

MAPPING THEOREMS ON k -SEMISTRATIFIABLE SPACES*

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Abstract. In this paper the mapping properties on k -semistratifiable spaces are discussed. The main results are

- (1) A closed map from a k -semistratifiable space is a compact-covering map.
- (2) An open and compact image of a k -semistratifiable space is a σ -space.
- (3) A perfect inverse image of a k -semistratifiable space is a k -semistratifiable space if and only if it has a KG -sequence.

1. Introduction

k -semistratifiable spaces as a generalization of stratifiable spaces and \aleph -spaces have many important properties [6, 10, 14–16]. The method by maps is a powerful tool for studying generalized metric spaces. In this paper we shall establish a closed mapping theorem, an open and compact mapping theorem, and a perfect inverse mapping theorem on k -semistratifiable spaces, which deepen some known results in [1, 3, 6, 11, 14], and answers the question 4.6 in [8].

In this paper all spaces are regular and T_1 , and a topology for a space X is denoted by $\tau(X)$. Maps are continuous and onto. N denotes the set of all natural numbers.

Keywords. k -semistratifiable spaces, σ -spaces, semistratifiable spaces, G_δ -diagonals, closed maps, compact maps, perfect maps.

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DEFINITION 1.1 [14]. A space X is said to be a k -semistratifiable space if there is a function F which assigns, to each $n \in N$ and each $U \in \tau(X)$, a closed subset $F(n, U)$ of X such that

- (1) $U = \bigcup_{n \in N} F(n, U)$;
- (2) $U_1 \subset U_2 \Rightarrow F(n, U_1) \subset F(n, U_2)$ for each $n \in N$;
- (3) If K is a compact subset in X and $K \subset U$, then $K \subset F(m, U)$ for some $m \in N$.

The correspondence $U \rightarrow \{F(n, U)\}_{n \in N}$ is said to be a k -semistratification for the space X . If a space X has a sequence $\{F(n, U)\}_{n \in N}$ of closed subsets of X for each $U \in \tau(X)$ satisfying the conditions (1) and (2), then X is said to be a semistratifiable space [2], and the correspondence $U \rightarrow \{F(n, U)\}_{n \in N}$ is said to be a semistratification for X .

2. Closed Images

The behaviour of many generalized metric spaces under closed maps has been studied. In particular, it is interesting to study whether closed maps from such spaces are compact-covering maps or not [3, 4, 10, 13, 14]. It is known that closed maps from normal isocompact spaces or spaces with point-countable bases are compact-covering maps [3, 4, 12]. The main result in this section is that a closed map from a k -semistratifiable space is compact-covering.

LEMMA 2.1 [5]. *A space X is a k -semistratifiable space if and only if there is a function $g: N \times X \rightarrow \tau(X)$ such that*

- (1) $x \in g(n+1, x) \subset g(n, x)$ for each $n \in N$ and each $x \in X$;
- (2) If $x_n \in g(n, a_n)$ for each $n \in N$ and $x_n \rightarrow p$ in X , then $a_n \rightarrow p$ in X .

LEMMA 2.2 [9, 13]. *Suppose $f: X \rightarrow Y$ is a closed map and each point of X is a G_δ -set in X . If a sequence $\{x_n\}$ in X satisfies that $\{f(x_n)\}$ is a convergent sequence in Y and $f(x_n) \neq f(x_m)$ when $n \neq m$, then $\{x_n\}$ contains a convergent subsequence in X .*

Guoshi Gao proved the following theorem.

THEOREM 2.3 [3]. *A closed map from a normal k -semistratifiable space is a compact-covering map.*

We show that the normality can be removed from his theorem.

THEOREM 2.4. *A closed map from a k -semistratifiable space is a compact-covering map.*

PROOF. Suppose $f: X \rightarrow Y$ is a closed map and X is a k -semistratifiable space. Since k -semistratifiable spaces are σ -spaces [5] and σ -spaces are preserved by closed maps, Y is a σ -space. Thus every compact subset of Y is metrizable. If K is a non empty compact subset of Y , then K is metrizable, thus K is separable. Let D be a countable dense subset of K , take a countable subset E of X such that $f|E: E \rightarrow D$ is a one-to-one map. Since f is a closed map, $f(\text{cl}(E)) = \text{cl}(f(E)) = K$. We show that $\text{cl}(E)$ is a compact subset of X , thus f is a compact-covering map.

Suppose that $\text{cl}(E)$ is not a compact subset of X . Since X is a σ -space, $\text{cl}(E)$ is not countably compact. Hence $\text{cl}(E)$ contains a countable closed discrete subset $C = \{c_n: n \in \mathbb{N}\}$. Assume that the set $C \cap E$ is infinite. Since $f|(C \cap E)$ is a one-to-one closed map, the set $f(C \cap E)$ is an infinite closed discrete subset of K , which contradicts the compactness of K . Thus $C \cap E$ is finite. Without loss of generality, we may assume that $C \cap E = \emptyset$. Let $g: \mathbb{N} \times X \rightarrow \tau(X)$ be a function satisfying the condition in Lemma 2.1. Since $C \subset \text{cl}(E) \setminus E$, the set $E \cap g(n, c_n)$ is infinite. So we can take $x_n \in E \cap g(n, c_n)$ such that all x_n 's are distinct. Since f is one-to-one on E , the sequence $\{f(x_n)\}$ of distinct points of a compact metrizable space K has a convergent subsequence $\{f(x_{n_i})\}$. Then by Lemma 2.2, $\{x_{n_i}\}$ contains a subsequence converging to a point p in X . Then by Lemma 2.1(2), $\{c_{n_i}\}$ converges to p , which contradicts our assumption that C is closed discrete.

COROLLARY 2.5 [6]. *k -semistratifiable spaces are preserved by closed maps.*

PROOF. Let $f: X \rightarrow Y$ be a closed map and X be a k -semistratifiable space. Suppose $U \rightarrow \{F(n, U)\}_{n \in \mathbb{N}}$ is a k -semistratification for X . For each $n \in \mathbb{N}$ and each $V \in \tau(Y)$, define $E(n, V) = f(F(n, f^{-1}(V)))$. Then $V \rightarrow \{E(n, V)\}_{n \in \mathbb{N}}$ is a k -semistratification for Y , hence Y is a k -semistratifiable space.

REMARK 2.6. Theorem 2.4 is different from Theorem 2.3 because k -semistratifiable spaces may not be normal spaces by [10, Example 3.7.22]. On the other hand, closed maps from Moore spaces may not be compact-covering maps by [12, Remark 3.5].

3. Open and Compact Images

The study about open and compact maps by Arhangel'skii, Chaber and many others has shown that the behaviour of topological properties under open and compact maps is bad [7]. In this section we shall prove that an open and compact image of a k -semistratifiable space is a σ -space, which answers the following question in [8]: Is an open and compact image of an \aleph -space is a σ -space?

LEMMA 3.1 [1]. *An open and compact image of a σ -space is a σ -space if and only if it is a subparacompact space.*

THEOREM 3.2. *An open and compact image of a k -semistratifiable space is a σ -space.*

PROOF. Suppose $f: X \rightarrow Y$ is an open and compact map and X is a k -semistratifiable space. First of all, we show that Y is a semistratifiable space. Let $U \rightarrow \{F(n, U)\}_{n \in N}$ is a k -semistratification for X . For each $n \in N$ and each $V \in \tau(Y)$, define $E(n, V) = Y \setminus f(X \setminus F(n, f^{-1}(V)))$. Then $E(n, V)$ contained in V is a closed set in Y , and if $V_1 \subset V_2$, then $E(n, V_1) \subset E(n, V_2)$. If $y \in V$, then $f^{-1}(y) \subset F(m, f^{-1}(V))$ for some $m \in N$, then $y \in E(m, V)$, thus $V = \bigcup_{n \in N} E(n, V)$. Hence $V \rightarrow \{E(n, V)\}_{n \in N}$ is a semistratification for Y , and Y is a semistratifiable space.

Since Y is a semistratifiable space, Y is a subparacompact space, thus Y is a σ -space by Lemma 3.1.

REMARK 3.3. We have known that [7]

- (1) An open and compact image of a metric space may not be a k -semistratifiable space.
- (2) An open and compact image of a Moore space may not be a σ -space.

4. Perfect Inverse Images

A perfect inverse image of a k -semistratifiable space may not be a k -semistratifiable space [7]. We have the following conjecture.

CONJECTURE 4.1. A perfect inverse image of a k -semistratifiable space is a k -semistratifiable space if and only if it has a G_δ -diagonal sequence.

DEFINITION 4.2. Suppose $\{\mathcal{U}_n\}$ is a sequence of open covers of a space X .

(1) $\{\mathcal{U}_n\}$ is said to be a G_δ -diagonal sequence for X if $\{x\} = \bigcap_{n \in N} \text{st}(x, \mathcal{U}_n)$ for each $x \in X$.

(2) $\{\mathcal{U}_n\}$ is said to be a KG -sequence for X if $x_n \in \text{st}(a_n, \mathcal{U}_n)$ for each $n \in N$, and $x_n \rightarrow p$, $a_n \rightarrow q$, then $p = q$.

(3) $\{\mathcal{U}_n\}$ is said to be a $K - G_\delta^*$ -diagonal sequence for X if $K = \bigcap_{n \in N} \text{cl}(\text{st}(K, \mathcal{U}_n))$ for each compact subset \mathcal{K} of X [11].

LEMMA 4.3. For a space X , $K - G_\delta^*$ -diagonal sequence $\Rightarrow KG$ -sequence $\Rightarrow G_\delta$ -diagonal sequence.

PROOF. It is obvious that a KG -sequence for X is a G_δ -diagonal sequence. Now, suppose $\{\mathcal{U}_n\}$ is a $K - G_\delta^*$ -diagonal sequence for X , we can assume that each \mathcal{U}_{n+1} refines \mathcal{U}_n . If $x_n \in \text{st}(a_n, \mathcal{U}_n)$ for each $n \in N$, and $x_n \rightarrow p$, $a_n \rightarrow q$, we shall show that $p = q$. Otherwise, there is an $m \in N$ such that $p \neq a_n$ when $n \geq m$. Put $K = \{q\} \cup \{a_n : n \geq m\}$. Then K is a compact set in X , and $p \in X \setminus K = X \setminus \bigcap_{n \in N} \text{cl}(\text{st}(K, \mathcal{U}_n))$. Thus $p \in X \setminus \text{cl}(\text{st}(K, \mathcal{U}_i))$ for some $i \in N$. Take $a_j \in N$ with $j \geq \max\{i, m\}$ and $x_j \in X \setminus \text{cl}(\text{st}(K, \mathcal{U}_i))$, then $x_j \in X \setminus \text{st}(a_j, \mathcal{U}_j)$, a contradiction. This show that $p = q$, hence $\{\mathcal{U}_n\}$ is a KG -sequence for X .

THEOREM 4.4. Suppose $f: X \rightarrow Y$ is a perfect map and Y is a k -semi-stratifiable space. Then X is a k -semistratifiable space if and only if X has a KG -sequence.

PROOF. If X is a k -semistratifiable space, then there is a function $g: N \times X \rightarrow \tau(X)$ satisfying the condition in Lemma 2.1. For each $n \in N$, put $\mathcal{U}_n = \{g(n, x) : x \in X\}$. Then $\{\mathcal{U}_n\}$ is a KG -sequence for X .

Conversely, suppose X has a KG -sequence. Let $\{\mathcal{U}_n\}$ be a KG -sequence for X and \mathcal{U}_{n+1} refine \mathcal{U}_n for each $n \in N$. Since Y is a k -semistratifiable space, there is a function $g: N \times Y \rightarrow \tau(Y)$ satisfying the condition in Lemma 2.1. Define $h: N \times X \rightarrow \tau(X)$ by $h(n, x) = \text{st}(x, \mathcal{U}_n) \cap f^{-1}(g(n, f(x)))$ for each $n \in N$ and each $x \in X$. We assert that h satisfies the condition in Lemma 2.1. Suppose $x_n \in h(n, a_n)$ for each $n \in N$, and $x_n \rightarrow p$ in X . Then $f(x_n) \in g(n, f(a_n))$ for each $n \in N$, and $f(x_n) \rightarrow f(p)$, thus $f(a_n) \rightarrow f(p)$. We show that $\{a_n\}$ has a convergent subsequence in X . In fact, if $\{f(a_n) : n \in N\}$ is an infinite subset, then $\{a_n\}$ has a convergent subsequence in X by Lemma 2.2. If $\{f(a_n) : n \in N\}$ is a

finite subset, then $\{a_n\}$ has a subsequence $\{a_{n_i}\}$ such that $f(a_{n_i}) = f(p)$ for each $i \in N$, thus each $a_{n_i} \in f^{-1}(f(p))$. Since $f^{-1}(f(p))$ is a compact space with a G_δ -diagonal sequence, it is a metrizable space, hence $\{a_{n_i}\}$ has a convergent subsequence in X .

Now, suppose $\{a_{n_j}\}$ is a convergent subsequence of $\{a_n\}$ with $a_{n_j} \rightarrow q$ in X , then $p = q$ because $x_{n_j} \in \text{st}(a_{n_j}, \mathcal{U}_{n_j}) \subset \text{st}(a_{n_j}, \mathcal{U}_j)$ for each $j \in N$ and $\{\mathcal{U}_j\}$ is a KG -sequence for X . Thus $a_{n_j} \rightarrow p$. This shows that every subsequence of the sequence $\{a_n\}$ has a subsequence which converges to p , thus $a_n \rightarrow p$ in X . Hence X is a k -semistratifiable space.

DEFINITION 4.5 [11]. A space X is said to be a submesocompact space if \mathcal{U} is an open cover of X , then there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that every \mathcal{U}_n refines \mathcal{U} and for each non empty compact subset K of X there is an $n \in N$ such that $(\mathcal{U}_n)_K$ is finite.

We have shown that a submesocompact space with a G_δ -diagonal sequence has a $K - G_\delta^*$ -diagonal sequence in [11]. By Lemma 4.3 and Theorem 4.4 we have

COROLLARY 4.6 [11]. *Suppose $f: X \rightarrow Y$ is a perfect map, and X is a submesocompact space, and Y is a k -semistratifiable space. Then X is a k -semistratifiable space if and only if X has a G_δ -diagonal sequence.*

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