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SIX-DIMENSIONAL QUASI-KÄHLER MANIFOLDS OF CONSTANT SECTIONAL CURVATURE

By

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Abstract. In this paper we prove that a six-dimensional quasi Kähler manifold of constant sectional curvature is a nearly Kähler manifold of positive sectional curvature or a flat Kähler manifold.

1. Introduction

In an almost Hermitian geometry, it is a natural question to ask the relationships between the properties of the curvatures and the ones of almost complex structures (for example, the integrability and so on). For general classes of almost Hermitian manifolds, we can not expect much concerning this question even for the case of spaces of constant sectional curvature. For example, F. Tricerri and L. Vanhecke ([7]) gave examples of locally flat almost Hermitian manifold which are not Kähler manifolds. On one hand, we may easily observe that there exist Hermitian structures on even-dimensional Hyperbolic spaces H^{2n} $(n \geq 1)$. On the contrary to this, T. Oguro and the second author ([6]) proved that $2n$ (≥ 4)-dimensional Hyperbolic space H^{2n} can not admit compatible almost Kähler structure. Further, T. Oguro ([5]) obtained the local version. Quite recently, the present authors proved that a four-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature is a Kähler manifold (that is, it is a locally complex 2-dimensional complex space form) ([1]).

It is also well-known that among even-dimensional spheres, only two- and six-dimensional spheres admit almost complex structure and further that a six-dimensional sphere S^6 equipped with the canonical Riemannian metric admits a

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nearly Kähler structure ([2]). An almost Hermitian manifold $M=(M,J,g)$ is called a quasi Kähler manifold if M satisfies the condition $(\nabla_X J)Y+(\nabla_{JX}J)JY=0$ for X, $Y\in\mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M). It is known that the classes of almost Kähler manifolds and nearly Kähler manifolds are subclasses of the class of quasi-Kähler manifolds. In the present paper, we consider six-dimensional quasi-Kähler manifolds of constant sectional curvature and prove the following

MAIN THEOREM. *A six-dimensional quasi-Kähler manifold of constant sectional curvature is a nearly Kähler manifold of positive sectional curvature or a flat Kähler manifold.*

2. Preliminaries

Let $M=(M^{2n},J,g)$ be a $2n$-dimensional quasi-Kähler manifold. The curvature tensor R is defined by

$$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z,$$

for $X,Y,Z\in\mathfrak{X}(M)$. Then it is known that the curvature tensor R satisfies the following identity (cf. [3]):

$$\begin{aligned}(2.1)\quad & R(x,y,z,w)-R(x,y,Jz,Jw)+R(Jx,Jy,Jz,Jw)-R(Jx,Jy,z,w)\\ &+R(Jx,y,Jz,w)+R(Jx,y,z,Jw)+R(x,Jy,z,Jw)+R(x,Jy,Jz,w)\\ &=-2\{g((\nabla_{(\nabla_x J)y}J)z,w)-g((\nabla_{(\nabla_y J)x}J)z,w)\}\end{aligned}$$

for $x,y,z,w\in T_pM$, $p\in M$. If M is of constant sectional curvature c, then (2.1) reduces to

$$\begin{aligned}(2.2)\quad & -2c\{g(y,z)g(x,w)-g(y,w)g(x,z)-g(y,Jz)g(x,Jw)+g(y,Jw)g(x,Jz)\}\\ &=\{g((\nabla_{(\nabla_x J)y}J)z,w)-g((\nabla_{(\nabla_y J)x}J)z,w)\}\end{aligned}$$

for $x,y,z,w\in T_pM$, $p\in M$. We denote by Ω the Kähler form of M defined by $\Omega(X,Y)=g(JX,Y)$, for $X,Y\in\mathfrak{X}(M)$. For any 1-form ϕ on M, we denote by $J\phi$ the 1-form defined by $(J\phi)(X)=-\phi(JX)$, for $X\in\mathfrak{X}(M)$. Now, in particular we assume that the dimension of the quasi-Kähler manifold M under consideration is six. The vector bundle Λ^2M of 2-forms over M is decomposed into the following form

$$(2.3)\qquad \Lambda^2M=\boldsymbol{R}\Omega\oplus\Lambda_0^{1,1}M\oplus LM \text{ (orthogonal direct sum)},$$

where $\Lambda_0^{1,1}M$ and LM are bundle of primitive J-invariant 2-forms and J-skew-invariant 2-forms of M, respectively. The bundle LM can be equipped with the canonical complex structure J defined by $(J\Phi)(X, Y) = -\Phi(JX, Y)$ for any section Φ of LM and X, $Y \in \mathfrak{X}(M)$ and the action of the group $U(3)$ on T_pM at each point $p \in M$ induces the action on LM (at p) defined by $(\sigma(h)(\Psi))(x, y) = \Psi(h^{-1}x, h^{-1}y)$ for $h \in U(3)$ and $x, y \in T_pM$. Further, we may easily check that the induced action of the group $U(3)$ on the set of all unitary bases of LM (at p) is transitive. Let $\{e_i\} = \{e_1, e_2, e_3, e_4 = Je_1, e_5 = Je_2, e_6 = Je_3\}$ be any unitary basis of T_pM at any point $p \in M$ and $\{e^i\}$ the dual basis. Then we see that $e^4 = Je^1$, $e^5 = Je^2$, $e^6 = Je^3$, and further, the covariant derivative $\nabla\Omega$ of the Kähler form Ω is expressed in the following form:

(2.4)
$$\nabla\Omega = \alpha_1 \otimes \Phi_1 - J\alpha_1 \otimes J\Phi_1 + \alpha_2 \otimes \Phi_2 - J\alpha_2 \otimes J\Phi_2 + \alpha_3 \otimes \Phi_3 - J\alpha_3 \otimes J\Phi_3,$$

where

$$\Phi_1 = (1/\sqrt{2})(e^1 \wedge e^2 - e^4 \wedge e^5), \quad J\Phi_1 = (1/\sqrt{2})(e^1 \wedge e^5 + e^4 \wedge e^2),$$
$$\Phi_2 = (1/\sqrt{2})(e^1 \wedge e^3 - e^4 \wedge e^6), \quad J\Phi_2 = (1/\sqrt{2})(e^1 \wedge e^6 + e^4 \wedge e^3),$$
$$\Phi_3 = (1/\sqrt{2})(e^2 \wedge e^3 - e^5 \wedge e^6), \quad J\Phi_3 = (1/\sqrt{2})(e^2 \wedge e^6 + e^5 \wedge e^3),$$

for some $\alpha_1, \alpha_2, \alpha_3 \in T_p^*M$. In the present paper, we shall adopt the following notational convention unless otherwise specified:

$$a, b, c, = 1, 2, 3, \quad \bar{a} = a + 3, \quad \bar{b} = b + 3, \quad \bar{c} = c + 3,$$
$$i, j, k, l = 1, 2, 3, 4, 5, 6.$$

Now we put

$$\Psi(u, v) = (1/2) \sum_{i,j} (\nabla_u J_{ij}) \nabla_v J_{ij},$$

for $u, v \in T_pM$. Then we see that Ψ satisfies

$$\Psi(u, v) = \Psi(v, u), \quad \Psi(Ju, Jv) = \Psi(u, v), \tag{2.5}$$

for $u, v \in T_pM$. We shall prove the following Lemma which plays an essential role in the proof of our Main Theorem.

Lemma 2.1. *Let $M = (M, J, g)$ be a six-dimensional quasi-Kähler manifold. Then, for each point $p \in M$, we may choose a unitary basis $\{e_i\}$ of T_pM such that the forms $\{\alpha_1, \alpha_2, \alpha_3, J\alpha_1, J\alpha_2, J\alpha_3\}$ in (2.4) are orthogonal to each other.*

PROOF. Taking account of (2.5), we may choose a unitary basis $\{e_i\} = \{e_a, e_{\bar{a}}\}$ $(e_{\bar{a}} = Je_a, a = 1,2,3)$ in such a way that

(2.6)
$$\Psi = A_1^2(e^1 \otimes e^1 + e^{\bar{1}} \otimes e^{\bar{1}}) + A_2^2(e^2 \otimes e^2 + e^{\bar{2}} \otimes e^{\bar{2}}) + A_3^2(e^3 \otimes e^3 + e^{\bar{3}} \otimes e^{\bar{3}}),$$

for some $A_1, A_2, A_3 \geq 0$. On one hand, from (2.4), we have also

$$\Psi = \alpha_1 \otimes \alpha_1 + \alpha_{\bar{1}} \otimes \alpha_{\bar{1}} + \alpha_2 \otimes \alpha_2 + \alpha_{\bar{2}} \otimes \alpha_{\bar{2}} + \alpha_3 \otimes \alpha_3 + \alpha_{\bar{3}} \otimes \alpha_{\bar{3}}. \tag{2.7}$$

Now we put

$$\begin{aligned} \alpha_a &= \sum_b T_{ab} e^b + \sum_b T_{a\bar{b}} e^{\bar{b}}, \\ \alpha_{\bar{a}} &= J\alpha_a = \sum_b T_{\bar{a}b} e^b + \sum_b T_{\bar{a}\bar{b}} e^{\bar{b}}. \end{aligned} \tag{2.8}$$

Then we have easily $T_{ab} = T_{\bar{a}\bar{b}}$, $T_{a\bar{b}} = -T_{\bar{a}b}$. From (2.5), (2.6), (2.7) and (2.8), we have also

$$\begin{aligned} &\sum_a T_{ab} T_{ac} + \sum_a T_{\bar{a}b} T_{\bar{a}c} = A_b^2 \delta_{bc}, \\ &\sum_a T_{ab} T_{a\bar{c}} + \sum_a T_{\bar{a}b} T_{\bar{a}\bar{c}} = 0, \\ &\sum_a T_{a\bar{b}} T_{a\bar{c}} + \sum_a T_{\bar{a}\bar{b}} T_{\bar{a}\bar{c}} = A_b^2 \delta_{bc}. \end{aligned} \tag{2.9}$$

By (2.5), we see that the rank of Ψ is even at each point of M. Thus, for the rank of Ψ, only the following four cases are possible: (I) rank $\Psi = 6$, (II) rank $\Psi = 4$, (III) rank $\Psi = 2$ and (IV) rank $\Psi = 0$ (i.e., $\Psi = 0$), at each point of M.

We shall check that the assertion of the Lemma is valid for all cases (I) $\sim$ (IV) above.

Case (I). Then we see that A_1, A_2 and A_3 are all positive, and hence from (2.9), we see that the 6×6 matrix $\tilde{T} = (\tilde{T}_{ij})$ given by

$$\tilde{T} = \begin{pmatrix} T_{ab}/A_b & T_{a\bar{b}}/A_b \\ T_{\bar{a}b}/A_b & T_{\bar{a}\bar{b}}/A_b \end{pmatrix} \tag{2.10}$$

is an orthogonal one. Since $\{e_i\} = \{e_a, Je_a\}$ is an orthonormal basis at p, we see that $\{A_b e_b, A_b J e_b\}$ is an orthogonal basis at p. From (2.8) we get

$$\begin{aligned}
\alpha_a &= \sum_b (T_{ab}/A_b)(A_b e^b) + \sum_b (T_{a\bar{b}}/A_b)(A_b e^{\bar{b}}) \\
&= \sum_b (\tilde{T}_{ab})(A_b e^b) + \sum_b (\tilde{T}_{a\bar{b}})(A_b e^{\bar{b}}),
\end{aligned}$$

(2.11)

$$\begin{aligned}
J\alpha_a &= \sum_b (T_{\bar{a}\bar{b}}/A_b)(A_b e^{\bar{b}}) + \sum_b (T_{\bar{a}b}/A_b)(A_b e^b) \\
&= \sum_b (\tilde{T}_{\bar{a}\bar{b}})(A_b e^{\bar{b}}) + \sum_b (\tilde{T}_{\bar{a}b})(A_b e^b).
\end{aligned}$$

From (2.4) and (2.11), we have

$$\begin{aligned}
\nabla\Omega &= \sum_b \left\{ A_b e^b \otimes \sum_a (\tilde{T}_{ab}\Phi_a - \tilde{T}_{\bar{a}b} J\Phi_a) \right\} \\
&\quad - \sum_b \left\{ A_b e^{\bar{b}} \otimes \sum_a (-\tilde{T}_{a\bar{b}}\Phi_a + \tilde{T}_{\bar{a}\bar{b}} J\Phi_a) \right\}.
\end{aligned}$$

Here, $\{\tilde{\Phi}_b = \sum_a \tilde{T}_{ab}\Phi_a - \sum_a \tilde{T}_{\bar{a}b} J\Phi_a,\ J\tilde{\Phi}_b = -\sum_a \tilde{T}_{a\bar{b}}\Phi_a + \sum_a \tilde{T}_{\bar{a}\bar{b}} J\Phi_a\}$ is a unitary basis of LM at p. Since the action of the group $U(3)$ on the set of all unitary bases of LM (at p) is transitive, there exists an element $h \in U(3)$ such that

$$\begin{aligned}
\tilde{\Phi}_1 &= 1/\sqrt{2}\{\iota(h(e_1)) \wedge \iota(h(e_2)) - \iota(h(e_4)) \wedge \iota(h(e_5))\}, \\
J\tilde{\Phi}_1 &= 1/\sqrt{2}\{\iota(h(e_1)) \wedge \iota(h(e_5)) + \iota(h(e_4)) \wedge \iota(h(e_2))\}, \\
\tilde{\Phi}_2 &= 1/\sqrt{2}\{\iota(h(e_1)) \wedge \iota(h(e_3)) - \iota(h(e_4)) \wedge \iota(h(e_6))\}, \\
J\tilde{\Phi}_2 &= 1/\sqrt{2}\{\iota(h(e_1)) \wedge \iota(h(e_6)) + \iota(h(e_4)) \wedge \iota(h(e_3))\}, \\
\tilde{\Phi}_3 &= 1/\sqrt{2}\{\iota(h(e_2)) \wedge \iota(h(e_3)) - \iota(h(e_5)) \wedge \iota(h(e_6))\}, \\
J\tilde{\Phi}_3 &= 1/\sqrt{2}\{\iota(h(e_2)) \wedge \iota(h(e_6)) + \iota(h(e_5)) \wedge \iota(h(e_3))\},
\end{aligned}$$

where ι denotes the duality $T_pM \to T_p^*M$ defined by means of the metric g. Thus we see that $\{\alpha_a, J\alpha_a\}$ is an orthogonal basis of T_p^*M.

Case (II). Without loss of generality we may assume that $A_3 = 0$ $(A_1, A_2 > 0)$. Then we have $(T_{a3}, T_{\bar{a}3}) = (T_{a\bar{3}}, T_{\bar{a}\bar{3}}) = 0$, $a = 1, 2, 3$. Thus, we may choose $(\tilde{T}_{a3}, \tilde{T}_{\bar{a}3})$, $(\tilde{T}_{a\bar{3}}, \tilde{T}_{\bar{a}\bar{3}}) \in \boldsymbol{R}^6$ in such a way that the 6×6 matrix $\tilde{T} = (\tilde{T}_{ij})$ given by

$$\tilde{T} = \begin{pmatrix} T_{ab}/A_b & \tilde{T}_{a3} & T_{a\bar{b}}/A_b & \tilde{T}_{a\bar{3}} \\ T_{\bar{a}b}/A_b & \tilde{T}_{\bar{a}3} & T_{\bar{a}\bar{b}}/A_b & \tilde{T}_{\bar{a}\bar{3}} \end{pmatrix} \tag{2.12}$$

($a = 1,2,3, b = 1,2$) is an orthogonal one. From (2.8) and (2.12), we have

$$\begin{aligned} \alpha_a &= \sum_{b=1}^{2}(T_{ab}/A_b)(A_b e^b) + \tilde{T}_{a3}(A_3 e^3) + \sum_{b=1}^{2}(T_{a\bar{b}}/A_b)(A_b e^{\bar{b}}) + \tilde{T}_{a\bar{3}}(A_3 e^{\bar{3}}) \\ &= \sum_{b=1}^{2}(\tilde{T}_{ab})(A_b e^b) + \tilde{T}_{a3}(A_3 e^3) + \sum_{b=1}^{2}(\tilde{T}_{a\bar{b}})(A_b e^{\bar{b}}) + \tilde{T}_{a\bar{3}}(A_3 e^{\bar{3}}), \\ J\alpha_a &= \sum_{b=1}^{2}(T_{\bar{a}\bar{b}}/A_b)(A_b e^{\bar{b}}) + \tilde{T}_{\bar{a}\bar{3}} A_3 e^{\bar{3}} + \sum_{b=1}^{2}(T_{\bar{a}b}/A_b)(A_b e^b) + \tilde{T}_{\bar{a}3} A_3 e^3 \\ &= \sum_{b=1}^{2}(\tilde{T}_{\bar{a}\bar{b}})(A_b e^{\bar{b}}) + \tilde{T}_{\bar{a}\bar{3}} A_3 e^{\bar{3}} + \sum_{b=1}^{2}(\tilde{T}_{\bar{a}b})(A_b e^b) + \tilde{T}_{\bar{a}3} A_3 e^3, \end{aligned} \tag{2.13}$$

for $a = 1,2,3$. From (2.4) and (2.13), we have

$$\begin{aligned} \nabla\Omega = &\sum_{b=1}^{2}\left\{A_b e^b \otimes \sum_a(\tilde{T}_{ab}\Phi_a - \tilde{T}_{\bar{a}b} J\Phi_a)\right\} + A_3 e^3 \otimes \sum_a(\tilde{T}_{a3}\Phi_a - \tilde{T}_{\bar{a}3} J\Phi_a) \\ &- \sum_{b=1}^{2}\left\{A_b e^{\bar{b}} \otimes \sum_a(-\tilde{T}_{a\bar{b}}\Phi_a + \tilde{T}_{\bar{a}\bar{b}} J\Phi_a)\right\} - A_3 e^{\bar{3}} \otimes \sum_a(-\tilde{T}_{a\bar{3}}\Phi_a + \tilde{T}_{\bar{a}\bar{3}} J\Phi_a). \end{aligned}$$

By applying the similar arguments as in the case (I), we see that $\{\alpha_a, J\alpha_a\}$ is an orthogonal basis of T_p^*M.

Similarly, in the case (III) we can also see that $\{\alpha_a, J\alpha_a\}$ is an orthogonal basis at p. The case (IV) is trivial since $\alpha_a = 0$, $J\alpha_a = 0$, $a = 1,2,3$. This completes the proof of Lemma 2.1. Q.E.D.

In the remainder of this section, we assume that M is a six-dimensional quasi-Kähler manifold of constant sectional curvature c. Then from (2.2) and (2.4) we have

$$\begin{aligned} &-2c\{g(y,z)g(x,w) - g(y,w)g(x,z) - g(y,Jz)g(x,Jw) + g(y,Jw)g(x,Jz)\} \\ &= \sum_i [\{\alpha_1(x)\Phi_1(y,e_i) - (J\alpha_1)(x)(J\Phi_1)(y,e_i) + \alpha_2(x)\Phi_2(y,e_i) \\ &\quad - (J\alpha_2)(x)(J\Phi_2)(y,e_i) + \alpha_3(x)\Phi_3(y,e_i) - (J\alpha_3)(x)(J\Phi_3)(y,e_i)\} \end{aligned} \tag{2.14}$$

$$
\begin{aligned}
&- \{\alpha_1(y)\Phi_1(x, e_i) - (J\alpha_1)(y)(J\Phi_1)(x, e_i) + \alpha_2(y)\Phi_2(x, e_i) \\
&- (J\alpha_2)(y)(J\Phi_2)(x, e_i) + \alpha_3(y)\Phi_3(x, e_i) - (J\alpha_3)(y)(J\Phi_3)(x, e_i)\}] \\
&\times [\alpha_1(e_i)\Phi_1(z, w) - (J\alpha_1)(e_i)(J\Phi_1)(z, w) + \alpha_2(e_i)\Phi_2(z, w) \\
&- (J\alpha_2)(e_i)(J\Phi_2)(z, w) + \alpha_3(e_i)\Phi_3(z, w) - (J\alpha_3)(e_i)(J\Phi_3)(z, w)],
\end{aligned}
$$

where $x, y, z, w \in T_pM$, $p \in M$. By substituting x, y, z, w by e_1, e_2, e_1, e_2 in (2.14) respectively, we have

$$
\begin{aligned}
(2.15) \qquad &\alpha_1(e_1)^2 + \alpha_1(e_4)^2 + \alpha_1(e_2)^2 + \alpha_1(e_5)^2 \\
&+ \alpha_1(e_3)(\alpha_2(e_2) - \alpha_3(e_1)) + \alpha_1(e_6)(\alpha_2(e_5) - \alpha_3(e_4)) = -4c.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(2.16) \qquad &\alpha_2(e_1)^2 + \alpha_2(e_4)^2 + \alpha_2(e_3)^2 + \alpha_2(e_6)^2 \\
&+ \alpha_2(e_2)(\alpha_3(e_1) + \alpha_1(e_3)) + \alpha_2(e_5)(\alpha_3(e_4) + \alpha_1(e_6)) = -4c,
\end{aligned}
$$

$$
\begin{aligned}
(2.17) \qquad &\alpha_3(e_2)^2 + \alpha_3(e_5)^2 + \alpha_3(e_3)^2 + \alpha_3(e_6)^2 \\
&+ \alpha_3(e_1)(\alpha_2(e_2) - \alpha_1(e_3)) + \alpha_3(e_4)(\alpha_2(e_5) - \alpha_1(e_6)) = -4c.
\end{aligned}
$$

By substituting x, y, z, w by e_1, e_2, e_1, e_5 in (2.14) respectively, we have

$$
(2.18) \qquad \alpha_1(e_3)(\alpha_2(e_5) - \alpha_3(e_4)) + \alpha_1(e_6)(\alpha_3(e_1) - \alpha_2(e_2)) = 0.
$$

Similarly, we have

$$
(2.19) \qquad \alpha_2(e_2)(\alpha_3(e_4) + \alpha_1(e_6)) + \alpha_2(e_5)(-\alpha_3(e_1) - \alpha_1(e_3)) = 0,
$$

$$
(2.20) \qquad \alpha_3(e_1)(\alpha_2(e_5) - \alpha_1(e_6)) + \alpha_3(e_4)(\alpha_1(e_3) - \alpha_2(e_2)) = 0.
$$

By substituting x, y, z, w by e_1, e_2, e_1, e_3 in (2.14) respectively, we have

$$
\begin{aligned}
(2.21) \qquad &\alpha_2(e_1)\alpha_1(e_1) + \alpha_2(e_4)\alpha_1(e_4) + \alpha_2(e_2)\alpha_1(e_2) + \alpha_2(e_5)\alpha_1(e_5) \\
&+ \alpha_2(e_3)(\alpha_2(e_2) - \alpha_3(e_1)) + \alpha_2(e_6)(\alpha_2(e_5) - \alpha_3(e_4)) = 0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(2.22) \qquad &\alpha_3(e_1)\alpha_1(e_1) + \alpha_3(e_4)\alpha_1(e_4) + \alpha_3(e_2)\alpha_1(e_2) + \alpha_3(e_5)\alpha_1(e_5) \\
&+ \alpha_3(e_3)(\alpha_2(e_2) - \alpha_3(e_1)) + \alpha_3(e_6)(\alpha_2(e_5) - \alpha_3(e_4)) = 0,
\end{aligned}
$$

$$
\begin{aligned}
(2.23) \qquad &\alpha_2(e_1)\alpha_1(e_4) - \alpha_2(e_4)\alpha_1(e_1) + \alpha_2(e_2)\alpha_1(e_5) - \alpha_2(e_5)\alpha_1(e_2) \\
&+ \alpha_2(e_3)(\alpha_2(e_5) - \alpha_3(e_4)) + \alpha_2(e_6)(\alpha_3(e_1) - \alpha_2(e_2)) = 0,
\end{aligned}
$$

$$\begin{aligned}(2.24)\qquad & \alpha_3(e_1)\alpha_1(e_4)-\alpha_3(e_4)\alpha_1(e_1)+\alpha_3(e_2)\alpha_1(e_5)-\alpha_3(e_5)\alpha_1(e_2)\\ & +\alpha_3(e_3)(\alpha_2(e_5)-\alpha_3(e_4))+\alpha_3(e_6)(\alpha_3(e_1)-\alpha_2(e_2))=0,\end{aligned}$$

$$\begin{aligned}(2.25)\qquad & \alpha_3(e_1)\alpha_2(e_1)+\alpha_3(e_4)\alpha_2(e_4)+\alpha_3(e_3)\alpha_2(e_3)+\alpha_3(e_6)\alpha_2(e_6)\\ & +\alpha_3(e_2)(\alpha_3(e_1)+\alpha_1(e_3))+\alpha_3(e_5)(\alpha_3(e_4)+\alpha_1(e_6))=0,\end{aligned}$$

$$\begin{aligned}(2.26)\qquad & \alpha_3(e_1)\alpha_2(e_4)-\alpha_3(e_4)\alpha_2(e_1)+\alpha_3(e_3)\alpha_2(e_6)-\alpha_3(e_6)\alpha_2(e_3)\\ & +\alpha_3(e_2)(\alpha_3(e_4)+\alpha_1(e_6))-\alpha_3(e_5)(\alpha_3(e_1)+\alpha_1(e_3))=0.\end{aligned}$$

In the forthcoming proof of the Main theorem, we assume that the 1-forms $\{\alpha_i\}$ and unitary basis $\{e_i\}$ in the equality (2.4) at any point $p \in M$ are chosen in such a way that the Lemma 2.1 holds for them.

3. Case of positive constant sectional curvature

In this section, we shall show that a six-dimensional quasi-Kähler manifold of positive sectional curvature is a nearly Kähler manifold of constant sectional curvature. We assume that M is a six-dimensional quasi-Kähler manifold of constant sectional curvature c. We denote by α_a^* the dual vectors to α_a $(a=1,2,3)$, and $\alpha_a^*(e) \in \boldsymbol{R}^6$ $(a=1,2,3)$ by

$$\alpha_1^*(e)=(\alpha_1(e_1),\alpha_1(e_2),\alpha_1(e_3),\alpha_1(e_4),\alpha_1(e_5),\alpha_1(e_6)),$$
$$\alpha_2^*(e)=(\alpha_2(e_1),\alpha_2(e_2),\alpha_2(e_3),\alpha_2(e_4),\alpha_2(e_5),\alpha_2(e_6)),$$
$$\alpha_3^*(e)=(\alpha_3(e_1),\alpha_3(e_2),\alpha_3(e_3),\alpha_3(e_4),\alpha_3(e_5),\alpha_3(e_6)).$$

We also set

$$\alpha_1^*(\lambda)=(\alpha_1(e_1),\alpha_1(e_2),\lambda\alpha_1(e_3),\alpha_1(e_4),\alpha_1(e_5),\lambda\alpha_1(e_6)),$$
$$\alpha_2^*(\mu)=(\alpha_2(e_1),\mu\alpha_2(e_2),\alpha_2(e_3),\alpha_2(e_4),\mu\alpha_2(e_5),\alpha_2(e_6)),$$
$$\alpha_3^*(\nu)=(\nu\alpha_3(e_1),\alpha_3(e_2),\alpha_3(e_3),\nu\alpha_3(e_4),\alpha_3(e_5),\alpha_3(e_6)).$$

Suppose that M is of positive constant sectional curvature, i.e., c is positive. Then from (2.15), (2.16) and (2.17), we have

$$\begin{aligned}(3.1)\qquad & (\alpha_1(e_3),\alpha_1(e_6))\neq(0,0),\quad (\alpha_2(e_2),\alpha_2(e_5))\neq(0,0),\\ & (\alpha_3(e_1),\alpha_3(e_4))\neq(0,0).\end{aligned}$$

Also, from (2.18), (2.19) and (2.20) and (3.1), we see that

$$(\alpha_2(e_2) - \alpha_3(e_1), \alpha_2(e_5) - \alpha_3(e_4)) = \lambda(\alpha_1(e_3), \alpha_1(e_6)), \tag{3.2}$$

$$(\alpha_3(e_1) + \alpha_1(e_3), \alpha_3(e_4) + \alpha_1(e_6)) = \mu(\alpha_2(e_2), \alpha_2(e_5)), \tag{3.3}$$

$$(\alpha_2(e_2) - \alpha_1(e_3), \alpha_2(e_5) - \alpha_1(e_6)) = \nu(\alpha_3(e_1), \alpha_3(e_4)), \tag{3.4}$$

for some constants λ, μ, ν. So, (2.15), (2.16) and (2.17) reduce respectively to

$$\alpha_1(e_1)^2 + \alpha_1(e_4)^2 + \alpha_1(e_2)^2 + \alpha_1(e_5)^2 + \lambda(\alpha_1(e_3)^2 + \alpha_1(e_6)^2) = -4c, \tag{3.5}$$

$$\alpha_2(e_1)^2 + \alpha_2(e_4)^2 + \mu(\alpha_2(e_2)^2 + \alpha_2(e_5)^2) + \alpha_2(e_3)^2 + \alpha_2(e_6)^2 = -4c, \tag{3.6}$$

$$\nu(\alpha_3(e_1)^2 + \alpha_3(e_4)^2) + \alpha_3(e_2)^2 + \alpha_3(e_5)^2 + \alpha_3(e_3)^2 + \alpha_3(e_6)^2 = -4c. \tag{3.7}$$

Further, we define $J\alpha_a^*(e) \in \boldsymbol{R}^6$ by making use of the equalities $J\alpha_a(x) = -\alpha_a(Jx)$, for $x \in T_pM$, $a = 1, 2, 3$. Then by the definitions of $\alpha_1^*(\lambda)$ and $J\alpha_1^*(e)$, we have immediately

$$\alpha_1^*(\lambda) \perp J\alpha_1^*(e). \tag{3.8}$$

From (2.23) and (2.24) we have

$$\alpha_1^*(\lambda) \perp J\alpha_2^*(e), \quad J\alpha_3^*(e). \tag{3.9}$$

Similarly, from (2.21) and (2.22), we have

$$\alpha_1^*(\lambda) \perp \alpha_2^*(e), \quad \alpha_3^*(e). \tag{3.10}$$

Thus, from (3.8) $\sim$ (3.10) and Lemma 2.1, we see that $\alpha_1^*(\lambda)//\alpha_1^*(e)$, and hence we may set

$$\alpha_1^*(\lambda) = t\alpha_1^*(e) \tag{3.11}$$

for some $t \in \boldsymbol{R}$. Since $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$, from (3.2) and (3.8) we have

$$\lambda = t. \tag{3.12}$$

If $(\alpha_1(e_1), \alpha_1(e_2), \alpha_1(e_4), \alpha_1(e_5)) \neq 0$, then, from (3.11) and (3.12) we get $t = \lambda = 1$. But, this and (3.5) yield a contradiction. Hence, we have

$$(\alpha_1(e_1), \alpha_1(e_2), \alpha_1(e_4), \alpha_1(e_5)) = 0. \tag{3.13}$$

Similarly, we have

$$(\alpha_2(e_1), \alpha_2(e_3), \alpha_2(e_4), \alpha_2(e_6)) = 0 \tag{3.14}$$

and

$$(\alpha_3(e_2), \alpha_3(e_3), \alpha_3(e_5), \alpha_3(e_6)) = 0. \tag{3.15}$$

Thus, from (3.5), (3.6) and (3.7), we have

$$\lambda\|\alpha_1\|^2 = \mu\|\alpha_2\|^2 = \nu\|\alpha_3\|^3 = -4c, \quad \lambda, \mu, \nu < 0. \tag{3.16}$$

Taking account of (3.13) $\sim$ (3.15), we may set

$$\begin{aligned} \alpha_1 &= \sqrt{4c/\lambda'}(\cos\xi e^3 + \sin\xi e^6), \\ \alpha_2 &= \sqrt{4c/\mu'}(\cos\eta e^2 + \sin\eta e^5), \\ \alpha_3 &= \sqrt{4c/\nu'}(\cos\zeta e^1 + \sin\zeta e^4), \end{aligned} \tag{3.17}$$

for some $\xi, \eta, \zeta \in \boldsymbol{R}$, here we put $\lambda' = -\lambda$, $\mu' = -\mu$, $\nu' = -\nu$. From (3.2) and (3.17), we get

$$\begin{aligned} \sqrt{1/\mu'}\cos\eta - \sqrt{1/\nu'}\cos\zeta &= -\sqrt{\lambda'}\cos\xi, \\ \sqrt{1/\mu'}\sin\eta - \sqrt{1/\nu'}\sin\zeta &= -\sqrt{\lambda'}\sin\xi. \end{aligned} \tag{3.18}$$

Similarly, from (3.3), (3.4) and (3.17), we get

$$\begin{aligned} \sqrt{1/\nu'}\cos\zeta + \sqrt{1/\lambda'}\cos\xi &= -\sqrt{\mu'}\cos\eta, \\ \sqrt{1/\nu'}\sin\zeta + \sqrt{1/\lambda'}\sin\xi &= -\sqrt{\mu'}\sin\eta, \end{aligned} \tag{3.19}$$

$$\begin{aligned} \sqrt{1/\mu'}\cos\eta - \sqrt{1/\lambda'}\cos\xi &= -\sqrt{\nu'}\cos\zeta, \\ \sqrt{1/\mu'}\sin\eta - \sqrt{1/\lambda'}\sin\xi &= -\sqrt{\nu'}\sin\zeta. \end{aligned} \tag{3.20}$$

From (3.18), (3.19) and (3.20), we have

$$\sqrt{1/\mu'}\cos(\xi-\eta) - \sqrt{1/\nu'}\cos(\zeta-\xi) = -\sqrt{\lambda'}, \tag{3.18$'$}$$

$$\sqrt{1/\nu'}\cos(\eta-\zeta) + \sqrt{1/\lambda'}\cos(\xi-\eta) = -\sqrt{\mu'}, \tag{3.19$'$}$$

$$\sqrt{1/\mu'}\cos(\eta-\zeta) - \sqrt{1/\lambda'}\cos(\zeta-\xi) = -\sqrt{\nu'}, \tag{3.20$'$}$$

$$\sqrt{1/\mu'} - \sqrt{1/\nu'}\cos(\eta-\zeta) = -\sqrt{\lambda'}\cos(\xi-\eta), \tag{3.18$''$}$$

$$\sqrt{1/\nu'} + \sqrt{1/\lambda'}\cos(\zeta - \xi) = -\sqrt{\mu'}\cos(\eta - \zeta), \tag{3.19''}$$

$$\sqrt{1/\mu'}\cos(\xi - \eta) - \sqrt{1/\lambda'} = -\sqrt{\nu'}\cos(\zeta - \xi). \tag{3.20''}$$

From (3.18′) ~ (3.20′), (3.18″) ~ (3.20″), we obtain also

$$-(\sqrt{\lambda'} + \sqrt{1/\lambda'})\cos(\xi - \eta) = \sqrt{\mu'} + \sqrt{1/\mu'}, \tag{3.21}$$

$$-(\sqrt{\mu'} + \sqrt{1/\mu'})\cos(\eta - \zeta) = \sqrt{\nu'} + \sqrt{1/\nu'}, \tag{3.22}$$

$$(\sqrt{\nu'} + \sqrt{1/\nu'})\cos(\zeta - \xi) = \sqrt{\lambda'} + \sqrt{1/\lambda'}. \tag{3.23}$$

From (3.18′) ~ (3.20′), we have

$$\begin{gathered} \cos(\xi - \eta) = -\sqrt{\lambda'\mu'}/2, \quad \cos(\eta - \zeta) = -\sqrt{\mu'\nu'}/2, \\ \cos(\zeta - \xi) = \sqrt{\nu'\lambda'}/2. \end{gathered} \tag{3.24}$$

Thus, from (3.21) ~ (3.23) and (3.24), we have

$$(\lambda' - 1)\mu' = 2, \quad (\mu' - 1)\nu' = 2, \quad (\nu' - 1)\lambda' = 2. \tag{3.25}$$

From (3.25), we can see that $\lambda' = \mu' = \nu' = 2$, and hence $\lambda = \mu = \nu = -2$. Thus, from (3.17) we have finally

$$\alpha_1 = \sqrt{2c}(\cos\xi e^3 + \sin\xi e^6), \quad \alpha_2 = \sqrt{2c}(-\cos\xi e^2 - \sin\xi e^5),$$
$$\alpha_3 = \sqrt{2c}(\cos\xi e^1 + \sin\xi e^4),$$

and hence

$$\begin{aligned} \nabla\Omega = \sqrt{c}[&\hat{e}^3 \otimes (e^1 \wedge e^2 - e^4 \wedge e^5) - \hat{e}^6 \otimes (e^1 \wedge e^5 + e^4 \wedge e^2) \\ &+ e^2 \otimes (\hat{e}^3 \wedge e^1 - \hat{e}^6 \wedge e^4) - e^5 \otimes (\hat{e}^3 \wedge e^4 + \hat{e}^6 \wedge e^1) \\ &+ e^1 \otimes (e^2 \wedge \hat{e}^3 - e^5 \wedge \hat{e}^6) - e^4 \otimes (e^2 \wedge \hat{e}^6 + e^5 \wedge \hat{e}^3)], \end{aligned} \tag{3.26}$$

where $\hat{e}^3 = \cos\xi e^3 + \sin\xi e^6$ and $\hat{e}^6 = -\sin\xi e^3 + \cos\xi e^6$. Therefore, by (3.26) we can see that M is a nearly Kähler manifold.

4. Case of negative constant sectional curvature

In this section, we shall show that there does not exist six-dimensional quasi-Kähler manifold of negative constant sectional curvature. Let M be a six-dimensional quasi-Kähler manifold of negative constant sectional curvature c. First of all, from (2.15) $\sim$ (2.17), we see that $\alpha_1, \alpha_2, \alpha_3 \neq 0$ at each point $p \in M$ (i.e., the rank of Ψ is 6 everywhere on M). We suppose that $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$. Then, by (2.18), we see that

$$\begin{aligned} \alpha_2(e_5) - \alpha_3(e_4) &= \lambda \alpha_1(e_6), \\ \alpha_3(e_1) - \alpha_2(e_2) &= -\lambda \alpha_1(e_3), \end{aligned} \tag{4.1}$$

for some real number λ. In this case, by the similar arguments as in section 3, we can see that $\alpha_1^*(\lambda) = \lambda \alpha_1^*(e)$ for some real number λ. If $(\alpha_1(e_1), \alpha_1(e_2), \alpha_1(e_4), \alpha_1(e_5)) \neq 0$, then we have $\lambda = 1$. Thus, from (4.1), we have

$$\begin{aligned} \alpha_1(e_3) - \alpha_2(e_2) + \alpha_3(e_1) &= 0, \\ \alpha_1(e_6) - \alpha_2(e_5) + \alpha_3(e_4) &= 0. \end{aligned} \tag{4.2}$$

From (2.4) and (4.2), by direct calculations, we can easily check that M is an almost Kähler manifold. However, by the result of T. Oguro ([4]), we see that this case can not occur. Thus, in this case, it must follow that $(\alpha_1(e_1), \alpha_1(e_2), \alpha_1(e_4), \alpha(e_5)) = 0$. Similarly, if $(\alpha_2(e_2), \alpha_2(e_5)) \neq 0$ (resp. $(\alpha_3(e_1), \alpha_3(e_4)) \neq 0$), then we have $(\alpha_2(e_1), \alpha_2(e_3), \alpha_2(e_4), \alpha_2(e_6)) = 0$ (resp. $(\alpha_3(e_2), \alpha_3(e_3), \alpha_3(e_5), \alpha_3(e_6)) = 0$). We here suppose that $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$, $(\alpha_2(e_2), \alpha_2(e_5)) \neq 0$, $(\alpha_3(e_1), \alpha_3(e_4)) \neq 0$. In this case, by modifying the arguments in section 3 slightly, we can see that M is a non-Kähler nearly Kähler manifold. However, it is known that a six-dimensional non-Kähler, nearly Kähler manifold is an Einstein manifold with positive scalar curvature ([4]). Therefore, we see that one of $\{(\alpha_1(e_3), \alpha_1(e_6)), (\alpha_2(e_2), \alpha_2(e_5)), (\alpha_3(e_1), \alpha_3(e_4))\}$ must be $0 \in \boldsymbol{R}^2$.

First, we suppose that $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$, $(\alpha_2(e_2), \alpha_2(e_5)) = (\alpha_3(e_1), \alpha_3(e_4)) = 0$. In this case, we have $(\alpha_1(e_1), \alpha_1(e_2), \alpha_1(e_4), \alpha_1(e_5)) = 0$ and further, $(\alpha_2(e_3), \alpha_2(e_6)) = (\alpha_3(e_3), \alpha_3(e_6)) = 0$ by virtue of Lemma 2.1. Since $\alpha_2, \alpha_3 \neq 0$, we have

$$\begin{aligned} \alpha_2^*(e) &= (\alpha_2(e_1), 0, 0, \alpha_2(e_4), 0, 0), \\ \alpha_3^*(e) &= (0, \alpha_3(e_2), 0, 0, \alpha_3(e_5), 0). \end{aligned} \tag{4.3}$$

Thus, from (2.25), (2.26) and (4.3), we have

$$\begin{aligned} \alpha_3(e_2)\alpha_1(e_3) + \alpha_3(e_5)\alpha_1(e_6) &= 0, \\ \alpha_3(e_2)\alpha_1(e_6) - \alpha_3(e_5)\alpha_1(e_3) &= 0. \end{aligned} \tag{4.4}$$

Since $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$, by (4.4) we have $(\alpha_3(e_2), \alpha_3(e_5)) = 0$ (and hence $\alpha_3^*(e) = 0$, by (4.3)), But, this is a contradiction. Similarly, we see that the case $(\alpha_2(e_2), \alpha_2(e_5)) \neq 0$, $(\alpha_1(e_3), \alpha_1(e_6)) = (\alpha_3(e_1), \alpha_3(e_4)) = 0$ (resp. $(\alpha_3(e_1), \alpha_3(e_4)) \neq 0$, $(\alpha_1(e_3), \alpha_1(e_6)) = (\alpha_2(e_2), \alpha_2(e_5)) = 0$) can not occur. Next, we suppose that $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$, $(\alpha_2(e_2), \alpha_2(e_5)) \neq 0$ and $(\alpha_3(e_1), \alpha_3(e_4)) = 0$. In this case we have $(\alpha_1(e_1), \alpha_1(e_2), \alpha_1(e_4), \alpha_1(e_5)) = 0$ and $(\alpha_2(e_1), \alpha_2(e_3), \alpha_2(e_4), \alpha_2(e_6)) = 0$. From Lemma 2.1, since $\alpha_3^*(e) \perp \alpha_1^*(e)$, $J\alpha_1^*(e)$, $\alpha_2^*(e)$, $J\alpha_2^*(e)$, we have therefore $(\alpha_3(e_2),\ \alpha_3(e_5)) = (\alpha_3(e_3), \alpha_3(e_6)) = 0$ (and hence $\alpha_3^*(e) = 0$). But, this is a contradiction. Similarly, we see that the case $(\alpha_2(e_2), \alpha_2(e_5)) \neq 0$, $(\alpha_3(e_1), \alpha_3(e_4)) \neq 0$ and $(\alpha_1(e_3), \alpha_1(e_6)) = 0$ (resp. $(\alpha_3(e_1), \alpha_3(e_4)) \neq 0$, $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$ and $(\alpha_2(e_2), \alpha_2(e_5)) = 0$) can not occur. Therefore, it must follow that $(\alpha_1(e_3), \alpha_1(e_6)) = (\alpha_2(e_2), \alpha_2(e_5)) = (\alpha_3(e_1), \alpha_3(e_4)) = 0$. Then, taking account of (2.4) and (4.2), we see that M is an almost Kähler manifold. Again, by the result of T. Oguro ([4]), we see that this can not occur. Summing up the arguments in this section, we see finally that there does not exist six-dimensional quasi-Kähler manifold of negative constant sectional curvature.

5. Locally flat case

In this section, we shall show that a six-dimensional locally flat ($c = 0$) quasi-Kähler manifold is a locally flat Kähler manifold. We assume that M is a six-dimensional locally flat quasi-Kähler manifold. At first, we show that there is no point in M such that the rank of Ψ is 6 or 4.

Suppose that there exist a point $p \in M$ such that the rank of Ψ is 6 at p. Then from (2.15) $\sim$ (2.17) (with $c = 0$), we see easily that $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$, $(\alpha_2(e_2), \alpha_2(e_5)) \neq 0$ and $(\alpha_3(e_1), \alpha_3(e_4)) \neq 0$. Thus, by (2.18) $\sim$ (2.20), we see that

$$
\begin{aligned}
&\alpha_2(e_5) - \alpha_3(e_4) = \lambda\alpha_1(e_6),\\
&\alpha_3(e_1) - \alpha_2(e_2) = -\lambda\alpha_1(e_3),\\
&\alpha_3(e_4) + \alpha_1(e_6) = \mu\alpha_2(e_5),\\
&\alpha_3(e_1) + \alpha_1(e_3) = \mu\alpha_2(e_2),\\
&\alpha_2(e_5) - \alpha_1(e_6) = v\alpha_3(e_4),\\
&\alpha_1(e_3) - \alpha_2(e_2) = -v\alpha_3(e_1),
\end{aligned}
\tag{5.1}
$$

holds for some real numbers λ, μ, v. Thus, again from (2.15) $\sim$ (2.17), taking account of (5.1), we have

$$\alpha_1(e_1)^2 + \alpha_1(e_4)^2 + \alpha_1(e_2)^2 + \alpha_1(e_5)^2 + \lambda(\alpha_1(e_3)^2 + \alpha_1(e_6)^2) = 0, \tag{5.2}$$

$$\alpha_2(e_1)^2 + \alpha_2(e_4)^2 + \mu(\alpha_2(e_2)^2 + \alpha_2(e_5)^2) + \alpha_2(e_3)^2 + \alpha_2(e_6)^2 = 0, \tag{5.3}$$

$$\nu(\alpha_3(e_1)^2 + \alpha_3(e_4)^2) + \alpha_3(e_2)^2 + \alpha_3(e_5)^2 + \alpha_3(e_3)^2 + \alpha_3(e_6)^2 = 0. \tag{5.4}$$

From (5.2) $\sim$ (5.4), we see that $\lambda, \mu, \nu \leq 0$. First, we suppose that $(\lambda, \mu, \nu) \neq (0,0,0)$. Without loss of essentiality, we may suppose that $\lambda < 0$. Then, from (2.18), (2.23) and (2.24), we have

$$\alpha_1^*(\lambda) \perp J\alpha_1^*(e), \quad J\alpha_2^*(e), \quad J\alpha_3^*(e). \tag{5.5}$$

Further, from (2.15), (2.21) and (2.22), we have also

$$\alpha_1^*(\lambda) \perp \alpha_1^*(e), \quad \alpha_2^*(e), \quad \alpha_3^*(e). \tag{5.6}$$

From (5.5) and (5.6), taking account of Lemma 2.1, we have $\alpha_1^*(\lambda) = 0$, and hence $\alpha_1^*(e) = 0$ (i.e., $\alpha_1^* = 0$), since $\lambda < 0$. But, this contradicts to the hypothesis that rank $\Psi = 6$ at p. Thus, it must follow that $(\lambda, \mu, \nu) = (0,0,0)$. Then, by (5.1) $\sim$ (5.4), we see easily that $\alpha_1^*(e) = \alpha_2^*(e) = \alpha_3^*(e) = 0$. But, this contradicts also to our hypothesis.

Next, we suppose that there exists a point $p \in M$ such that the rank of Ψ at p is 4. Then taking account of (2.7) and Lemma 2.1, without loss of essentiality, we may assume that $\alpha_3 = J\alpha_3 = 0$ and $\alpha_1, \alpha_2 \neq 0$. Then, by (2.18), we have

$$\alpha_1(e_3)\alpha_2(e_5) - \alpha_1(e_6)\alpha_2(e_2) = 0. \tag{5.7}$$

Since $\alpha_1 \neq 0$, from (2.15), we see that $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$. Thus, from (2.16) and (5.7), we have

$$(\alpha_2(e_2), \alpha_2(e_5)) = \lambda(\alpha_1(e_3), \alpha_1(e_6)), \tag{5.8}$$

for some real number $\lambda < 0$. Thus, from (2.15) and (5.8), we have

$$\alpha_1^*(e) \perp \alpha_1^*(\lambda). \tag{5.9}$$

From (2.21), (2.23) and (5.8), we have also

$$\alpha_1^*(\lambda) \perp \alpha_2^*(e), \quad J\alpha_2^*(e). \tag{5.10}$$

Since $\alpha_1^*(\lambda) \perp J\alpha_1^*(e)$, from (5.9) and (5.10), it follows that

$$\alpha_1^*(\lambda) \perp \alpha_1^*(e), \quad J\alpha_1^*(e), \quad \alpha_2^*(e), \quad J\alpha_2^*(e). \tag{5.11}$$

Next, we consider $\alpha_2^*(1/\lambda)$. By (2.16) and (5.8) with $\alpha_3 = 0$, we have

$$\alpha_2^*(1/\lambda) \perp \alpha_2^*(e). \tag{5.12}$$

By substituting x, y, z, w by e_1, e_6, e_1, e_2 in (2.14) respectively, and taking account of $\alpha_3 = J\alpha_3 = 0$, we have

$$\begin{aligned}(5.13)\qquad & \alpha_2(e_1)\alpha_1(e_4) - \alpha_2(e_4)\alpha_1(e_1) - \alpha_1(e_6)\alpha_1(e_2) \\ & + \alpha_1(e_3)\alpha_1(e_5) - \alpha_2(e_6)\alpha_1(e_3) + \alpha_2(e_3)\alpha_1(e_6) = 0.\end{aligned}$$

Thus, from (5.13), taking account of (5.8), we have

$$(5.14)\qquad J\alpha_2^*(1/\lambda) \perp \alpha_1^*(e) \quad \text{and hence } \alpha_2^*(1/\lambda) \perp J\alpha_1^*(e).$$

Similarly, substituting x, y, z, w by e_4, e_6, e_1, e_2 in (2.14) respectively, and taking account of $\alpha_3 = J\alpha_3 = 0$, we have

$$(5.15)\qquad \alpha_2^*(1/\lambda) \perp \alpha_1^*(e).$$

Since $\alpha_2^*(1/\lambda) \perp J\alpha_2^*(e)$, from (5.12), (5.14) and (5.15), we have therefore

$$(5.16)\qquad \alpha_2^*(1/\lambda) \perp \alpha_1^*(e), \quad J\alpha_1^*(e), \quad \alpha_2^*(e), \quad J\alpha_2^*(e).$$

From (5.11), since $\alpha_1^*(\lambda) \perp \alpha_2^*(e)$, we have

$$\begin{aligned}(5.17)\qquad & \alpha_1(e_1)\alpha_2(e_1) + \alpha_1(e_2)\alpha_2(e_2) + \lambda(\alpha_1(e_3)\alpha_2(e_3) + \alpha_1(e_6)\alpha_2(e_6)) \\ & + \alpha_1(e_4)\alpha_2(e_4) + \alpha_1(e_5)\alpha_2(e_5) = 0.\end{aligned}$$

From (5.17), taking account of $\alpha_1^*(e) \perp \alpha_2^*(e)$, we have

$$(\lambda - 1)(\alpha_1(e_3)\alpha_2(e_3) + \alpha_1(e_6)\alpha_2(e_6)) = 0,$$

and hence

$$(5.18)\qquad \alpha_1(e_3)\alpha_2(e_3) + \alpha_1(e_6)\alpha_2(e_6) = 0.$$

From (5.16), since $\alpha_2^*(1/\lambda) \perp \alpha_1^*(e)$, we have also

$$(5.19)\qquad \alpha_1(e_2)\alpha_2(e_2) + \alpha_1(e_5)\alpha_2(e_5) = 0.$$

Since $\alpha_1^*(e) \perp \alpha_2^*(e)$, from (5.18) and (5.19), we have further

$$(5.20)\qquad \alpha_1(e_1)\alpha_2(e_1) + \alpha_1(e_4)\alpha_2(e_4) = 0.$$

Taking account of (5.18) $\sim$ (5.20), we have immediately

$$(5.21)\qquad \alpha_2^*(1/\lambda) \perp \alpha_1^*(\lambda).$$

Similarly, taking account of $\alpha_1^*(e) \perp J\alpha_2^*(e)$, $\alpha_1^*(\lambda) \perp J\alpha_2^*(e)$ and $\alpha_2^*(1/\lambda) \perp J\alpha_1^*(e)$, we have also

$$(5.22)\qquad J\alpha_2^*(1/\lambda) \perp \alpha_1^*(\lambda)$$

and hence $\alpha_2^*(1/\lambda) \perp J\alpha_1^*(\lambda)$. Since $\alpha_1^*(e) \neq 0$, it follows that $\alpha_1^*(\lambda) \neq 0$. Thus, by (5.16), (5.21) and (5.22), it must follow that $\alpha_2^*(1/\lambda) = 0$ and hence $\alpha_2^*(e) = 0$. But, this is a contradiction.

Finally, we suppose that rank $\Psi \leq 2$ at each point in M and there is a point at which the rank of Ψ is 2. Then we see that the subset W of M on which the rank of Ψ is 2 is a non-empty open set in M. We discuss our arguments on a neighborhood $N(p)$ of a point p of W. Then, taking account of (2.7) and Lemma 2.1, without loss of essentiality, we may assume that $\alpha_2 = J\alpha_2 = \alpha_3 = J\alpha_3 = 0$ and $\alpha_1 \neq 0$. Then, by (2.15), we have

$$(\alpha_1(e_1), \alpha_1(e_2), \alpha_1(e_4), \alpha_1(e_5)) = 0, \tag{5.23}$$

and $(\alpha_1(e_3), \alpha_1(e_6)) \neq 0$. From (5.23), we see that there exists a local unitary frame field $\{e_i\}$ on $N(p)$ such that

$$\nabla\Omega = \|\nabla J\|/2\sqrt{2}\{e^3 \otimes (e^1 \wedge e^2 - e^4 \wedge e^5) - e^6 \otimes (e^4 \wedge e^2 + e^1 \wedge e^5)\}. \tag{5.24}$$

We put

$$\nabla_i\Omega_{jk} = \nabla_i J_{jk} = g((\nabla_{e_i}J)e_j, e_k)$$

and

$$\nabla^2_{hi}J_{jk} = g((\nabla^2_{e_h,e_i}J)e_j, e_k),$$

where $(\nabla^2_{e_h,e_i}J)e_j = \nabla_{e_h}((\nabla_{e_i}J)e_j) - (\nabla_{\nabla_{e_h}e_i}J)e_j - (\nabla_{e_i}J)\nabla_{e_h}e_j$. Then, since M is locally flat, by using the Ricci identity we have

$$\nabla^2_{hi}J_{jk} - \nabla^2_{ih}J_{jk} = 0. \tag{5.25}$$

Further, since M is a quasi-Kähler manifold we have

$$(\nabla_{Jx})Jy = -(\nabla_x J)y \tag{5.26}$$

for any tangent vector field x and y on $N(p)$. Now we define two holomorphic distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$ on $N(p)$ by

$$\mathfrak{D} = \operatorname{span}\{e_3, e_6 = Je_3\}, \quad \mathfrak{D}^\perp = \operatorname{span}\{e_1, e_2, e_4 = Je_1, e_5 = Je_2\}.$$

Then from (5.24), (5.25) and (5.26) we can show after some long but straightforward computations that both $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are parallel distributions. So, W is a locally product space of a 2-dimensional flat Kähler manifold and a 4-dimensional flat quasi-Kähler manifold. But it is known that a 4-dimensional quasi-Kähler manifold is an almost Kähler manifold and further that a flat almost Kähler manifold is a flat Kähler manifold. Thus it follows that W is a locally flat Kähler manifold, which is a contradiction. Therefore we see that the rank of Ψ is 0 whole on M, and hence M is locally flat Kähler manifold.

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