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VANISHING THEOREM FOR 2-TORSION INSTANTON INVARIANTS

By

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Abstract. For any closed, oriented, simply connected spin 4-manifold X with $b_X^+ > 1$ and even, by Fintushel and Stern [7], differential-topological polynomial invariants with its values in $\boldsymbol{Z}_2$ are defined. These invariants are analogues of Donaldson polynomial invariants. In [7], it is proved that if $X = X' \sharp S^2 \times S^2$, then these invariants do not always vanish. But in this paper, it is proved that these invariants vanish for a large class of connected sums.

1. Introduction

For any smooth oriented closed simply connected 4-manifold X with b_X^+ greater than one and odd, in [4] Donaldson polynomial invariants are defined, where b_X^+ is the dimension of the maximal positive subspace for the intersection form on $H_2(X)$. They are evaluations of certain elements of the rational cohomology of the space $\mathfrak{B}_{X,l}^*$ of equivalence classes of irreducible connections in a $SU(2)$-bundle over X of charge l on a homology class represented by the moduli space of anti-self-dual connections. In the case when b_X^+ is even, analogues of these invariants are defined by using the cohomology $H^*(\mathfrak{B}_{X,l}^*; R)$ for some coefficient ring R ([5]). In particular, in [7], torsion invariants are defined when $R = \boldsymbol{Z}_2$, and some properties of these are proved. The main properties, which are in contrast with the vanishing theorem for the Donaldson polynomials for connected sums, are that the torsion invariants do not always vanish for $X \sharp S^2 \times S^2$, where X is spin with $b_X^+ > 1$ odd. But in this article, it is proved that the torsion invariants vanish for large class of connected sums.

Let $\mathscr{M}_{X,l}$ be the moduli space of anti-self-dual $SU(2)$ connections of charge l. In case $b_X^+ > 1$ is even, for a generic metric, the moduli space $\mathscr{M}_{X,l}$ is the smooth manifold and we have $\dim \mathscr{M}_{X,l} = 8l - 3(1 + b_X^+) = 2d + 1$. So if there is a non-

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trivial 1-dimensional cohomology class u in $\mathfrak{B}^*_{X,l}$, then, for homology classes $z_1, \dots, z_d \in H_2(X; \boldsymbol{Z})$, similar polynomial invariants can be defined by evaluating the class $\mu(z_1) \cup \cdots \cup \mu(z_d) \cup u$ on the fundamental class of $\mathcal{M}_{X,l}$, where $\mu(z_i) \in H^2(\mathfrak{B}^*_{X,l}; \boldsymbol{Q})$ (for the definition of homomorphism μ, see [6]). In the case l is even and X is spin, there is a non-trivial class $u_1 \in H^1(\mathfrak{B}^*_{X,l}; \boldsymbol{Z}_2)$. Thus in this case, for large enough l, there is a polynomial invariant $q_{l,u_1,X}$ of degree d in $H_2(X; \boldsymbol{Z})$ with values in $\boldsymbol{Z}_2$. Like as Donaldson invariants, $q_{l,u_1,X}$ is an invariant of the smooth structure of X. (See section 2 for a more complete description.)

Our main theorem, whose proof will be given in section 3, is

THEOREM 1.1. *Let X_1, X_2 be closed simply connected spin 4-manifolds with $b^+_{X_i} > 1$ and with $b^+_{X_1} + b^+_{X_2}$ even. Suppose that $l_i > 3(1 + b^+_{X_i})/4$ and $l_1 + l_2$ even. If we set $d_i = 4l_i - 3(1 + b^+_{X_i})/2$, for $r \neq 0$, $d_1 + d_2 + 1$*

$$q_{l_1+l_2,u_1,X_1 \sharp X_2}(z_1, \dots, z_r, w_1, \dots, w_{d_1+d_2+1-r}) = 0,$$

where $z_1, \dots, z_r \in H_2(X_1; \boldsymbol{Z})$ and $w_1, \dots, w_{d_1+d_2+1-r} \in H_2(X_2; \boldsymbol{Z})$.

In [7], Fintushel and Stern considered the torsion invariants of the connected sums $X \sharp S^2 \times S^2$, in the case when X is closed simply connected oriented spin 4-manifold with b^+_X odd. One of the main theorems in that article is the "nonvanishing" theorem.

THEOREM 1.2 (Fintushel–Stern [7]). *Let X be a closed simply connected spin 4-manifold with $b^+_X > 1$ and odd. Suppose l is odd and in stable range (i.e. $l > 3(1 + b^+_X)/4$).*

Then $q_{l+1,u_1,X\sharp S^2\times S^2}$ is defined and for any classes $z_1, \dots, z_d \in H_2(X; \boldsymbol{Z})$ and for $x = [S^2 \times \{point\}]$ and $y = [\{point\} \times S^2]$ in $H_2(S^2 \times S^2; \boldsymbol{Z})$ we have

$$q_{l,X}(z_1, \dots, z_d) \equiv q_{l+1,u_1,X\sharp S^2\times S^2}(z_1, \dots, z_d, x, y) \mod 2$$

Here $q_{l,X}$ in the left-hand side is the ordinary Donaldson polynomial and we set $2d = 8l - 3(1 + b^+_X)$.

Note that $b^+_{S^2\times S^2} = 1$. Thus, in general, Theorem 1.1 dose not hold in the case when one of the X_i has $b^+_{X_i} = 1$.

2. Torsion invariants

Let X be a closed, oriented, simply connected smooth 4-manifold, and P be a $SU(2)$-bundle over X with the second Chern class $c_2(P) = l \geq 0$. We write

$\mathscr{A} = \mathscr{A}(P)$ for the space of connections on P and $\mathscr{G} = \mathscr{G}(P)$ for the group of gauge transformations on P. The gauge group $\mathscr{G}$ acts on $\mathscr{A}$ by pullback and its quotient $\mathscr{A}/\mathscr{G}$ is denoted by $\mathfrak{B}(P) = \mathfrak{B}_{X,l}$. Denote by $\mathfrak{B}^*_{X,l} \subset \mathfrak{B}_{X,l}$ the subset of irreducible connections. The moduli space of equivalence classes of anti-self-dual connections on P is denoted by $\mathscr{M}_{X,l}$, i.e.

$$\mathscr{M}_{X,l}(g) = \{A \in \mathscr{A} | *_g F_A = -F_A\}/\mathscr{G},$$

where F_A is the curvature of A, g is a Riemaniann metric on X, and $*_g$ is the Hodge star operator induced from g. The transversality argument and the Atiyah-Singer index theorem show that the moduli space is, if non-empty, an $8l - 3(1 + b_X^+)$ dimensional smooth manifold for a generic metric, when $b_X^+ > 0$ and $l > 0$.

To define torsion invariants of X with $b_X^+ > 1$ and even, we need a non-trivial torsion class $u_1 \in H^1(\mathfrak{B}^*_{X,l}; \mathbf{Z}_2)$. But it is proved in [10] (see also [7], [2]) that

$$\pi_1(\mathfrak{B}^*_{X,k}) \simeq \begin{cases} \mathbf{Z}_2, & (X \text{ is spin and k is even}) \\ 0, & (\text{otherwise}). \end{cases}$$

So we can define the class u_1 as a generator of $H^1(\mathfrak{B}^*_{X,l}; \mathbf{Z}_2)$. This class is identified with the first Stiefel-Whitney class of a (real) determinant line bundle of a family of (real) twisted Dirac operators parametrized by $\mathfrak{B}^*_{X,l}$ ([3], [7], [2]). Here suppose b_X^+ is greater than one and even. The moduli space of anti-self-dual connections on the $SU(2)$ bundle over X with the second Chern class $c_2 = k$ has formal dimension $8k - 3(1 + b_X^+) = 2d + 1$. Let homology classes $z_1, \dots, z_d \in H_2(X; \mathbf{Z})$ be represented by generic embedded surfaces $\Sigma_1, \dots, \Sigma_d \subset M$, and $V_{\Sigma_1}, \dots, V_{\Sigma_d} \subset \mathfrak{B}^*_{X,k}$ be codimension 2 geometric representatives associated with $\mu(\Sigma_i)$, where the μ-map $\mu : H_2(X, \mathbf{Z}) \to H^2(\mathfrak{B}^*_{X,k}; \mathbf{Z})$ defined in [3]. By the usual dimension counting argument as in the case of Donaldson invariants, if $k > (3(1 + b_X^+) + 1)/4$ (which is "stable range" condition), then the intersection

$$\mathscr{I} = \mathscr{M}_{X,k} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$$

is a 1-dimensional compact manifold (for generic metrics on X).

Definition 2.1.

$$q_{k,u_1,X}(\Sigma_1, \dots, \Sigma_d) = \langle V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap \mathscr{M}_{X,k}, u_1 \rangle.$$

As the Donaldson invariants, next conditions hold.

(i) $q_{k,u_1,X}(\Sigma_1, \dots, \Sigma_d)$ depends on Σ_i only through its homology class $[\Sigma_i] \in H_2(X; \mathbf{Z})$,

(ii) $q_{k,u_1,X}(\Sigma_1,\dots,\Sigma_d)$ is multilinear and symmetric in $[\Sigma_1],\dots,[\Sigma_d]$,

(iii) $q_{k,u_1,X}$ is natural with respect to orientation preserving diffeomorphism.

Condition (iii) contains the assertion that $q_{k,u_1,X}$ is independent of the choice of generic metric and is an invariant of the oriented diffeomorphism type of X.

3. The proof of Theorem 1.1

Now, we prove Theorem 1.1, which is some version of vanishing theorem.

Proof of Theorem 1.1. We only prove the case when both $b_{X_1}^+$ and $b_{X_2}^+$ are odd. (If both $b_{X_1}^+$ and $b_{X_2}^+$ are even, it can be proved by exactly the same argument.) First, we must see that $q_{l_1+l_2,u_1,X_1\sharp X_2}$ is well-defined. Since $b_{X_1\sharp X_2}^+ = b_{X_1}^+ + b_{X_2}^+ > 1$ and $l_i > 3(1+b_{X_i}^+)/4$, we have $l_1+l_2 > (3(1+b_{X_1}^+ + b_{X_2}^+)+3)/4 > 3(1+b_{X_1\sharp X_2}^+)/4 + 1/4$, so l_1+l_2 is in the stable range to $X_1\sharp X_2$. On the other hand, the dimension of the moduli space $\mathcal{M}_{X_1\sharp X_2,l_1+l_2}$ equals to $8(l_1+l_2) - 3(1+b_{X_1\sharp X_2}) = 2(d_1+d_2)+3 = 2(d_1+d_2+1)+1$. So $q_{l_1+l_2,u_1,X_1\sharp X_2}$ is well-defined and in $Sym_{\mathbf{Z}_2}^{d_1+d_2+1}(H_2(X_1\sharp X_2;\mathbf{Z}))$.

Now, suppose first that $0<r<d_1+2$. Let $X = X_1\sharp X_2$. We will consider a sequence of metrics $\{g_\nu\}$ on X, as $\nu\to 0$, whose limit is the one point union $(X_1\vee X_2, g_{X_1}\vee g_{X_2})$ where g_{X_1} and g_{X_2} are generic.

For each ν, we may assume that $\mathcal{I}_X(\nu) = \mathcal{M}_{X,l_1+l_2}(g_\nu)\cap V_1\cap\cdots\cap V_{d_1+d_2+1}$ is non-empty, where the V_i's are the codimension 2 submanifolds in $\mathfrak{B}_{X,l_1+l_2}^*\cup\{[\theta]\}$ ($[\theta]$ is the gauge equivalence class of the trivial connection on X) corresponding to generic embedded surfaces representing the homology classes z_i and w_j. Then we take a sequence $\{[A_\nu]\}$, where $[A_\nu]\in\mathcal{I}_X(\nu)$. By Uhlenbeck's theorem on compactness and removability of singularities, there is a subsequence of $\{[A_\nu]\}$, which has a weak limit $([A_1], [A_2]; (x_1,\dots,x_p,y_1,\dots,y_\sigma))$ as $\nu\to 0$, here, $[A_1]\in\mathcal{M}_{X_1,m}(g_{X_1})$, $[A_2]\in\mathcal{M}_{X_2,n}(g_{X_2})$, and $x_i\in X_1$, $y_j\in X_2$ are the bubble points. Note that by counting charges $m+n+p+\sigma\le l_1+l_2$.

Lemma 3.1. *For the sequence $\{[A_\nu]\}$ ($[A_\nu]\in\mathcal{I}_X(\nu)$), if ν converges to 0, we have only the weak limits of the following form,*

$$([\Theta],[A_2];(x_1,\dots,x_p,y_1,\dots,y_\sigma)),$$

where Θ is a trivial connection on X_1, and $[A_2]\in\mathcal{M}_{X_2,n}(g_{X_2})$ for $n\neq 0$.

Proof. As the same way in [7], we can prove this lemma by using counting argument as follows.

Let $([A_1], [A_2]; (x_1, \dots, x_\rho, y_1, \dots, y_\sigma))$ be the weak limit of the sequence $\{[A_\nu]\}$.

If both m and n are greater than 0, it means that both $[A_1]$ and $[A_2]$ are not trivial connections, $[A_1]$ must lie on at least $r - 2\rho$ of $V_1, \dots, V_r$, when we choose the surfaces, which represent the homology classes z_i and w_j, in general position. Hence we have a inequality $8m - 3(1 + b^+_{X_1}) = 2d_1 - 8(l_1 - m) \geq 2(r - 2\rho)$. Similarly, by the condition of $[A_2]$, we deduce $2d_2 - 8(l_2 - n) \geq 2(d_1 + d_2 + 1 - r - 2\sigma)$. Conbining these inequalities, we have $8m + 8n + 4\rho + 4\sigma \geq 8(l_1 + l_2) + 2$. Since $\rho \geq 0$ and $\sigma \geq 0$, this inequality contradicts the charge count.

If m is greater than 0 and n equals to 0, for $[A_1]$, we have the same inequality as above. On the other hand, in this case, A_2 is the trivial connection on X_2. Here, recall the definition of the representative V_*. For a homology class $w_j \in H_2(X_2; \boldsymbol{Z})$, we choose a generic embedded surface Σ_j which represents w_j. Then we can construct a line bundle $\mathscr{L}_{\Sigma_j}$ over $\mathfrak{B}^*_{X_2}$ and extend it to $\tilde{\mathscr{L}}_{\Sigma_j}$ over $\mathfrak{B}^*_{X_2} \cup [\Theta']$, where $[\Theta']$ is the gauge equivalence class of the trivial connection on X_2. (For the precise construction of $\mathscr{L}_{\Sigma_j}$ and $\tilde{\mathscr{L}}_{\Sigma_j}$, see [3], [6].) We have a section $s_j : \mathfrak{B}^*_{X_2} \cup [\Theta'] \to \tilde{\mathscr{L}}_{\Sigma_j}$ such that $[\Theta'] \notin s_j^{-1}(0)$. We define the representative V_j as $s_j^{-1}(0)$. So $A_2 = \Theta'$ is not in any of $V_{r+1}, \dots, V_{d_1+d_2+1}$. So we have $d_1 + d_2 + 1 - r - 2\sigma \leq 0$. To sum up these inequalities and the charge count, $r \geq d_1 + d_2 - 3(1 + b^+_{X_2})/2 + 2\rho + 2$. Since l_2 is in the stable range, we have $d_1 + d_2 - 3(1 + b_{X_2})/2 > d_1$. So we get $r > d_1 + 2\rho + 2$, and this contradicts our assumption $0 < r < d_1 + 2$.

Suppose that $m = n = 0$. In this case, as above, we have $r \leq 2\rho$ and $d_1 + d_2 + 1 - r \leq 2\sigma$. From these inequalities and the charge count $\rho + \sigma \leq l_1 + l_2$, the inequality $2(l_1 + l_2) \leq 3(1 + b^+_X)/2 + 1/2$ holds. This contradicts the stable range condition. □

Next, for sufficient small ν, we construct the obstruction bundle model which gives the finite dimensional model of $\mathscr{I}_X(\nu)$.

Definition. Fix an $\varepsilon > 0$. A gauge orbit $[A] \in \mathfrak{B}^*_{X, l_1+l_2}$ is in U if and only if there are disjoint balls B_i, the number of these balls is finite, with centers p_i and radii λ_i in $X'_1 = X_1 \backslash B_{X_1}(p_0, \nu)$, where $B_{X_1}(p_0, \nu)$ is a geodesic ball with center $p_0 \in X_1$ and radius ν which is removed when we make the connected sum $X_1 \sharp X_2$, such that

(i) $\int_{X'_1 \cup B_i} |F_A|^2 < \varepsilon$.

(ii) $\sum \lambda_i^2 < \varepsilon$.

(iii) $|\int_{B_i} |F_A^-|^2 - 8\pi^2 m_i| < \varepsilon$ for some positive integers m_i.

From the definition above we see that, if we fix an ε, then for small enough v, $\mathscr{I}_X(v) \subset U$. Since $\mathscr{I}_X(v)$ is 1-dimensional and codimension of the subset U' in U which consists of connections for which some m_i is greater than one is at least four, to see the homology class $[\mathscr{I}_X(v)]$ vanish, we can modify the condition (iii) defining U so that

(iii)′ $|\int_{B_i} |F_A^-|^2 - 8\pi^2| < \varepsilon$, for each i.

Since we can decompose U to disjoint union $\bigcup U_p$ (U_p is the subset of U which consists of the connections with the property that the number of the points $\{p_i\}$ is equal to p), in order to prove that the homology class of $\mathscr{I}_X(v)$ is trivial it suffices to verify that $\mathscr{I}_{X,p}(v) = \mathscr{I}_X(v) \cap U_p$ is homologically trivial in U_p for each p. Note that by counting argument and hypothesis, $p \geq r/2 > 0$. By the same way, we define an open subset $U_{p,X_1} \subset \mathfrak{B}^*_{X_1,p}$, to be the connections which satisfy the conditions of (i), (ii) and (iii)′ above.

Fix a $p > 0$. Let $\pi : F^+_{X_1} \to X_1$ be the oriented orthonormal frame bundle of $\bigwedge^+_{X_1}$, and $G^p(F^+_{X_1} \times \boldsymbol{R}^+) \subset S^p(F^+_{X_1} \times \boldsymbol{R}^+)$ be the preimage of $\{((x_1, \lambda_1), \dots, (x_p, \lambda_p)) \in S^p(X_1 \times \boldsymbol{R}^+) \mid x_i \neq x_j (i \neq j)\}$, where S^p means unordered p-tuple. If we set $\mathscr{N}_p^0 = \{((f_1, \lambda_1), \dots, (f_p, \lambda_p)) \in G^p(F^+_{X_1} \times \boldsymbol{R}^+) | \sum \lambda_i^2 < \varepsilon\}$, we consider the diagonal action of $SO(3)$ on $\mathscr{N}_p^0$ (where $SO(3)$ acts trivially on the $\boldsymbol{R}^+$ factors). Then there is a smooth embedding

$$\gamma : \mathscr{N}_p = \mathscr{N}_p^0 / SO(3) \to \mathfrak{B}^*_{X_1,p},$$

whose image is in U_{p,X_1}. We identify the image $\gamma(\mathscr{N}_p)$ with $\mathscr{N}_p$ and we can take a section ψ of a rank $3b^+_{X_1}$ vector bundle

$$\eta_{X_1} = \bigoplus^{b^+_{X_1}} [\mathscr{N}_p^0 \times_{SO(3)} \boldsymbol{R}^3] \to \mathscr{N}_p$$

such that $\gamma(\psi^{-1}(0)) = U_{p,X_1} \cap \mathscr{M}_{p,X_1}$. Roughly speaking, the map γ is given by gluing the standard 1-instantons on S^4 with the trivial connection on X_1 by gluing parameter $\mathscr{N}_p$ (actually γ is its small perturbation in the ASD equation), and the section ψ is the obstruction of image of γ to be ASD. (For a precise definition, see [11], [6])

Likewise, $U_p \cap \mathscr{M}_{X,l_1+l_2}(g_v)$ has the following description. We consider a vector bundle

$$\eta = \bigoplus^{b^+_{X_1}} [(\mathscr{N}_p^0 \times \mathscr{M}^0_{X_2,a}) \times_{SO(3)} \boldsymbol{R}^3] \to \mathscr{N}_p' = (\mathscr{N}_p^0 \times \mathscr{M}^0_{X_2,a}) / SO(3)$$

where a equals to $l_1 + l_2 - p$, $\mathscr{M}^0_{X_2,a}$ is the based moduli space (i.e. the space of the ASD connections modulo gauge transformations with their restrictions to a base point in X_2 to be identity), and $U_p \cap \mathcal{M}_{X, l_1+l_2}(g_v)$ can be identified with the zero set of a section $\sigma(v)$ of this bundle.

Let $Fr(\mathcal{N}_p')$ be the end of $\mathcal{N}_p'$ corresponding $\sum \lambda_i^2 = \varepsilon$. Since $\mathcal{I}_X(v) \subset Int\, \mathcal{N}_p'$, $\sigma(v)|_{Fr(\mathcal{N}_p') \cap V_1 \cap \cdots \cap V_{d_1+d_2+1}}$ has no zeros.

When v converges to 0, the bundle η splits to the pullbacks of η_{X_1} and $\eta_{X_2} = \bigoplus^{b_{X_1}^+} (\mathcal{M}_{X_2,a}^0 \times_{SO(3)} \boldsymbol{R}^3)$, and, since for any element in $\mathcal{M}_{X_2,a}$ and the standard 1-instantons on S^4, the second cohomology groups of their Atiyah-Hitchin-Singer complexes vanish, we can apply the multiple connected sum construction ([6], section 7.2.8) to our situation. Then, for sufficiently small v, the restriction of the section $\sigma(v)$ to $Fr(\mathcal{N}_p')$ is $\sigma_{X_1}(v)$ which is a small perturbation of the pullback of the section $\psi|_{Fr(\mathcal{N}_p)}$. We fix such a small v and drop it. Let $\mathcal{W}_p$ be the intersection $\mathcal{N}_p' \cap V_1 \cap \cdots \cap V_{d_1+d_2+1}$. Then

$$\mathcal{I}_{X,p} = \sigma^{-1}|_{\mathcal{W}_p}(0). \tag{1}$$

Thus the section σ is nonvanishing on $Fr(\mathcal{W}_p) = Fr(\mathcal{N}_p') \cap V_1 \cap \cdots \cap V_{d_1+d_2+1}$. Since $\mathcal{W}_p$ consists of almost anti-self-dual connections, the notion of Uhlenbeck compactification $\overline{\mathcal{W}_p}$ makes sense. (See [8]). The section σ has no zeros of the singular set of $\overline{\mathcal{W}_p}$ by compactness of $\mathcal{I}_{X,p}$.

Now we can extend the constructions above (the obstruction bundles and so on) to $\overline{\mathcal{W}_p}$ and consider as follows: η is the vector bundle of rank $3b_{X_1}^+$ over $\overline{\mathcal{W}_p}$ with the section σ which is nonvanishing over the singular set S of $\overline{\mathcal{W}_p}$ (which is the lower strata of the Uhlenbeck type compactification, its codimension is more than 4), and the section σ is the small perturbation of the pullback of $\psi|_{Fr(\mathcal{W}_{p,X_1})}$ over the boundary $Fr(\mathcal{W}_p)$. Here we denote that $\mathcal{W}_{p,X_1} = \mathcal{N}_p \cap V_1 \cap \cdots \cap V_r$ and $Fr(\mathcal{W}_{p,X_1}) = Fr(\mathcal{N}_p) \cap V_1 \cap \cdots \cap V_r$. So from the setting above, we have its relative Euler class $e \in H^{3b_{X_1}^+}(\overline{\mathcal{W}_p}, Fr(\mathcal{W}_p); \boldsymbol{Z})$.

Lemma 3.2. *The cohomology class e equals to zero if and only if the homology class $[\mathcal{I}_{X,p}] \in H_1(\mathcal{W}_p; \boldsymbol{Z})$ is zero.*

Proof. Since $\dim \overline{\mathcal{W}_p} = 3b_{X_1}^+ + 1$, by Poincaré duality and the fact that the codimension of S is greater than 2, the following is hold;

$$H_1(\mathcal{W}_p; \boldsymbol{Z}) \simeq H_c^{3b_{X_1}^+}(\mathcal{W}_p, Fr(\mathcal{W}_p); \boldsymbol{Z}) \simeq H^{3b_{X_1}^+}(\overline{\mathcal{W}_p}, Fr(\mathcal{W}_p); \boldsymbol{Z}).$$

So from the equation (1) and the isomorphism above, the statement is immediate. □

Now we have the following two inequalities by the counting argument.

$$\begin{cases} \rho \geq r/2 \\ 2d_2 - 8(l_2 - a) \geq 2(d_1 + d_2 + 1 - r). \end{cases}$$

On the other hand, the following equalities hold;

$$\dim \overline{\mathscr{W}}_{\rho, X_1} = 8\rho - 3 - 2r,$$

$$\dim \overline{\mathscr{W}}_{\rho} = 8\rho - 8(l_2 - a) - 2(d_1 + 1).$$

So we have

$$3b_{X_1}^+ + 1 \geq \dim \overline{\mathscr{W}}_{\rho, X_1} + 3.$$

From this inequality, the dimension of $\mathscr{W}_{\rho, X_1}$ is at least 2 less than $3b_{X_1}^+$, the rank of η_{X_1}. So we have the following lemma.

LEMMA 3.3. *Any nonvanishing section of* $\eta_{X_1}|_{Fr(\mathscr{W}_{\rho, X_1})}$ *is homotopic through nonvanishing sections to* $\psi|_{Fr(\mathscr{W}_{\rho, X_1})}$.

PROOF. The obstructions to such homotopies lie in $H^i(Fr(\mathscr{W}_{\rho, X_1}); \pi_i(S^{3b_{X_1}^+ - 1}))$. But, by the argument above, we have $H^i(Fr(\mathscr{W}_{\rho, X_1}); \pi_i(S^{3b_{X_1}^+ - 1})) = 0$ for any i. □

By the Lemma 3.2, to prove Theorem 1.1, it suffices to show that

CLAIM. *The relative Euler class e is zero.*

PROOF OF CLAIM. This claim is equivalent to the existance of a nonvanishing section of the bundle η, whose restriction to $Fr(\mathscr{W}_\rho)$ is the pullback of $\sigma_{X_1}|_{Fr(\mathscr{W}_{\rho, X_1})}$. We will make such a section.

CASE 1. $3b_{X_1}^+ + 1 > \dim \overline{\mathscr{W}}_{\rho, X_1} + 3 = 8\rho - 2r$

In this case, we will make a nonvanishing section τ of η over $\overline{\mathscr{W}}_\rho$ as the following procedure. We construct nonvanishing sections τ_1 of η_{X_1} over $\mathscr{W}_{\rho, X_1}$ and τ_2 of η_{X_2} over $\mathscr{Y} = \mathscr{M}_{X_2, a} \cap V_{r+1} \cap \cdots \cap V_{d_1 + d_2 + 1}$ and combine them to get τ. The section τ will be nonvanishing over S and it will be pulled back from τ_1 over $Fr(\mathscr{W}_\rho)$. But, since $\tau_1|_{Fr(\mathscr{W}_{\rho, X_1})}$ is homotopic to $\sigma_{X_1}|_{Fr(\mathscr{W}_{\rho, X_1})}$ through nonvanishing sections, by the Lemma 3.3, this section is the one which we want.

First, the section τ_1 is constracted as follows. Let k be the unique integer such that

$$8p - 2r - 3 \leq 3k < 8p - 2r \leq 3b_{X_1}^+.$$

We consider the subbundle of $\eta_{X_1}|_{\mathscr{W}_{p,X_1}}$

$$\bigoplus^k (\mathscr{W}_{p,X_1}^0 \times_{SO(3)} \boldsymbol{R}^3) \to \mathscr{W}_{p,X_1}.$$

here $\mathscr{W}_{p,X_1}^0$ is the preimage of $\mathscr{W}_{p,X_1}$ by the base point fibration. Then obstructions to the existence of nonvanishing sections of this bundle lie in $H^i(\mathscr{W}_{p,X_1};\pi_{i-1}(S^{3k-1}))$. So, from the definition of k, the obstruction arises only in the case $\dim \overline{\mathscr{W}}_{p,X_1} = 8p - 2r - 3 = 3k$. In this case, the obstruction is $c^k \in H^{3k}(\mathscr{W}_{p,X_1};\boldsymbol{Z})$, where $c \in H^3(\mathscr{W}_{p,X_1};\boldsymbol{Z})$ is the Euler class of the vector bundle $\mathscr{W}_{p,X_1}^0 \times_{SO(3)} \boldsymbol{R}^3$. Since this bundle has odd rank, c is 2-torsion, so c^k is 2-torsion, too. But, $H^{3k}(\mathscr{W}_{p,X_1};\boldsymbol{Z})$ is torsion free, so this obstruction vanishes. Here, we see that $\bigoplus^k(\mathscr{W}_{p,X_1}^0 \times_{SO(3)} \boldsymbol{R}^3) \simeq (\mathscr{W}_{p,X_1}^0 \times \boldsymbol{R}^{3k})/SO(3)$, so equivalently, we have a nonvanishing $SO(3)$-equivariant map $\tau_1^0 : \mathscr{W}_{p,X_1}^0 \to \boldsymbol{R}^{3k}$.

Next, we construct a section τ_2. Since $a = l_1 + l_2 - p$, we have

$$\begin{aligned}\dim \mathscr{Y} &= 2d_2 - 8(l_2 - a) - 2(d_1 + d_2 + 1 - r) \\ &= 3b_{X_1}^+ + 1 - (8p - 2r) \\ &< 3b_X^+ + 1 - 3k.\end{aligned}$$

So we have $\dim \mathscr{Y} \leq 3b_{X_1}^+ - 3k$. We want to find a nonvanishing section τ_2 of the rank $3(b_{X_1}^+ - k)$ vector bundle

$$\bigoplus^{b_{X_1}^+ - k} (\mathscr{Y}^0 \times_{SO(3)} \boldsymbol{R}^3) \to \mathscr{Y}$$

here, $\mathscr{Y}^0$ is the preimage of $\mathscr{Y}$ by the base point fibration. Note here that $b_{X_1}^+ > k$. This is a subbundle of $\eta_{X_2}|_{\mathscr{Y}}$. Since obstructions to the existance of nonvanishing sections of this bundle lie in $H^i(\mathscr{Y};\pi_{i-1}(S^{3(b_{X_1}^+ - k)-1}))$, the obstruction arises only in the case $\dim \mathscr{Y} = 3(b_{X_1}^+ - k)$. In this case, the obstruction vanishes by the same argument as the case of τ_1. So there is a nonvanishing $SO(3)$-equivariant map $\tau_2^0 : \mathscr{Y}^0 \to \boldsymbol{R}^{3(b_{X_1}^+ - k)}$.

To sum up, we construct τ which is desired as follows. Let

$$W = [((f_1,\lambda_1),\dots,(f_p,\lambda_p));(A,\xi)] \in \mathscr{W}_p^0 \subset \mathscr{N}_p^0 \times \mathscr{M}_{X_2,a}^0,$$

and we define an $SO(3)$-equivariant map

$$\tau^0 : \mathscr{W}_p^0 \to \boldsymbol{R}^{3b_{X_1}^+},$$

by

$$\tau^0(W) = \left(\sum \lambda_i^2\right)\tau_1^0(((f_1,\lambda_1),\dots,(f_p,\lambda_p))) + \left(\varepsilon - \sum \lambda_i^2\right)\tau_2^0(A,\xi).$$

Here we regard $\boldsymbol{R}^{3b^+_{X_1}} = \boldsymbol{R}^{3k} \oplus \boldsymbol{R}^{3(b^+_{X_1}-k)}$, and the image of τ_1^0 lies in the $\boldsymbol{R}^{3k}$-summand and the image of τ_2^0 in the $\boldsymbol{R}^{3(b^+_{X_1}-k)}$. The section τ^0 is $SO(3)$-equivariant, nonvanishing, and nonvanishing when extended to the singular set of $\mathscr{W}_p^0$. So the quotient of τ^0 is the desired section.

CASE 2. $\dim \overline{\mathscr{W}}_{p,X_1} + 3 = 8p - 2r = 3b^+_{X_1} + 1$

In this case, we have

$$\dim \mathscr{W}_p = 8p - 8(l_2 - a) - 2(d_1 + 1),$$
$$\dim \mathscr{Y} = 0.$$

So $\mathscr{Y}$ consists of a finite number of points. Then $\mathscr{I}_{X,p}$ is decomposed into unions of connected components corresponding to these points. Thus, it is sufficient to see each component. We may assume that $\mathscr{Y}$ is one point. Then $\mathscr{Y}^0$ is $SO(3)$. So

$$\mathscr{W}_p^0 = \mathscr{W}_{p,X_1}^0 \times SO(3)$$

$$\eta = \bigoplus^{b^+_{X_1}}(\overline{\mathscr{W}}_p^0 \times_{SO(3)} \boldsymbol{R}^3) \simeq \overline{\mathscr{W}}_{p,X_1}^0 \times \boldsymbol{R}^{3b^+_{X_1}} \to \overline{\mathscr{W}}_p$$
$$= (\overline{\mathscr{W}}_{p,X_1}^0 \times SO(3))/SO(3) \simeq \overline{\mathscr{W}}_{p,X_1}^0.$$

Since η is the trivial bundle, any nonvanishing section on η is homotopic through nonvanishing sections to a constant nonvanishing section, (when we interpret the nonvanishing sections as nonvanishing maps from $\overline{\mathscr{W}}_{p,X_1}^0$ to $\boldsymbol{R}^{3b^+_{X_1}}$, this means that all such maps are homotopic through nonvanishing maps to a nonzero constant map,) and the restriction to $Fr(\overline{\mathscr{W}}_{p,X_1}^0)$ of it is, too. Conversely, if there is a nonvanishing section on $Fr(\overline{\mathscr{W}}_{p,X_1}^0)$ which is homotopic through nonvanishing sections to a nonvanishing constant section, we have nonvanishing extention of it to $\overline{\mathscr{W}}_{p,X_1}^0$.

On the other hand, since the bundle η_{X_1} is isomorphic to $\eta/SO(3)$, if a section $Fr(\overline{\mathscr{W}}_{p,X_1}^0) \to \boldsymbol{R}^{3b^+_{X_1}}$ of $\eta|_{Fr(\overline{\mathscr{W}}_{p,X_1}^0)}$ is pulled back from a section of $\eta_{X_1}|_{Fr(\overline{\mathscr{W}}_{p,X_1})}$, then this map has to be $SO(3)$-equivariant, here the $SO(3)$-action on $\boldsymbol{R}^{3b^+_{X_1}}$ is isomorphic to $\bigoplus^{b^+_{X_1}} so(3)$ which is a direct sum of the adjoint representation.

Now, we want to construct a nonvanishing section of the vector bundle η over $\overline{\mathscr{W}}_p \simeq \overline{\mathscr{W}}_{p,X_1}^0$ which is the pullback of $\sigma_{X_1}|_{Fr(\overline{\mathscr{W}}_{p,X_1})}$ on $Fr(\overline{\mathscr{W}}_p)$. But, as in the previous case, we may consider the pullback of τ_{∂} instead of $\sigma_{X_1}|_{Fr(\overline{\mathscr{W}}_{p,X_1})}$. Here τ_{∂} is a nonvanishing section of a rank $3(b^+_{X_1}-1)$ subbundle of η_{X_1} over $Fr(\overline{\mathscr{W}}_{p,X_1})$, which can be found by the same argument given in the construction of τ_1.

The pullback is the same as a nonvanishing $SO(3)$-equivariant map

$$\tau'_{\partial} : Fr(\overline{\mathscr{W}}^0_{p,X_1}) \to \boldsymbol{R}^{3(b^+_{X_1}-1)} \subset \boldsymbol{R}^{3b^+_{X_1}}.$$

Thus τ'_{∂} is homotopic through nonvanishing maps to a constant nonvanishing map and therefore it extends to a nonvanishing map $\overline{\mathscr{W}}^0_{p,X_1} \to \boldsymbol{R}^{3b^+_{X_1}}$. So in this case, relative Euler class vanishes as in the previous case. □

Note here that, if $b^+_{X_1}=1$ and $8p-2r=3b^+_{X_1}+1=4$, then $p=1$ and $r=2$, since we have $p \geq r/2 > 0$ and $p \in \boldsymbol{Z}$ (this case corresponds to Theorem 1.2). Thus $\dim Fr(\overline{\mathscr{W}}_{p,X_1})$ is zero and $Fr(\overline{\mathscr{W}}^0_{p,X_1})$ is a disjoint union of $SO(3)$. So any $SO(3)$-equivariant map $Fr(\overline{\mathscr{W}}^0_{p,X_1}) \to \boldsymbol{R}^3 \simeq so(3)$ is not homotopic through nonvanishing maps to a nonvanishing constant map. From the argument above, in this case, we cannot extend these maps to nonvanishing maps $\overline{\mathscr{W}}^0_{p,X_1} \to \boldsymbol{R}^{3b^+_{X_1}}$. This means that the relative Euler class e is not zero.

The proof of Theorem 1.1 in the case when $0 < r < d_1+2$ is completed.

On the other hand, if $d_1+d_2+1 > r > d_1+2$, then $0 < d_1+d_2+1-r < d_2+2$. So we set $r' = d_1+d_2+1-r$, and, from the same argument above, we get conclusion. □

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