

ON ALMOST KÄHLER MANIFOLDS OF CONSTANT CURVATURE

By

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§ 1. Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the corresponding Kähler form is closed (or equivalently $\mathfrak{S}_{X, Y, Z} g((\nabla_X J)Y, Z) = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, where \mathfrak{S} and $\mathfrak{X}(M)$ denotes the cyclic sum and the Lie algebra of all differentiable vector fields on M respectively). A Kähler manifold, which is defined by $\nabla J = 0$, is necessarily an almost Kähler manifold. It is well-known that an almost Kähler manifold with integrable almost complex structure is a Kähler manifold. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. Concerning the integrability of almost Kähler manifolds, the following conjecture by S. I. Goldberg is known ([2]):

CONJECTURE. A compact almost Kähler Einstein manifold is a Kähler manifold.

K. Sekigawa proved the above conjecture is true for the case where the scalar curvature is nonnegative ([7]). However, the above conjecture is still open in the case where the scalar curvature is negative.

Concerning the above conjecture, Z. Olszak proved that, in dimensions ≥ 8 , an almost Kähler manifold of constant curvature is a flat Kähler manifold ([6]). In dimension 4, D. E. Blair claimed that the same assertion is valid by making use of quaternionic analysis. However, there is a gap in the final step of his proof. The statement “each $a_i = 0$ ” is not correct ([1], p. 1038). Recently, K. Sekigawa and the author proved that a $2n(\geq 4)$ -dimensional complete almost Kähler manifold of constant sectional curvature is a flat Kähler manifold ([5]). The proof in [5] is essentially dependent on the completeness.

The aim of the present paper is to prove that the hypothesis of completeness in the above result is needless, namely we prove the following.

THEOREM. *In dimensions ≥ 4 , there are no almost Kähler manifolds of constant curvature unless the constant is 0, in which case the manifold is Kählerian.*

In dimensions ≥ 8 , above Theorem is nothing but the result of Z. Olszak, but we shall give a proof which does not depend on the dimension.

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§2. Preliminaries

Let $M = (M, J, g)$ be a $2n$ -dimensional almost Kähler manifold. We denote by ∇ and R the Riemannian connection and the curvature tensor of M with respect to the Riemannian metric g . Here, we assume that the curvature tensor R is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ for $X, Y \in \mathfrak{X}(M)$. Further, we assume that M is oriented by the volume form $dM = (-1)^n \Omega^n / n!$, where Ω is the Kähler form defined by $\Omega(X, Y) = g(X, JY)$ for $X, Y \in \mathfrak{X}(M)$. We recall a curvature identity for almost Kähler manifold due to A. Gray ([3]):

$$(2.1) \quad \begin{aligned} & R(w, x, y, z) - R(w, x, Jy, Jz) - R(Jw, Jx, y, z) + R(Jw, Jx, Jy, Jz) \\ & \quad + R(Jw, x, Jy, z) - R(Jw, x, Jz, y) - R(Jx, w, Jy, z) + R(Jx, w, Jz, y) \\ & = 2g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y) \end{aligned}$$

for $w, x, y, z \in T_p M, p \in M$. If M is also a space of constant curvature c , then the equality (2.1) becomes

$$\begin{aligned} & 2c\{g(x, y)g(w, z) - g(x, z)g(w, y) - g(x, Jy)g(w, Jz) + g(x, Jz)g(w, Jy)\} \\ & = g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y) \end{aligned}$$

and hence, we have

$$(2.2) \quad \|\nabla J\|^2 = -8cn(n-1).$$

Since we may assume that $n \geq 2$, this implies that $c \leq 0$ and that $c = 0$ if and only if M is a flat Kähler manifold.

In the present paper, unless otherwise specified, we assume that all manifolds are connected and of class C^∞ and that all tensor fields are of class C^∞ .

§ 3. Proof of the theorem

If there exists a strictly almost Kähler structure on a space of constant curvature, then we have, in the view of the argument in section 2, that locally hyperbolic space must carry such a structure. We denote by H^{2n} the $2n$ -dimensional hyperbolic space of constant curvature -1 . As a model of H^{2n} , we take the upper half space $R_+^{2n} = \{(x_1, \dots, x_{2n}) \in R^{2n} | x_1 > 0\}$ of R^{2n} and the metric g given by

$$g = \frac{1}{x_1^2} \sum_{i=1}^{2n} dx_i \otimes dx_i.$$

Let $\{X_i = x_1(\partial/\partial x_i)\}_{i=1, \dots, 2n}$ be a global orthonormal frame field. Then

$$[X_1, X_i] = -[X_i, X_1] = X_i \quad \text{for } i = 2, \dots, 2n,$$

and are otherwise zero. If we put $\Gamma_{ijk} = g(\nabla_{X_i} X_j, X_k)$, then

$$(3.1) \quad \Gamma_{i11} = -\Gamma_{11i} = 1 \quad \text{for } i = 2, \dots, 2n,$$

and are otherwise zero.

Now, we assume that there exists a compatible almost Kähler structure (J, g) on a connected open neighborhood U of a point $p \in H^{2n}$. If we put $J_{ij} = g(JX_i, X_j)$, then

$$J_{ij} = -J_{ji}, \quad \sum_{u=1}^{2n} J_{iu} J_{ju} = \delta_{ij}.$$

We can choose isometries $\phi_{(1)}, \dots, \phi_{(2n)}$ of a neighborhood of p in U such that

- (1) $\phi_{(a)}(p) = p$ for $a = 1, \dots, 2n$;
- (2) $(\phi_{(1)})_{*p}$ is the identity mapping of the tangent space at p ;
- (3) $(\phi_{(a)})_{*p}(X_1) = (X_a)_p$, $(\phi_{(a)})_{*p}(X_a) = (X_1)_p$ and $(\phi_{(a)})_{*p}(X_i) = (X_i)_p$ ($i \neq 1, a$) for $a = 2, \dots, 2n$.

We note that $\phi_{(1)}$ is the identity mapping. For brevity, we shall write $\phi_{(a)}$ instead of $(\phi_{(a)})_*$. We put $\phi_{(a)}(X_i) = \sum_{j=1}^{2n} B_{ij}^{(a)} X_j$. Then, from (2) and (3), we have

$$(B_{ij}^{(1)}(p)) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

From this equality, by direct calculation, we have

$$\begin{aligned}
 (3.5) \quad \sum_{i,j=1}^{2n} (X_1 J_{ij}^{(a)})^2 &= \sum_{i,j=1}^{2n} (X_1 J_{ij})^2 + 2 \sum_{i,u=1}^{2n} (X_1 B_{iu}^{(a)})^2 \\
 &+ 2 \sum_{i,j,u,u',v,v'=1}^{2n} (X_1 B_{iu}^{(a)})(X_1 B_{jv'}^{(a)}) B_{jv}^{(a)} B_{iu'}^{(a)} J_{uv} J_{u'v'} \\
 &+ 4 \sum_{j,u,v,v'=1}^{2n} (X_1 B_{jv}^{(a)})(X_1 J_{uv'}) B_{jv'}^{(a)} J_{uv}.
 \end{aligned}$$

From (3.4), we have

$$\sum_{i,u=1}^{2n} (X_1 B_{iu}^{(a)})^2(p) = (X_1 B_{11}^{(a)})^2(p) + \sum_{i,u \geq 2} (X_1 B_{iu}^{(a)})^2(p) = 2(\delta_1^a - 1)^2.$$

Now, we set

$$\xi^{(a)} = \sum_{u,v=1}^{2n} B_{1u}^{(a)} J_{1u} B_{1v}^{(a)} J_{1v}, \quad \eta^{(a)} = \sum_{u,v=1}^{2n} (X_1 J_{1u}) B_{1v}^{(a)} J_{uv}.$$

Then, from (3.2) and (3.4), by direct calculation, we can derive

$$\begin{aligned}
 &\sum_{i,j,u,u',v,v'=1}^{2n} (X_1 B_{iu}^{(a)})(p)(X_1 B_{jv'}^{(a)})(p) B_{jv}^{(a)}(p) B_{iu'}^{(a)}(p) J_{uv}(p) J_{u'v'}(p) \\
 &= \sum_{u',v=1}^{2n} (X_1 B_{11}^{(a)})(p)(X_1 B_{11}^{(a)})(p) B_{1v}^{(a)}(p) B_{1u'}^{(a)}(p) J_{1v}(p) J_{u'1}(p) \\
 &+ \sum_{u',v=1}^{2n} \sum_{j,v' \geq 2} (X_1 B_{11}^{(a)})(p)(X_1 B_{jv'}^{(a)})(p) B_{jv}^{(a)}(p) B_{1u'}^{(a)}(p) J_{1v}(p) J_{u'v'}(p) \\
 &+ \sum_{u',v=1}^{2n} \sum_{i,u \geq 2} (X_1 B_{iu}^{(a)})(p)(X_1 B_{11}^{(a)})(p) B_{1v}^{(a)}(p) B_{iu'}^{(a)}(p) J_{uv}(p) J_{u'1}(p) \\
 &+ \sum_{u',v=1}^{2n} \sum_{i,j,u,v' \geq 2} (X_1 B_{iu}^{(a)})(p)(X_1 B_{jv'}^{(a)})(p) B_{jv}^{(a)}(p) B_{iu'}^{(a)}(p) J_{uv}(p) J_{u'v'}(p) \\
 &= -(\delta_1^a - 1)^2 \xi^{(a)}(p) + (1 - \delta_1^a)^2 \sum_{u,v \geq 2} B_{1u}^{(a)}(p) B_{1v}^{(a)}(p) J_{u1}(p) J_{1v}(p) \\
 &= -2(\delta_1^a - 1)^2 \xi^{(a)}(p),
 \end{aligned}$$

and

$$\begin{aligned}
& \sum_{j,u,v,v'=1}^{2n} (X_1 B_{jv}^{(a)})(p)(X_1 J_{uv})(p) B_{jv'}^{(a)}(p) J_{uv}(p) \\
&= \sum_{u,v'=1}^{2n} (X_1 B_{11}^{(a)})(p)(X_1 J_{uv})(p) B_{1v'}^{(a)}(p) J_{u1}(p) \\
&\quad + \sum_{u,v'=1}^{2n} \sum_{j,v \geq 2} (X_1 B_{jv}^{(a)})(p)(X_1 J_{uv})(p) B_{jv'}^{(a)}(p) J_{uv}(p) \\
&= (\delta_1^a - 1)\eta^{(a)}(p) + (1 - \delta_1^a) \sum_{u=1}^{2n} \sum_{v \geq 2} (X_1 J_{u1})(p) B_{1v}^{(a)}(p) J_{uv}(p) \\
&= 2(\delta_1^a - 1)\eta^{(a)}(p).
\end{aligned}$$

Therefore, from (3.5), we obtain

$$\begin{aligned}
(3.6) \quad \sum_{i,j=1}^{2n} (\nabla_1 J_{ij}^{(a)})^2(p) &= \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) + 4(\delta_1^a - 1)^2 \\
&\quad - 4(\delta_1^a - 1)^2 \xi^{(a)}(p) + 8(\delta_1^a - 1)\eta^{(a)}(p).
\end{aligned}$$

Thus, taking account of

$$\xi^{(a)}(p) = (J_{1a})^2(p), \quad \eta^{(a)}(p) = - \sum_u (\nabla_1 J_{1u})(p) J_{au}(p),$$

we have

$$\begin{aligned}
(3.7) \quad \sum_{a,i,j=1}^{2n} (\nabla_1 J_{ij}^{(a)})^2(p) &= 2n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) + 8(n-1) \\
&\quad + 8 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u})(p) J_{au}(p).
\end{aligned}$$

Since

$$(3.8) \quad g((\nabla_X J^{(a)})Y, Z) = g((\nabla_{\phi_{(a)}(X)} J)\phi_{(a)}(Y), \phi_{(a)}(Z)),$$

we have

$$\sum_{i,j} (\nabla_1 J_{ij}^{(a)})^2(p) = \sum_{i,j} (\nabla_a J_{ij})^2(p).$$

Thus, (3.7) becomes

$$\|\nabla J\|^2(p) = 2n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) + 8(n-1) + 8 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u})(p) J_{au}(p).$$

Therefore, from (2.2), we have

$$(3.9) \quad n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) = 4(n-1)^2 - 4 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u})(p) J_{au}(p).$$

Since the above argument does not depend on the choice of the almost complex structure J , corresponding to (3.9), we can obtain

$$(3.10) \quad n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij}^{(c)})^2(p) = 4(n-1)^2 - 4 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u}^{(c)})(p) J_{au}^{(c)}(p).$$

for $c = 2, \dots, 2n$. Therefore, from (3.9) and (3.10), we have

$$n \sum_{c,i,j=1}^{2n} (\nabla_1 J_{ij}^{(c)})^2(p) = 8(n-1)^2 - 4 \sum_{c,a,u=1}^{2n} (\nabla_1 J_{1u}^{(c)})(p) J_{au}^{(c)}(p).$$

Again from (3.8), the above equality becomes

$$n \sum_{c,i,j=1}^{2n} (\nabla_c J_{ij})^2(p) = 8n(n-1)^2 - 4 \sum_{c,a,u=1}^{2n} (\nabla_c J_{cu})(p) J_{au}(p),$$

namely,

$$(3.11) \quad n\|\nabla J\|^2(p) = 8n(n-1)^2 - 4 \sum_{c,a,u} (\nabla_c J_{cu})(p) J_{au}(p).$$

Since an almost Kähler manifold is necessarily a semi-Kähler manifold, the second term in the right-hand-side of (3.11) must vanish. Therefore, (3.11) yields

$$\|\nabla J\|^2(p) = 8(n-1)^2.$$

Hence, from (2.2), we have

$$8n(n-1) = 8(n-1)^2.$$

This implies $n = 1$.

Therefore, we have finally our Theorem.

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