}$, and $G_k(z,-a_1,-a_2,N_1,N_2) = (-1)^k G_k(z,a_1,a_2,N_1,N_2)$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, there holds $G_k(z,a_1,a_2,N_1,N_2)|_M = G_k(z,aa_1,da_2,N_1,N_2)$. If e is a common divisor of a_1,N_1 (resp. a_2,N_2), then $G_k(z,a_1,a_2,N_1,N_2)$ is equal to $G_k(ez,a_1/e,a_2,N_1/e,N_2)$ (resp. $e^{k-1}G_k(ez,a_1,a_2/e,N_1,N_2/e)$).

Let $\chi' \in (\boldsymbol{Z}/N_1)^*, \chi \in (\boldsymbol{Z}/N_2)^*$. We define an arithmetic function $\sigma_{k,\chi}^{\chi'}$ by setting

$$\sigma_{k,\chi}^{\chi'}(n) = \sum_{0<d|n} \chi'(n/d)\chi(d)d^k, \quad n \in \boldsymbol{N}.$$

Further we define $\sigma_{k,\chi}^{\chi'}(n)$ to be 0 if $n \notin \boldsymbol{N} \cup \{0\}$. If $N_1 = 1$ (resp. $N_2 = 1$), then we denote it by $\sigma_{k,\chi}$ (resp. $\sigma_k^{\chi'}$). Now assume that $\chi\chi'$ has the same parity as k, namely $\chi\chi'(-1) = (-1)^k$. Then we put

$$\begin{aligned} G_{k,\chi}^{\chi'}(z) &:= \frac{1}{2} \sum_{\substack{a_1:(\boldsymbol{Z}/N_1) \\ a_2:(\boldsymbol{Z}/N_2)}} \chi'(a_1)\chi(a_2)G_k(z,a_1,a_2,N_1,N_2) \\ &= L(1-k,\chi)(N_1 = 1) + \frac{1}{4\pi N_1 y} \sum_{a_1:(\boldsymbol{Z}/N_1)} \chi'(a_1) \quad (k = 2 \text{ and } N_2 = 1) \\ &\quad + (\sqrt{-1}\pi)^{-1}L(1,\chi') \quad (k = 1 \text{ and } N_2 = 1) + 2\sum_{n=1}^{\infty} \sigma_{k-1,\chi}^{\chi'}(n)e(nz). \end{aligned}$$

For $k = 1, G_{1,\chi}^{\chi'}(z) = G_{1,\chi'}^{\chi}(z)$. If $N_1 = 1$ (resp. $N_2 = 1$), then we denote the Eisenstein series by $G_{k,\chi}(z)$ (resp. $G_k^{\chi'}(z)$). If $N_1 = N_2 = 1$, then it is denoted by $G_k(z)$. We define $\sigma_{k-1,\chi}^{\chi'}(0)$ to be the half of the constant term of the Fourier expansion of $G_{k,\chi}^{\chi'}(z)$ at the cusp $\sqrt{-1}\infty$. Hence it is 0 if $N_1 > 0$ and $k > 1$. Let us set

$$Y_{k,N_1,N_2} := \{G_{k,\chi}^{\chi'}(ez) \mid e|N_2, \chi' \in (\boldsymbol{Z}/N_1)^*, x \in (\boldsymbol{Z}/(N_2/e))^*, \chi\chi'(-1) = (-1)^k\}.$$

The Eisenstein series $G_{k,\chi}^{\chi'}(ez)$ in Y_{k,N_1,N_2} is holomorphic if and only if $k \neq 2$ or it is not in the form $G_2^{\chi'}(N_2 z)$ with $\chi' = 1_{N_1}$.

LEMMA 2. *Let $k \geq 2$. Let $N = N_1N_2$. Then the $\boldsymbol{C}$-span of Y_{k,N_1,N_2} is equal to the $\boldsymbol{C}$-span of X_{k,N_1,M_2}'s, $M_2|N_2$.*

PROOF. By definition, the former is obviously contained in the latter. We must show the converse. Let $a_1 \in (\boldsymbol{Z}/N_1), a \in \boldsymbol{Z}$. Then

$$\begin{aligned}
&G_k(z, a_1, a, N_1, N_2) \\
&\quad = (a, N_2)^{k-1} G_k((a, N_2)z, a_1, a/(a, N_2), N_1, N_2/(a, N_2)) \\
&\quad = 2(a, N_2)^{k-1} \varphi(N_1)^{-1} \varphi(N_2)^{-1} \\
&\quad\quad \times \sum_{\substack{\chi' \in (\boldsymbol{Z}/N_1)^* \\ \chi \in (\boldsymbol{Z}/(N_2/(a, N_2)))^*}} \chi'(a_1)^{-1} \chi(a/(a, N_2))^{-1} G_{k,\chi}^{\chi'}((a, N_2)z).
\end{aligned}$$

Hence the $\boldsymbol{C}$-span of Y_{k,N_1,N_2} contains $G_k(z, a_1, a, N_1, N_2), a_1 \in (\boldsymbol{Z}, N_1), a \in \boldsymbol{Z}$. For $a_2 \in \boldsymbol{Z}$ we have

$$E_k(z, -a_1, a_2/N_2, N_1) = \frac{(2\sqrt{-1}\pi)^k}{2(k-1)!} \sum_{a \in \boldsymbol{Z}/N_2} e\left(-\frac{aa_2}{N_2}\right) G_k(z, a_1, a, N_1, N_2).$$

This shows our assertion. q.e.d.

COROLLARY. *Let $k \geq 2$. Let $N = N_1 N_2$.*

(1) *An Eisenstein series in Y_{k,N_1,N_2} vanishes at cusps $[*, *, N_2'; N]$ for N_2' with $N_2' \nmid N_2$.*

(2) *The elements of Y_{k,N_1,N_2} separate cusps $[a_1, a_2, N_2; N]$ $(a_1 \in (\boldsymbol{Z}/N_1), a_2 \in (\boldsymbol{Z}/N_2)^\times)$ of $\Gamma_1(N)$.*

PROOF. Let M_2 be a divisor of N_2. An Eisenstein series in X_{k,N_1,M_2} vanishes at $[*, *, N_2'; N_1 M_2]$ with $N_2' \nmid M_2$ by Proposition 1 (2). Hence by Lemma 1 (2), it does at $[*, *, N_2'; N]$ with $N_2' \nmid N_2$. This shows the first assertion. The second assertion follows from Proposition 1 (1). q.e.d.

LEMMA 3. *Let $G_{k,\chi}^{\chi'}(ez) \in Y_{k,N_1,N_2}$. Suppose that χ is not a primitive character* (mod N_2/e). *Then one of the following holds;*

(1) *$G_{k,\chi}^{\chi'}(ez)$ vanishes at cusps $[*, *, N_2; N]$, or*

(2) *there is $G_{k,\omega}^{\chi'}(ee'z) \in Y_{k,N_1,N_2}$ with $e' > 1$, $\omega \in (\boldsymbol{Z}/(N_2/ee'))^*$ whose constant multiple takes the same value as $G_{k,\chi}^{\chi'}(ez)$ at each cusp $[*, *, N_2; N]$.*

PROOF. Let M_2 be a conductor of χ, which is a proper divisor of N_2/e, and let ω be the primitive character (mod M_2) associated with χ. There are two cases that (i) $v_p(M_2) > 0$ for any prime factor p of N_2/e, and that (ii) there is a prime factor p of N_2/e with $v_p(M_2)=0$. Let us consider the case (i). Then $G_{k,\chi}^{\chi'}(ez)=G_{k,\omega}^{\chi'}(ez)$, and it is in Y_{k,N_1,eM_2}, or in the linear span of X_{k,N_1,cM_2}. Then by Proposition 1 (1), it vanishes at cusps $[*,*,N_2';N]$ with $N_2'\nmid eM_2$, in particular at $[*,*,N_2;N]$. Now we consider the case (ii). Let $\{p_1,\dots,p_s\}$ be all the prime factors of N_2/e relatively prime to M_2. Then we have equalities

$$\begin{aligned}\sigma_{k,\omega}^{\chi'}(n)&=\sigma_{k,\chi}^{\chi'}(n)+\sum_{i=1}^{s}p_i^{k-1}\omega(p_i)\sigma_{k,\omega}^{\chi'}(n/p_i)\\&\quad-\sum_{1\le i,j\le s}(p_ip_j)^{k-1}\omega(p_ip_j)\sigma_{k,\omega}^{\chi'}(n/p_ip_j)\\&\quad+\cdots+(-1)^s(p_1\cdots p_s)^{k-1}\omega(p_1\cdots p_s)\sigma_{k,\omega}^{\chi'}(n/p_1\cdots p_s)\end{aligned}$$

for $n\in N$ and

$$\begin{aligned}G_{k,\chi}^{\chi'}(ez)&=G_{k,\omega}^{\chi'}(ez)-\sum_{i=1}^{s}p_i^{k-1}\omega(p_i)G_{k,\omega}^{\chi'}(p_iez)\\&\quad+\sum_{1\le i,j\le s}(p_ip_j)^{k-1}\omega(p_ip_j)G_{k,\omega}^{\chi'}(p_ip_jez)\\&\quad-\cdots+(-1)^{s+1}(p_1\cdots p_s)^{k-1}\omega(p_1\cdots p_s)G_{k,\omega}^{\chi'}(p_1\cdots p_sez).\end{aligned}$$

Except for the last one, Eisenstein series of the right hand side vanish at cusps $[*,*,N_2;N]$ since they are in $Y_{k,N_1,N_2''}$ with proper divisors N_2'' of N_2. If the last one vanishes, the $G_{k,\chi}^{\chi'}(ez)$ does also. The last one does not vanish only when $N_2=p_1\cdots p_seM_2$. Since $G_{k,\chi}^{\chi'}(ez)$ is equal to $G_{k,\omega}^{\chi'}(p_1\cdots p_sez)\in Y_{k,N_1,N}$ up to a constant multiple, our assertion is proved. q.e.d.

Let us set

$$Z_{k,N_1,N_2}:=\{G_{k,\chi}^{\chi'}(ez)\mid e|N_2,\chi'\in(\boldsymbol{Z}/N_1)^*,$$
$$\text{primitive}\quad\chi\in(\boldsymbol{Z}/(N_2/e))^*,(\chi'\chi)(-1)=(-1)^k\}.$$

The set Z_{k,N_1,N_2} is a subset of Y_{k,N_1,N_2}. However as Lemma 3 shows, the separations of cusps $[*,*,N_2;N]$ by elements in Z_{k,N_1,N_2}, and by elements in Y_{k,N_1,N_2}, are the same.

Here we note that if $v_2(N_2/e)=1$, then no characters in $(\boldsymbol{Z}/(N_2/e))^*$ are primitive.

PROPOSITION 2. *Let $k \geq 2$ and let $k_0 \geq 1$. Let S be a set of some divisors of N. Let f_{e,N_2} $(N_2 \in S, e|N_2)$ be modular forms for $\Gamma_1(N)$ of weight k_0 such that f_{e,N_2} takes nonzero values at cusps $[*,*,N_2;N]$. Then $\{f_{e,N_2}(z)G_{k,\chi}^{\chi'}(ez)|N_1N_2=N$, $N_2 \in S, G_{k,\chi}^{\chi'}(ez) \in Z_{k,N_1,N_2}\}$ separates the cusps $[*,*,N_2;N]$ $(N_2 \in S)$. If S is the set of all divisors of N, then it separates all the cusps of $\Gamma_1(N)$.*

PROOF. Let r, s be two distinct cusps of $\Gamma_1(N)$. At first suppose that they are in the form $r=[*,*,N_2;N], s=[*,*,N_2';N]$ with $N_2 \neq N_2', N_2, N_2' \in S$. Replacing N_2 and N_2' if necessary, we may assume that $N_2' \nmid N_2$. Then Corollary to Lemma 2 shows that there is an element of Y_{k,N_1,N_2} which vanishes at s and does not vanish at r. By Lemma 3 such an element exists also in Z_{k,N_1,N_2}. Then the assertion immediately follows in this case. Now suppose that r, s are both in the form $[*,*,N_2;N]$. Let e be a maximal divisor of N_2 so that $\rho_e(r) \neq \rho_e(s)$ (see Lemma 1 (3)). By Proposition 1 (1), $E_k(z,-a_1,a_2/(N_2/e),N_1), a_1 \in (\boldsymbol{Z}/N_1), a_2 \in (\boldsymbol{Z}/(N_2/e))$, separate $\rho_e(r)$ and $\rho_e(r)$. They are written as linear combinations of $G_{k,\chi}^{\chi'}(e''z), \chi' \in (\boldsymbol{Z}/N_1)^*, e''|(N_2/e), \chi \in (\boldsymbol{Z}/(N_2/ee''))^*$. The maximality of e implies that each $G_{k,\chi}^{\chi'}(e''z)$ $(e''>1)$ takes the same value at $\rho_e(r)$ and at $\rho_e(s)$. By Lemma 3 $G_{k,\chi}^{\chi'}(z)$ with $\chi \in (\boldsymbol{Z}/(N_2/e))^*$ not primitive, also takes the same value. It follows that $G_{k,\chi}^{\chi'}(z)$'s, with primitive $\chi \in (\boldsymbol{Z}/(N_2/e))^*$, separate these two cusps. Hence $G_{k,\chi}^{\chi'}(ez)$'s, with $\chi' \in (\boldsymbol{Z}/N_1)^*$ and primitive $\chi \in (\boldsymbol{Z}/(N_2/e))^*$, separate r and s. Since $f_{e,N_2}(z)$ vanishes at neither r nor s, $f_{e,N_2}(z)G_{k,\chi}^{\chi'}(ez)$'s separate r and s. This shows our assertion. q.e.d.

Take characters $\chi' \in (\boldsymbol{Z}/N_1)^*$ and $\chi \in (\boldsymbol{Z}/(N_2/e))^*$. Let $\chi_0 := \chi'\chi$ be a character $(\bmod N)$. Then obviously $G_{k,\chi}^{\chi'}(ez)$ satisfies

$$G_{k,\chi}^{\chi'}(ez)|_{M,\chi_0} = G_{k,\chi}^{\chi'}(eMz), \quad M \in \Gamma_0(N),$$

namely it is in $M_k(N,\chi_0)$. As a corollary to the above proof, we obtain the following;

COROLLARY 1. (1) *Suppose that $k \neq 2$ or $\chi_0 \neq 1_N$. Then $\{G_{k,\chi}^{\chi'}(ez)|N_1N_2=N$, $e|N_2, \chi' \in (\boldsymbol{Z}/N_1)^*$, primitive $\chi \in (\boldsymbol{Z}/(N_2/e))^*, (\chi'\chi)^{(e)} = \chi_0\}$ forms a basis of $E_k(N,\chi_0)$.*

(2) *Let $k=2$. Then $\{G_{2,\chi}^{\chi'}(ez)|N_1N_2=N, e|N_2, e \neq N_2, \chi' \in (\boldsymbol{Z}/N_1)^*$, primitive $\chi \in (\boldsymbol{Z}/(N_2/e))^*, (\chi'\chi)^{(e)} = 1_N\} \cup \{G_2(N_2z)-(1/N)_2G_2(z) \mid N_2|N\}$ is a basis of $E_2(N,1_N)$.*

PROOF. In the proof of Lemma 2 we showed that $E_k(z, -a_1, a_2/N_2, N)$ is written as a linear combination of $G_{k,\chi}^{\chi'}(ez)$ in $\bigcup_{N_1N_2=N} Y_{k,N_1,N_2}$. In the proof of Lemma 3, we actually showed that $G_{k,\chi}^{\chi'}(ez)$ with χ not primitive, is written as a linear combinations of ones with primitive χ. The argument is valid also for $k=1$. By Proposition 1, $\bigcup_{N_1N_2=N} Z_{k,N_1,N_2}$ spans $E_k(\Gamma_1(N))$ for $k \neq 2$. For $k=2$, linear combinations of elements in Z_{2,N_2,N_2}'s which are holomorphic, span $E_2(\Gamma_1(N))$. In the former case $\bigcup_{N_1N_2=N} Z_{k,N_1,N_2}$ forms a basis since the number of elements equals the dimension, and in the latter, $\bigcup_{N_1N_2=N} [Z_{2,N_1,N_2} - \{G_2^{1_{N_1}}(N_2z)\}] \cup \{G_2(N_2z) - (1/N_2)G_2(z) \mid N_2|N\}$ forms a basis. The space $E_k(N,\chi_0)$ is the invariant subspace of $E_k(\Gamma_1(N))$ under the action of $f \to f|_{M,\chi_0}$, $M \in \Gamma_0(N)$, and there is the decomposition $E_k(\Gamma_1(N)) = \bigoplus_{\chi_0} E_k(N,\chi_0)$. If $k \neq 2$, then each element of $\bigcup_{N_1N_2=N} Z_{k,N_1,N_2}$ belongs to some $E_k(N,\chi_0)$. This shows our assertion. The case $k=2$ is similar. q.e.d.

3. In what follows, we always assume that N is divisible by 4. Let

$$\theta(z) := \sum_{n \in \boldsymbol{Z}} e(n^2 z)$$

be a theta series, which is a modular form for $\Gamma_0(4)$ of weight $1/2$. We denote by $j(M,z)$ $(M \in \Gamma_0(4))$, the automorphy factor of the theta series. Its forth power $j(M,z)^4$ is equal to $(cz+d)^2$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $k \in \boldsymbol{N}$, $M_{k+1/2}(\Gamma_1(N))$ denotes the space of modular forms for $\Gamma_1(N)$ with an automorphy factor $j(M,z) \cdot (cz+d)^k$, and $S_{k+1/2}(\Gamma_1(N))$ denotes the subspace consisting of cusp forms. Let χ_0 be a character (mod N). Then $M_{k+1/2}(N,\chi_0)$ (resp. $S_{k+1/2}(N,\chi_0)$) denotes the space of modular forms (resp. cusp forms) f such that $(f|_{M,\chi_0})(z) = f(z)$ for $M \in \Gamma_0(N)$ where $(f|_{M,\chi_0}) = \chi_0(d)^{-1} j(M,z)^{-1}(cz+d)^{-k} f(Mz)$. It should be noted that our automorphy factor differs from Shimura's in [7] by $\chi_{-4}(d)^k$ where χ_{-4} is the Kronecker-Jacobi-Legendre symbol. Let $e \in \boldsymbol{N}$. Then $\theta(ez)$ is in $M_{1/2}\left(4e, \left(\frac{e}{|\ |}\right)\right)$, indeed we have

$$j\left(\begin{pmatrix} a & eb \\ c/e & d \end{pmatrix}, ez\right) = \left(\frac{e}{|d|}\right) j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4e).$$

The group $\Gamma_0(4)$ has three cusps $0, 1/2, 1/4$, and $\theta(z)$ vanishes only at the cusp $1/2$. Then $\theta(ez)$ vanishes only at the cusps $i/2e, (i, 2e) = 1$, of $\Gamma_0(4e)$, or equivalently only at cusps $[*, *, 2e; 4e]$ of $\Gamma_1(4e)$.

Let $v_2(N) = 2$. Then cusps in the form $[*, *, N_2; N]$ with $v_2(N_2) = 1$ is irregular for $\Gamma_1(N)$ and for an automorphy factor $j(M, z)(cz + d)^k$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$ (cf. Shimura [6]), and so any modular forms for $\Gamma_1(N)$ with the automorphy factor vanishes at those cusps. Then other cusps are all regular. If $v_2(N) > 2$, then all cusps are regular.

PROPOSITION 3. *Let* $4|N$. *Let* ϕ_4 *denote an arithmetic function defined by* $\phi_4(e) = e/4(v_2(e) \geq 2), e(v_2(e) \leq 1)$.

(1) *Let* $k \geq 2$. *If* $v_2(N) = 2$, *then* $\{\theta(\phi_4(e)z)G_{k,\chi}^{\chi'}(ez) | N_1N_2 = N, v_2(N_2) \neq 1, G_{k,\chi}^{\chi'}(ez) \in Z_{k,N_1,N_2}\}$ *separates all the regular cusps of* $\Gamma_1(N)$. *If* $v_2(N) > 2$, *then* $\{\theta(\phi_4(e)z)G_{k,\chi}^{\chi'}(ez) | N_1N_2 = N, G_{k,\chi}^{\chi'}(ez) \in Z_{k,N_1,N_2}\}$ *separates all the cusps of* $\Gamma_1(N)$.

(2) *Let* $k = 2$. *If* $v_2(N) = 2$, *then* $\{\theta(\phi_4(e)z)G_{2,\chi}^{\chi'}(ez) | N_1N_2 = N, v_2(N_2) \neq 1, G_{2,\chi}^{\chi'}(ez) \in Z_{2,N_1,N_2}, e < N_2$ *or* $\chi' \neq 1_{N_1}\} \cup \{\theta(\phi_4(N_2)z)(G_2^{1_{N_1}}(N_2z) - G_2^{1_{2N_1}}(N_2/2)z)) | N_1N_2 = N, 4|N_2\} \cup \{\theta(N_2z)(2G_2^{1_{(N_1/2)}}(2N_2z) - G_2^{1_{N_1}}(N_2z)) | N_1N_2 = N, 2 \nmid N_2\}$ *separates the regular cusps of* $\Gamma_1(N)$. *If* $v_2(N) > 2$, *then* $\{\theta(\phi_4(e)z)G_{2,\chi}^{\chi'}(ez) | N_1N_2 = N, G_{2,\chi}^{\chi'}(ez) \in Z_{2,N_1,N_2}, e < N_2$ *or* $\chi' \neq 1_{N_1}\} \cup \{(\theta(\phi_4(N_2)z)(2G_2^{1_{N_1}}(N_2z) - G_2^{1_{2N_1}}((N_2/2)z)) | N_1N_2 = N, 4|N_2\} \cup \{\theta(N_2z)(2G_2^{1_{(N_1/2)}}(2N_2z) - G_2^{1_{N_1}}(N_2z)) | N_1N_2 = N, v_2(N_2) \leq 1\}$ *separates all the cusps of* $\Gamma_1(N)$. *All the elements in the sets are holomorphic.*

PROOF. (1) If $G_{k,\chi}^{\chi'}(ez) \in Z_{k,N_1,N_2}$, then $v_2(N_2/e) \neq 1$. So theta series $\theta(\phi_4(e)z)$ does not vanish at cusps i/N_2 $((i, N_2) = 1)$. Then we can apply Proposition 2 to our case. Then the assertion follows.

(2) Let $v_2(N_2) > 0$. Let b be 1 or 2 according as N_1 is odd or even. Then $G_2^{1_{N_1}}(N_2z)$ and $1/2\{bG_2^{1_{N_1}}(N_2z) - G_2^{1_{2N_1}}((N_2/2)z))\}$ takes the same value at cusps $[*, *, N_2; N]$ because $G_2^{1_{2N_1}}((N_2/2)z))$ vanishes there. The latter vanishes at cusps $[*, *, N_2'; N]$ with $N_2 \nmid N_2'$ as well as the former. This shows that we can replace $\theta(\phi_4(N_2)z)G_2^{1_{N_1}}(N_2z)$ by $\theta(\phi_4(N_2)z)(bG_2^{1_{N_1}}(N_2z) - G_2^{1_{2N_1}}((N_2/2)z)))$ in the argument (1). Let $v_2(N_2) \leq 1$. Then the values at cusps $[*, *, N_2; N]$ of $G_2^{1_{N_1}}(N_2z)$ and of $2G_2^{1_{(N_1/2)}}(2N_2z) - G_2^{1_{N_1}}(N_2z)$ are proportional to each other, indeed they are modular forms for $\Gamma_0(N)$ with trivial character and each of them is a non-zero constant on those cusps (see Lemma 1 (1)). The modular form $G_2^{1_{(N_1/2)}}(2N_2z)$ does not vanish at cusps $[*, *, 2N_2; N]$. However a theta series $\theta(N_2z)$ vanishes at cusps

$[*,*,2N_2;N]$ and so $\theta(N_2z)(2G_2^{1(N_1/2)}(2N_2z)-G_2^{1_{N_1}}(N_2z))$ vanishes at cusps $[*,*,N_2';N]$ with $N_2' \nmid N_2$. Since the values at cusps $[*,*,N_2;N]$, of $\theta(N_2z)\cdot G_2^{1_{N_1}}(N_2z))$ and of $\theta(N_2z)(2G_2^{1(N_1/2)}(2N_2z)-G_2^{1_{N_1}}(N_2z))$ are proportional, we can replace the former by the latter in the argument (1). q.e.d.

Let $G_{k+1/2}(\Gamma_1(N))$ denote the subspace of $M_{k+1/2}(\Gamma_1(N))$ generated by modular forms in Proposition 3 (1) if $k=1$ or $k>2$, and by those in Proposition 3 (2) if $k=2$. Proposition 3 implies that

$$M_{k+1/2}(\Gamma_1(N)) = G_{k+1/2}(\Gamma_1(N)) \oplus S_{k+1/2}(\Gamma_1(N))$$

for $k \geq 2$. For $k \neq 2$, let $G_{k+1/2}(N,\chi_0)$ denote the linear span of $\left\{\theta(\phi_4(e)z)G_{k,\chi}^{\chi'}(ez) \mid N_1N_2=N, e|N_2, \chi' \in (\boldsymbol{Z}/N_1)^*, \text{ primitive } \chi \in (\boldsymbol{Z}/(N_2/e))^*, \chi_0 = \left(\frac{\phi_4(e)}{}\right)\chi'\chi \text{ on } (\boldsymbol{Z}/N)^\times\right\}$ where the condition $v_2(N_2) \neq 1$ should be added if $v_2(N)=2$. It is a subspace of $M_{k+1/2}(N,\chi_0)$. Since the direct sum of $G_{k+1/2}(N,\chi_0)$'s with $\chi_0 \in (\boldsymbol{Z}/N)^*$ is equal to $G_{k+1/2}(\Gamma_1(N))$, we have $M_{k+1/2}(N,\chi_0) = G_{k+1/2}(N,\chi_0) \oplus S_{k+1/2}(N,\chi_0)$ for $k>2$. By the similar way we construct $G_{k+1/2}(N,\chi_0)$ also for $k=2$ by replacing $\theta(\phi_4(N_2)z)G_2^{1_{N_1}}(N_2z)$ by $\theta(\phi_4(N_2)z)\{2G_2^{1_{N_1}}(N_2z)-G_2^{1_{2N_1}}((N_2/2)z)\}$ $(v_2(N_2)\geq 2)$, and $\theta(N_2z)G_2^{1_{N_1}}(N_2z))$ by $\theta(N_2z)\{2G_2^{1(N_1/2)}(2N_2z)-G_2^{1_{N_1}}(N_2z)\}$ $(v_2(N_2)\leq 1)$ as in the proof of Proposition 3. Then the above equality holds also for $k=2$. We obtain;

COROLLARY. *Let $k \geq 2$. Then we have*

$$M_{k+1/2}(N,\chi_0) = G_{k+1/2}(N,\chi_0) \oplus S_{k+1/2}(N,\chi_0).$$

4. Let $4|N, \chi_0 \in (\boldsymbol{Z}/N)^*$. Let p be prime. A Hecke operator $T_{\chi_0}(p)$ defines a $\boldsymbol{C}$-linear endomorphism on $M_k(N,\chi_0)$ such that for $f(z)=\sum_{n=0}^\infty c_n e(nz)$, $(T_{\chi_0}(p)f)(z) = \sum_{n=0}^\infty (c_{pn} + \chi_0(p)p^{k-1}c_{p/n})e(nz)$ where $c_{p/n}=0$ if p/n is not integral. If $p|N$, then it is denoted also by U_p. If $v_p(N)\geq 2$, then $U_p(f)$ is in $M_k(N/p,\chi_0)$. A Hecke operator $T_{\chi_0}(p^2)$ on $M_{k+1/2}(N,\chi_0)$ is such that for $f(z)=\sum_{n=0}^\infty c_n e(nz)$, $(T_{\chi_0}(p^2)f)(z) = \sum_{n=0}^\infty (c_{p^2n} + \chi_0(p)(n/p)p^{k-1}c_n + \chi_0(p^2)p^{2k-1}c_{n/p^2})\cdot e(nz)$ (Shimura [7, Prop. 1.7]). For $p|N$, the Hecke operator is denoted also by U_{p^2}. For $e \in N$, B_e is defined to be an operator of the space of Fourier series sending $f(z)$ to $f(ez)$. We have inclusions $B_e(M_k(N,\chi_0)) \subset M_k(eN,\chi_0^{(e)})$ and

$B_e(M_{k+1/2}(N,\chi_0)) \subset M_{k+1/2}(eN,(e/|\cdot|)\chi_0)$ where $\chi_0^{(c)}$ is as in the beginning of the section one.

Let a be a squarefree natural number. Let a^* denotes a or $4a$ according as $a \equiv 1 \pmod 4$ or not. We note that a^* is a discriminant of a real quadratic field except for the case $a = a^* = 1$. Let χ_{a^*} be as in the beginning of the section one. Let $f(z) = \sum_{n=0}^{\infty} c_n e(nz) \in M_{k+1/2}(N,\chi_0)$. Then we define a Shimura lifting $\mathcal{S}_{a,\chi_0}(f)$ of f to be

$$\mathcal{S}_{a,\chi_0}(f)(z) = C + \sum_{n=1}^{\infty}\left(\sum_{0<d|n} \chi_{a^*}(d)\chi_0(d)d^{k-1}c_{an^2/d^2}\right)e(nz),$$

provided that there is a constant C so that $\mathcal{S}_{a,\chi_0}(f)$ is a modular form for some congruence subgroup. The constant C is obviously unique if it exists. Shimura [7] and Niwa [4] showed that $\mathcal{S}_{a,\chi_0}$ is a well-defined linear map of $S_{k+1/2}(N,\chi_0)$ to $M_{2k}(N/2,\chi_0^2)$, and that $\mathcal{S}_{a,\chi_0}(f)$ is a cusp form in $S_{2k}(N/2,\chi_0^2)$ for $f \in S_{k+1/2}(N,\chi_0)$ and for $k \geq 2$, in particular $C = 0$.

The lifting maps allow us to make formal computations for higher terms of the Fourier expansions of modular forms. In principle, higher terms determine the modular form, namely the constant term. In this sense the formal computation of the lifting map is possible as well as the Hecke operators and the operator B_e. By simple computations, we obtain the following;

LEMMA 4. *The Shimura lifting map $\mathcal{S}_{a,\chi_0}$ is Hecke equivariant map, namely there holds*

$$\mathcal{S}_{a,\chi_0} \circ T_{\chi_0}(p^2) = T_{\chi_0^2}(p) \circ \mathcal{S}_{a,\chi_0},$$

$$\mathcal{S}_{a,\chi_0} \circ U_{p^2} = U_p \circ \mathcal{S}_{a,\chi_0} \quad (p|N).$$

In particular if $f \in M_{k+1/2}(N,\chi_0)$ is a common eigen-function of $T_{\chi_0}(p^2)$ for all p, then $\mathcal{S}_{a,\chi_0}(f) \in M_{2k}(N/2,\chi_0^2)$ is a common eigen-function of $T_{\chi_0^2}(p)$, provided that $\mathcal{S}_{a,\chi_0}(f)$ is well-defined.

There is an obvious inclusion $M_{k+1/2}(N,\chi_0) \subset M_{k+1/2}(eN,\chi_0^{(e)})$. So element f of $M_{k+1/2}(N,\chi_0)$ has another lifting $\mathcal{S}_{a,\chi_0^{(e)}}(f)$. The relation between $\mathcal{S}_{a,\chi_0}$ and $\mathcal{S}_{a,\chi_0^{(e)}}$ is obtained by a formal computation as follows;

LEMMA 5. *Let $p_1,\dots,p_s$ be all the prime factors of e. Then we have*

$$\mathscr{S}_{a,\chi_0^{(e)}} = \left\{ 1 - \sum_{i=1}^{s} \chi_{a^*}(p_i)\chi_0(p_i)p_i^{k-1}B_{p_i} + \sum_{1\le i,j\le s} \chi_{a^*}(p_ip_j)\chi_0(p_ip_j)(p_ip_j)^{k-1}B_{p_ip_j} \right.$$

$$\left. + \cdots + (-1)^s\chi_{a^*}(p_1\cdots p_s)\chi_0(p_1\cdots p_s)(p_1\cdots p_s)^{k-1}B_{p_1\cdots p_s} \right\} \circ \mathscr{S}_{a,\chi_0},$$

provided that $\mathscr{S}_{a,\chi_0}$ *is well-defined.*

LEMMA 6. *Let* $4|N, \chi_0 \in (\boldsymbol{Z}/N)^*$. *Let* $a, e \in \boldsymbol{N}$ *where* a *is square-free. Let* $a/e = a_0/e_0^2$ *with* a_0 *square-free. Then*

$$\mathscr{S}_{a,\chi_0'} \circ B_e = B_{e_0} \circ \mathscr{S}_{a_0,\chi_0^{(e)}}$$

with $\chi_0' = \chi_0(e/|\cdot|)$, *provided that* $\mathscr{S}_{a_0,\chi_0^{(e)}}$ *is well-defined.*

PROOF. Let $f(z) = \sum_{n=0}^{\infty} c_n e(nz) \in M_{k+1/2}(N,\chi_0)$. Then $B_e(f)$ is in $M_{k+1/2}(eN,\chi_0')$. We have

$$\mathscr{S}_{a,\chi_0'}(B_e(f))(z) = C + \sum_{n=1}^{\infty}\sum_{d|n} \chi_{a^*}(d)\chi_0(d)\left(\frac{e}{|d|}\right)d^{k-1}c_{a_0n^2/e_0^2d^2}\,e(nz).$$

Since N is divisible by 4 and χ_0 is a character (mod N), $\chi_0(d)$ and $\chi_0^{(e_0)}(d)$ vanish for d even. It is easy to see that the equality $\chi_{a^*}(d)\chi_0(d)(e/|d|) = \chi_{a_0^*}(d)\chi_0^{(e)}(d)$ holds for d odd. Then

$$\begin{aligned} \mathscr{S}_{a,\chi_0'}(B_e(f))(z) &= C + \sum_{n=1}^{\infty}\sum_{d|n} \chi_{a_0^*}(d)\chi_0^{(e)}(d)d^{k-1}c_{a_0n^2/d^2}\,e(e_0nz) \\ &= B_{e_0}(\mathscr{S}_{a_0,\chi_0^{(e)}}(f))(z). \end{aligned}$$

q.e.d.

COROLLARY. *Let* $\{f_{i,\chi_0}(z)\}_{i\in I(\chi_0)}$ *be elements in* $M_{k+1/2}(N,\chi_0)$ *for* $\chi_0 \in (\boldsymbol{Z}/N)^*$. *Let* M *be the subspace of* $M_{k+1/2}(\Gamma_1(N))$ *generated by* $S_{k+1/2}(N)$ *and by elements in the form* $f_{i,\chi_0}(ez) \in M_{k+1/2}(\Gamma_1(N))$ *with* $e|N, i \in I(\chi_0), \chi_0 \in (\boldsymbol{Z}/N)^*$. *Suppose that* $\mathscr{S}_{a,\chi_0}(f_{i,\chi_0})$ *are well-defined for all square-free* a *and for all* $f_{i,\chi_0}, \chi_0 \in (\boldsymbol{Z}/N)^*$. *Then all the lifting maps* $\mathscr{S}_{a,\chi_0}$ *are well-defined on* $M \cap M_{k+1/2}(N,\chi_0)$.

PROOF. Since M is spanned by $S_{k+1/2}(N)$ and by elements $f_{i,\chi_0}(ez)$, it is enough to show that $B_e(f_{i,\chi_0}) \in M_{k+1/2}(N,\chi_0(e/|\cdot|))$ has liftings associated with all square-free a. Combining Lemma 5 with Lemma 6, we see that they exist if

$\mathscr{S}_{a,\chi_0}(f_{i,\chi_0})$ exist for all a and all $\chi_0 \in (\boldsymbol{Z}/N)^*$, which is our assumption. Then our assertion follows. q.e.d.

5. In our previous paper [8, Prop. 4, Lemma 5, Cor. to Theorem 2], we have shown the following;

THEOREM ([8]). *Let $k \in \boldsymbol{N}$. Let $N = N_1N_2 \in \boldsymbol{N}, \chi' \in (\boldsymbol{Z}/N_1)^*, \chi \in (\boldsymbol{Z}/N_2)^*$, $\chi_0 := \chi\chi' \in (\boldsymbol{Z}/N)^*$ where N is not necessarily divisible by 4 and χ is not necessary primitive. Suppose that a character $\chi_0 \in (\boldsymbol{Z}/N)^*$ has the same parity as k. Let $a \in \boldsymbol{N}$ be square-free. Put $l = 2^{\sharp\{i | v_2((a^*, N_i)) \geq 2\}}$. Then*

$$\lambda^{\chi'}_{2k,a^*,\chi}(z) := C + 4\sum_{n=1}^{\infty}\sum_{0<d|n}\chi_{a^*}(d)\chi_0(d)d^{k-1}\sum_{m\in\boldsymbol{Z}}\sigma^{\chi'}_{k-1,\chi}\left(\frac{(n/d)^2a^* - m^2}{4}\right)e(nz)$$

with a suitable constant C, is a modular form in $M_{2k}(N/l, \chi_0^2)$. If $k > 1$, then $C = L(1-k,\chi)L(1-k,\chi\chi_{a^*})$ $(N_1 = 1)$ and $C = 0$ $(N_1 > 1)$.

To construct the modular form in the theorem, we have made use of Hilbert Eisenstein series of real quadratic fields. We put $\lambda_{2k,a^*,\chi}(z) := \lambda^{\chi'}_{2k,a^*,\chi}(z)$ $(N_1 = 1)$, $\lambda^{\chi'}_{2k,a^*}(z) := \lambda^{\chi'}_{2k,a^*,\chi}(z)$ $(N_2 = 1)$.

LEMMA 7. *Let $a, N, N_1, N_2, \chi' \in (\boldsymbol{Z}/N_1)^*, \chi \in (\boldsymbol{Z}/N_2)^*, \chi_0 \in (\boldsymbol{Z}/N)^*$ be as in the above theorem.*

(1) *Suppose that $G^{\chi'}_{k,\chi}$ is holomorphic.*

(i) *Let $a \equiv 1 \pmod 4$. Then*

$$\mathscr{S}_{a,\chi_0}(\theta(z)G^{\chi'}_{k,\chi}(z)) = U_2(\lambda^{\chi'}_{2k,a^*,\chi}(z)),$$

$$\mathscr{S}_{a,\chi_0}(\theta(z)G^{\chi'}_{k,\chi}(4z)) = \lambda^{\chi'}_{2k,a^*,\chi}(z),$$

$$\mathscr{S}_{a,\chi_0}(\theta(z)G^{\chi'}_{k,\chi}(2z)) = 2^{k-1}\chi(2)\lambda^{\chi'}_{2k,a^*,\chi}(z) \quad (v_2(N_1) > 0),$$

$$\mathscr{S}_{a,\chi_0}(\theta(z)G^{\chi'}_{k,\chi}(2z)) = 2^{k-1}\chi'(2)\lambda^{\chi'}_{2k,a^*,\chi}(z) \quad (v_2(N_2) > 0),$$

which are modular forms in $M_{2k}(N, \chi_0^2)$ where the first one is in $M_{2k}(N/2, \chi_0^2)$ if $v_2(N) \geq 2$.

(ii) *Let $a \not\equiv 1 \pmod 4$. Then*

$$\mathscr{S}_{a,\chi_0}(\theta(z)G^{\chi'}_{k,\chi}(z)) = \lambda^{\chi'}_{2k,a^*,\chi}(z),$$

$$\mathscr{S}_{a,\chi_0}(\theta(z)G^{\chi'}_{k,\chi}(4z)) = B_2(\lambda^{\chi'}_{2k,a^*,\chi}(z))$$

where the former is in $M_{2k}(N/l,\chi_0^2)$ *and the latter in* $M_{2k}(2N/l,\chi_0^2)$ *with* $l=2^{\sharp\{i|v_2(N_i)\geq 2\}}$. *Suppose that* $2|N_1$ *and* $2\nmid N_2$. *Then*

$$\mathscr{S}_{a,\chi_0}(\theta(z)G_{k,\chi}^{\chi'}(2z))=2^{-k+1}\chi(2)^{-1}(\lambda_{2k,a^*,\chi}^{\chi'}(z)-\lambda_{2k,a^*,\chi^{(2)}}^{\chi'}(z)),$$

which is in $M_{2k}(2N/l,\chi_0^2)$ *with* $l=1$ $(v_2(N_1)\leq 1)$, 2 (*otherwise*).

(2) *If* b *denotes* 1 *or* 2 *according as* N_1 *is odd or even, then a function* $bG_{2,\chi}^{\chi'}(2z)-G_{2,\chi}^{\chi'(2)}(z)$ *is holomorphic even when* $k=2$. *Let* N_2 *be odd. Then* $\mathscr{S}_{a,\chi_0}(\theta(z)(bG_{2,\chi}^{\chi'}(4z)-G_{2,\chi}^{\chi'(2)}(2z))$ *is equal to*

$$b\lambda_{2k,a^*,\chi}^{\chi'}(z)-2^{k-1}\chi(2)\lambda_{2k,a^*,\chi}^{\chi'(2)}(z)\quad(a\equiv 1\ (\mathrm{mod}\ 4)),$$

$$bB_2(\lambda_{2k,a^*,\chi}^{\chi'}(z))-2^{-k+1}\chi(2)^{-1}(\lambda_{2k,a^*,\chi}^{\chi'(2)}(z)-\lambda_{2k,a^*,\chi^{(2)}}^{\chi'(2)}(z))\quad(a\not\equiv 1\ (\mathrm{mod}\ 4)),$$

which is in $M_{2k}(2N,\chi_0^2)$ $(a\equiv 1\ (\mathrm{mod}\ 4))$, *or in* $M_{2k}(4N/b,\chi_0^2)$ $(a\not\equiv 1\ (\mathrm{mod}\ 4))$. *Let* $2|N_1$. *Then* $\mathscr{S}_{a,\chi_0}(\theta(z)(2G_{2,\chi}^{\chi'}(2z)-G_{2,\chi}^{\chi'(2)}(z))$ *is equal to*

$$2^k\chi(2)\lambda_{2k,a^*,\chi}^{\chi'}(z)-U_2(\lambda_{2k,a^*,\chi}^{\chi'}(z))\quad(a\equiv 1\ (\mathrm{mod}\ 4)),$$

$$2^{-k+2}\chi(2)^{-1}(\lambda_{2k,a^*,\chi}^{\chi'}(z)-\lambda_{2k,a^*,\chi^{(2)}}^{\chi'}(z))-\lambda_{2k,a^*,\chi}^{\chi'}(z)\quad(a\not\equiv 1\ (\mathrm{mod}\ 4)),$$

which is in $M_{2k}(N,\chi_0^2)$ $(a\equiv 1\ (\mathrm{mod}\ 4))$, *or in* $M_{2k}(2N/l,\chi_0^2)$ $(a\not\equiv 1\ (\mathrm{mod}\ 4))$ *with* $l=1(v_2(N_1)\leq 1)$, 2 (*otherwise*).

PROOF. The first and second equalities in (i) of (1) are easily obtained. We prove the third one. We have

$$\mathscr{S}_{a,\chi_0}(\theta(z)G_{k,\chi}^{\chi'}(2z))=C+\sum_{n=1}^{\infty}\sum_{0<d|n}\chi_a(d)\chi_0(d)d^{k-1}\sigma_{k-1,\chi}^{\chi'}\left(\frac{a(n/d)^2-m^2}{2}\right)e(nz).$$

Since $a\equiv 1\ (\mathrm{mod}\ 4)$, $(a(n/d)^2-m^2)/2$ is even whenever it is integral. Then $\sigma_{k-1,\chi}^{\chi'}((a(n/d)^2-m^2)/2)=2^{k-1}\chi(2)\sigma_{k-1,\chi}^{\chi'}((a(n/d)^2-m^2)/4)$, and hence the equality follows. The fourth one is proved similarly.

The first and second equations in (ii) of (1) is again immediate. We prove the third one. Note that $\chi_0=\chi'\chi=\chi'\chi^{(2)}$. Then

$$\lambda_{2k,a^*,\chi}^{\chi'}(z)-\lambda_{2k,a^*,\chi^{(2)}}^{\chi'}(z)$$

$$=C+4\sum_{n=1}^{\infty}\sum_{0<d|n}\chi_{4a}(d)\chi_0(d)d^{k-1}\sum_{\substack{m\in Z\\ 2|(a(n/d)^2-m^2)}}\sigma_{k-1,\chi}^{\chi'}(a(n/d)^2-m^2)e(nz)$$

$$= C + 4\cdot 2^{k-1}\chi(2)\sum_{n=1}^{\infty}\sum_{0<d|n}\chi_{4a}(d)\chi_0(d)d^{k-1}\sum_{m\in Z}\sigma^{\chi'}_{k-1,\chi}\left(\frac{a(n/d)^2-m^2}{2}\right)e(nz)$$

$$= 2^{k-1}\chi(2)\mathscr{S}_{a,\chi_0}(\theta(z)G^{\chi'}_{k,\chi}(2z)).$$

The equalities in (2) follow from the computations in (1). q.e.d.

THEOREM 1. *Let $k \geq 1$, $4|N, \chi_0 \in (Z/N)^*$. Let $a \in N$ be square-free. Then the Shimura lifting map $\mathscr{S}_{a,\chi_0}$ is a Hecke equivariant linear map of $G_{k+1/2}(N,\chi_0) \oplus S_{k+1/2}(N,\chi_0)$ to $M_{2k}(N/2,\chi_0^2)$.*

PROOF. Let $k \neq 2$. By Lemma 7, $\theta(z)G^{\chi'}_{k,\chi}(z)$ $(N = N_1N_2, e|N_2, 4|N_1$ or $4|(N_2/e), \chi' \in (Z/N_1)^*, \chi \in (Z/(N_2/e))^*)$ and $\theta(z)G^{\chi'}_{k,\chi}(4z)$ $(N = N_1N_2, 4|e|N_2, \chi' \in (Z/N_1)^*, \chi \in (Z/(N_2/e))^*)$ have liftings in $M_{2k}(N/2e,(\chi\chi')^2)$ and in $M_{2k}(N/e,(\chi\chi')^2)$ respectively for any square-free a. We can take $G_{k+1/2}(N,\chi_0) \oplus S_{k+1/2}(N,\chi_0)$, as M in Corollary to Lemma 6. Hence the lifting map is well-defined on $G_{k+1/2}(N,\chi_0) \oplus S_{k+1/2}(N,\chi_0)$. By Lemma 6, it is easy to see that the image is in $M_{2k}(N/2,\chi_0^2)$. This shows our assertion.

Let $k = 2$. Then we must take into account modular forms $\theta(z) \times (bG_2^{1_{N_1}}(4z) - G_2^{1_{2N_1}}(2z)), \theta(z)(2G_2^{1_{(N_1/2)}}(2z) - G_2^{1_{N_1}}(z))$ in Proposition 3 (2). To this case we can apply Lemma 7 (2) and our assertion follows in the same manner as in the case $k \neq 2$. q.e.d.

From Corollary to Proposition 3, we obtain the main theorem of the paper.

THEOREM 2. *Let $k \geq 2$. Let a, χ_0 be as in Theorem 1. Then the Shimura lifting map $\mathscr{S}_{a,\chi_0}$ is a Hecke equivariant linear map of $M_{k+1/2}(N,\chi_0)$ to $M_{2k}(N/2,\chi_0^2)$.*

REMARK. Let $k \geq 2$. Let $E_{k+1/2}(N,\chi_0)$ denote the orthogonal complement of $S_{k+1/2}(N,\chi_0)$ in $M_{k+1/2}(N,\chi_0)$ with respect to the Petersson product. Then the space $M_{k+1/2}(N,\chi_0)$ is decomposed into $E_{k+1/2}(N,\chi_0) \oplus S_{k+1/2}(N,\chi_0)$ as Hecke modules. An eigen-function of all Hecke operators in $E_{k+1/2}(N,\chi_0)$ is mapped to an eigen-function in $M_{2k}(N,\chi_0^2)$. By the growth condition of Fourier coefficients, it is shown that the eigen-function in $M_{2k}(N,\chi_0^2)$ is in $E_{2k}(N,\chi_0^2)$. Hence the lifting map sends Eisenstein series of half-integral weight to Eisenstein series of integral weight.

The case $k = 1$ of Theorem 2 will be investigated in our later paper.

6. We give two applications of our theorem.

(1) Let S be a positive definite symmetric integral matrix of size $2k+1$. Let N be a multiple of 4 such that $1/(4N)S^{-1}$ is integral. Let

$$\theta_S(z) := \sum_{g \in \mathbf{Z}^{2k+1}} e({}^t gSgz).$$

Then $\theta_S(z) \in M_{k+1/2}(\Gamma_0(N), \chi_{-4}^k \cdot (\det(S)/\ \ ))$. Let $r_s(n), n \in \mathbf{N}$, denote the number of integral solutions of an equation ${}^tXSX = n$ where X is a variable vector with $2k+1$ elements. We have

$$\theta_S(z) = 1 + \sum_{n=1}^{\infty} r_S(n)e(nz).$$

By Theorem 2, the lifting of this modular form is in $M_{2k}(N/2, 1_{N/2})$. We state this as a proposition.

PROPOSITION 4. *Let S, N be as above with $k \geq 2$. Let a be any square-free natural number. Let $a_0 a_1^2 = (-4)^k a \cdot det(S)$ where a_0 is* 1 *or the discriminant of a quadratic field. Then for a suitable constant C,*

$$C + \sum_{n=1}^{\infty} \left(\sum_{0<d|n} \chi_{a_0}^{(a_1)}(d) d^{k-1} r_S(a(n/d)^2) \right) e(nz)$$

is a modular form in $M_{2k}(N/2, 1_{N/2})$.

In case that S has an odd prime determinant, the result was first obtained by van Asch [1, Theorem 2], where the constant term C is written explicitly in terms of S and a.

(2) We derive several relations among arithmetic functions or special values of L-functions.

(i) Since $j(M,z)^2 = \chi_{-4}(d)(cz+d)$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we have an inclusion $\theta(z)M_{7/2}(4, \chi_{-4}) \subset M_4(4)$. This shows in particular, that $S_{7/2}(4, \chi_{-4}) = \{0\}$ because $S_4(4) = \{0\}$. Thus $\dim_{\mathbf{C}} M_{7/2}(4, \chi_{-4})$ is at most 2, the number of regular cusps. The modular forms $31\theta(z)G_{3,\chi_{-4}}(z) - 16\theta(z)G_3^{\chi_{-4}}(z)$, $\theta(z)G_3^{\chi_{-4}}(z)$ spans $M_{7/2}(4, \chi_{-4})$. Hence $M_{7/2}(4, \chi_{-4}) = G_{7/2}(4, \chi_{-4})$, and the space is of dimension two. It is easy to see only be applying U_4, that they are common Hecke eigen-functions. The eigenvalue of $\theta(z)G_3^{\chi_{-4}}(z)$ for U_4 is 2^5. Since $G_6(z) - G_6(2z)$ is the unique function in $M_2(2, 1_2)$, up to constant factor, such that

$U_2(G_6(z)-G_6(2z))=2^5(G_6(z)-G_6(2z))$, we have

$$\mathscr{S}_{a^*,\chi_{-4}}(\theta(z)G_3^{\chi_{-4}}(z))=\sum_{m\in Z}\sigma_2^{\chi_{-4}}(a-m^2)(G_6(z)-G_6(2z))$$

by comparing the first Fourier coefficients. Then comparing higher terms, we obtain

$$\sum_{m\in Z}\sigma_2^{\chi_{-4}}(a-m^2)\sigma_5^{1_2}(n)-\sum_{\substack{0<d|n\\ d:odd}}\chi'(d)d^2\sum_{m\in Z}\sigma_2^{\chi_{-4}}((n/d)^2a-m^2)=0$$

for any $n\in N$, where $\chi'=\chi_{-4a}$ $(a\equiv 1,2\,(\mathrm{mod}\,4))$, χ_{-a} (otherwise).

(ii) Let $k\geq 4$. The Shimura lifting $\mathscr{S}_{a,1_4}(\theta(z)G_k(4z))$ $(a\equiv 1\,(\mathrm{mod}\,4))$ or $\mathscr{S}_{4a,1_4}(\theta(z)G_k(z))$ $(a\equiv 2,3\,(\mathrm{mod}\,4))$, namely, $\lambda_{2k,a^*,1}$ is in $M_{2k}(1)$. Let $k=4$. Then $M_8(1)$ is of dimension one, and hence

$$\lambda_{8,a^*,1}(z)=\zeta(-3)L(-3,\chi_{a^*})+4\sum_{n=1}^{\infty}\sum_{0<d|n}\chi_{a^*}(d)d^3\sum_{m\in Z}\sigma_3\left(\frac{(n/d)^2a^*-m^2}{4}\right)e(nz)$$

is equal to

$$G_8(z)=\zeta(-7)+2\sum_{n=1}^{\infty}\sigma_7(n)e(nz)$$

up to a constant multiple. Then we have an identity

$$L(-3,\chi_{a^*})\sigma_7(n)-\sum_{0<d|n}\chi_{a^*}(d)d^3\sum_{m\in Z}\sigma_3\left(\frac{(n/d)^2a^*-m^2}{4}\right)=0$$

for any $a\in N$, square-free. When $a=1$, this gives the well-known formula

$$\sigma_7(n)-120\sum_{m=0}^{n}\sigma_3(m)\sigma_3(n-m)=0,$$

because $L(-3,\chi_1)=\zeta(-3)=1/120$, $\sigma_3(0)=1/2\zeta(-3)=1/240$ and $\sum_{0<d|n}d^3\cdot\sum_{m\in Z}\sigma_3\left(\frac{(n/d)^2-m^2}{4}\right)=\sum_{m=0}^{n}\sigma_3(n)\sigma_3(n-m)$ (for example, see [8], the section 3).

(iii) Let us consider the case $k=6$ in (ii). Two modular forms

$$\lambda_{12,a^*,1}(z)=-\frac{1}{252}L(-5,\chi_{a^*})+4\sum_{n=1}^{\infty}\sum_{0<d|n}\chi_{a^*}(d)d^5\sum_{m\in Z}\sigma_5\left(\frac{(n/d)^2a^*-m^2}{4}\right)e(nz),$$

$$G_{12}(z)=\frac{691}{32760}+2\sum_{n=1}^{\infty}\sigma_{11}(n)e(nz)$$

are in $M_{12}(1)$. Let τ denote the Ramanujan function, namely $\Delta(z)=\sum_{n=1}^{\infty}\tau(n)e(nz), \tau(1)=1$, where $\Delta(z)$ denotes the cusp form in $M_{12}(1)$. Then $-1/252L(-5,\chi_{a^*})G_{12}(z)-691/32760\lambda_{12,a^*}(z)=-1/126\{L(-5,\chi_{a^*})+691/65\sum_{m\in\boldsymbol{Z}}\sigma_5((a^*-m^2)/4)\}\Delta(z)$, and we have

$$\left\{65L(-5,\chi_{a^*})+691\sum_{m\in\boldsymbol{Z}}\sigma_5\left(\frac{a^*-m^2}{4}\right)\right\}\tau(n)$$
$$=65L(-5,\chi_{a^*})\sigma_{11}(n)+691\sum_{0<d|n}\chi_{a^*}(d)d^5\sum_{m\in\boldsymbol{Z}}\sigma_5\left(\frac{(n/d)^2a^*-m^2}{4}\right).$$

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