

A NOTE ON KAEHLERIAN METRICS WITH CERTAIN PROPERTY FOR ∇Ric

By

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0. Introduction

Let M^{2n} be a $2n$ real dimensional Kaehlerian manifold, whose complex structure is given by a parallel tensor field $F = (F_j^i)$ satisfying

$$F_j^r F_r^i = \delta_j^i, \quad g_{ji} F_\rho^j F_k^i = g_{\rho k},$$

where $g = (g_{ji})$ is the Riemannian metric tensor of M^{2n} . Let $K = (K_{kji}^h)$ be the Bochner curvature tensor and $\hat{K} = (K_{kji})$ a tensor given by

$$(0.1) \quad K_{kji} = \nabla_k R_{ji} - \nabla_j R_{ki} \\ + \frac{1}{4(n+1)} (g_{ki} \delta_j^h - g_{ji} \delta_k^h + F_{ki} F_j^h - F_{ji} F_k^h + 2F_{kj} F_i^h) r_h,$$

where $R = (R_{kji}^h)$ is the Riemannian curvature tensor, $\text{Ric} = (R_{ji}) = (R_{kji}^k)$ the Ricci tensor, and $r = R_k^k$ the scalar curvature.

Let us consider the condition

$$(\#) \quad \nabla_k R_{ji} = \frac{1}{4(n+1)} (2r_k g_{ji} + r_j g_{ki} + r_i g_{kj} + \tilde{r}_j F_{ik} + \tilde{r}_i F_{jk}),$$

where $\tilde{r}_j = F_j^h r_h$ and $r_j = \nabla_j r$. This condition gives a necessary and sufficient condition for equality in the inequality

$$\frac{1}{m+1} |dr|^2 \leq |\nabla \text{Ric}|^2$$

which was proved in [2].

If M^{2n} satisfies $(\#)$ \hat{K} vanishes [2]. But the converse is not true. The example of metric satisfying $(\#)$ is unknown, except the case where r is constant.

On the other hand, if M^{2n} is Bochner-flat, i.e. $K = 0$, then \hat{K} vanishes. The

examples of non-flat Bochner-flat metric have been found by Tachibana and Liu [1]. The purpose of this paper is to show that Tachibana and Liu's metrics just give examples of non-flat metrics satisfying (#).

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1. Preliminaries

Throughout this paper, the complex coordinate $\{z^\lambda, z^{\lambda^*}\}$ shall be used, where $z^{\lambda^*} = \bar{z}^\lambda$, the conjugate of z^λ . We adopt the following ranges of indices:

$$1 \leq i, j, k, \dots \leq 2n,$$

$$1 \leq \lambda, \mu, \nu, \dots \leq n, \quad \lambda^* = \lambda + n.$$

With respect to the complex coordinate, the metric tensor g_{ji} and the complex structure F_j^i of M^{2n} satisfies

$$(1.1) \quad g_{\mu\lambda} = g_{\mu^*\lambda^*} = 0, \quad g_{\mu\lambda^*} = g_{\lambda^*\mu} = \bar{g}_{\lambda\mu^*} = \bar{g}_{\mu^*\lambda},$$

$$(1.2) \quad F_\mu^\lambda = i\delta_\mu^\lambda, \quad F_\mu^{\lambda^*} = 0,$$

$$F_{\mu\lambda} = g_{\alpha^*\lambda} F_\mu^{\alpha^*} = 0, \quad F_{\mu^*\lambda} = F_{\mu^*}^{\alpha^*} g_{\alpha^*\lambda} = -ig_{\mu^*\lambda},$$

and the Ricci tensor $\text{Ric} = (R_{ji})$ and scalar curvature $r = R_\lambda^\lambda + R_{\lambda^*}^{\lambda^*} = 2R_\lambda^\lambda$ satisfy

$$(1.3) \quad R_{\mu\lambda} = R_{\mu^*\lambda^*} = 0, \quad R_{\mu^*\lambda} = R_{\lambda\mu^*},$$

$$\tilde{r}_\mu = F_\mu^\alpha r_\alpha = i\delta_\mu^\alpha r_\alpha = ir_\mu,$$

$$\tilde{r}_\mu F_{\nu^*\lambda} = (ir_\mu)(-ig_{\lambda\nu^*}) = r_\mu g_{\lambda\nu^*}.$$

Putting $K_{kjih} = K_{kji}^r g_{rh}$, the Bochner curvature tensor is given by

$$K_{\lambda\mu^*\nu\rho^*} = R_{\lambda\mu^*\nu\rho^*}$$

$$- \frac{1}{n+2} (g_{\lambda\mu^*} R_{\nu\rho^*} + g_{\lambda\rho^*} R_{\nu\mu^*} + g_{\nu\rho^*} R_{\lambda\mu^*} + g_{\nu\mu^*} R_{\lambda\rho^*})$$

$$+ \frac{R}{2(n+1)(n+2)} (g_{\lambda\mu^*} g_{\nu\rho^*} + g_{\lambda\rho^*} g_{\nu\mu^*}).$$

By virtue of (1.1)–(1.3), the condition (#) is reduced to the following simple form:

$$(\#') \quad \nabla_\lambda R_{\mu\nu^*} = \frac{1}{2(n+1)} (r_\lambda g_{\mu\nu^*} + r_\mu g_{\lambda\nu^*}).$$

2. Metrics with Vanishing Bochner Curvature Tensor

Let \mathbb{C}^n be the complex number space with complex coordinate $\{z_\lambda\}$. In the following of this paper, we denote coordinates by z_λ instead of z^λ . A real valued holomorphic function $\phi = \phi(z, \bar{z})$ of $\{z_\lambda, \bar{z}_\lambda\}$ gives a Kaehlerian metric $g_{\mu\lambda^*} = \partial^2 \phi / \partial z_\mu \partial \bar{z}_\lambda$ to \mathbb{C}^n or its subdomain. Under the assumption that ϕ is a function of $t = \sum_{\alpha=1}^n z_\alpha \bar{z}_\alpha$, Tachibana and Liu found $\phi = f(t)$ so that the corresponding Kaehlerian metric has the vanishing Bochner curvature tensor. In this case, the metric tensor $g_{\mu\lambda^*}$, the Christoffel symbols $\Gamma_{\mu\lambda}^\nu$, the Ricci tensor $R_{\mu\lambda^*}$ and the scalar curvature r are as follows (' means differentiation with respect to t):

$$(2.1) \quad g_{\mu\lambda^*} = f' \delta_{\mu\lambda} + f'' \bar{z}_\mu z_\lambda,$$

$$(2.2) \quad \Gamma_{\mu\lambda}^\nu = \frac{f''}{f'} (\bar{z}_\mu \delta_{\nu\lambda} + \bar{z}_\lambda \delta_{\nu\mu}) + \sigma z_\nu \bar{z}_\mu \bar{z}_\lambda,$$

where

$$(2.3) \quad \sigma = \frac{f' f''' - 2f''^2}{f'(f' + t f'')},$$

$$(2.4) \quad R_{\mu\lambda^*} = \lambda \bar{z}_\mu z_\lambda + \mu \delta_{\mu\lambda},$$

where

$$(2.5) \quad \lambda = -\frac{(n+1)(f' f''' - f''^2)}{f'^2} - \sigma' t - \sigma = \mu'$$

and

$$(2.6) \quad \mu = -\frac{(n+1)f''}{f'} - \sigma t,$$

$$(2.7) \quad r = \frac{2}{f'} \left(t\lambda + n\mu - \frac{t f''(t\lambda + \mu)}{f' + t f''} \right).$$

For convenience sake we put

$$(2.8) \quad \Delta = \frac{r}{2(n+1)(n+2)}.$$

On account of vanishing Bochner curvature tensor, the function f satisfies the differential equation

$$(2.9) \quad 2\sigma f'' = \sigma' f',$$

which induces by integration

$$(2.10) \quad \sigma = af'^2 \quad (a \text{ is a constant})$$

or equivalently

$$(2.11) \quad f'f''' - 2f''^2 = af'^3(f' + tf'').$$

From this equation, Tachibana and Liu obtained the result: that $f(t)$ gives a non-flat Bochner-flat Kaehlerian metric satisfying $f''(0) = 0$ is equivalent to that f takes one of the following two forms;

$$(2.12) \quad f(t) = \frac{1}{c} \sin^{-1} \left(\frac{c}{b} t \right) + k,$$

$$(2.13) \quad f(t) = \frac{1}{c} \sinh^{-1} \left(\frac{c}{b} t \right) + k,$$

where b and c are positive constants and k is any constant.

For these metrics the following formulae hold [1]:

$$(2.14) \quad \sigma' = 2af'f''$$

$$(2.15) \quad f'' = atf'^3,$$

$$(2.16) \quad f'^2 \Delta = -\frac{nf'' + 2f'\sigma t}{n+2},$$

$$(2.17) \quad \lambda = -(n+2)af'^2(1 + 2at^2f'^2),$$

$$(2.18) \quad \mu = -(n+2)atf'^2$$

$$(2.19) \quad g_{\mu\nu} = f'(\sigma_{\mu\nu} + atf'^2 \bar{z}_\mu z_\nu).$$

3. Metrics Satisfying the Condition (#)

Now we shall show that above-mentioned metrics are examples of the space satisfying (#).

First we calculate each term of

$$\nabla_\lambda R_{\mu\nu} = \partial_\lambda R_{\mu\nu} - \Gamma_{\lambda\mu}^\rho R_{\rho\nu} - \Gamma_{\lambda\nu}^{\rho*} R_{\mu\rho}.$$

From (2.4) and $\mu' = \lambda$, the first term is

$$(3.1) \quad \partial_\lambda R_{\mu\nu} = \lambda' \bar{z}_\lambda \bar{z}_\mu z_\nu + \lambda(\bar{z}_\mu \delta_{\lambda\nu} + \bar{z}_\lambda \delta_{\mu\nu}).$$

From (2.2), (2.9), (2.10) and (2.13), we have

$$(3.2) \quad \Gamma_{\lambda\mu}^{\rho} = af'^2 \{z_{\rho}\bar{z}_{\lambda}\bar{z}_{\mu} + t(\bar{z}_{\lambda}\delta_{\rho\mu} + \bar{z}_{\mu}\delta_{\rho\lambda})\},$$

from which the second term is

$$(3.3) \quad \begin{aligned} \Gamma_{\lambda\mu}^{\rho} R_{\rho\nu} &= \sum_{\rho} af'^2 \{z_{\rho}\bar{z}_{\lambda}\bar{z}_{\mu} + t(\bar{z}_{\lambda}\delta_{\rho\mu} + \bar{z}_{\mu}\delta_{\rho\lambda})\} \{\lambda\bar{z}_{\rho}z_{\nu} + \mu\delta_{\rho\nu}\} \\ &= af'^2 \{(\lambda t + \mu)\bar{z}_{\lambda}\bar{z}_{\mu}z_{\nu} + 2\lambda t\bar{z}_{\lambda}\bar{z}_{\mu}z_{\nu} + \mu t(\bar{z}_{\mu}\delta_{\lambda\nu} + \bar{z}_{\lambda}\delta_{\mu\nu})\}. \end{aligned}$$

Hence

$$(3.4) \quad \nabla_{\lambda} R_{\mu\nu} = \{\lambda' - af'^2(3\lambda t + \mu)\}\bar{z}_{\lambda}\bar{z}_{\mu}z_{\nu} + \{\lambda - af'^2\mu t\}(\bar{z}_{\lambda}\delta_{\mu\nu} + \bar{z}_{\mu}\delta_{\lambda\nu})$$

from (3.1) and (3.3) on account of $\Gamma_{\lambda\nu}^{\rho} = 0$, where insides of $\{ \}$ is calculated as follows: differentiating (2.15), and using (2.13), we have

$$(3.5) \quad \begin{aligned} \lambda' &= -(n+2)a\{2f'f''(1+2at^2f'^2) + f'^2 2a(2tf'^2 + 2t^2f'f'')\} \\ &= -2(n+2)a^2tf'^4(3+4at^2f'^2), \end{aligned}$$

and from (2.15) and (2.16)

$$(3.6) \quad 3\lambda t + \mu = -2(n+2)atf'^2(2+3at^2f'^2).$$

Hence

$$(3.7) \quad \begin{aligned} \lambda' - af'^2(3t\lambda + \mu) &= -2(n+2)a^2tf'^4(3+4at^2f'^2) \\ &\quad + 2(n+2)a^2tf'^4(2+3at^2f'^2) \\ &= -2(n+2)a^2tf'^4(1+at^2f'^2), \end{aligned}$$

and from (2.15) and (2.16),

$$(3.8) \quad \begin{aligned} \lambda - af't\mu &= -(n+2)af'^2(1+2at^2f'^2) + (n+2)a^2t^2f'^4 \\ &= -(n+2)af'^2(1+at^2f'^2). \end{aligned}$$

Substituting (3.7) and (3.8) into (3.4), we obtain

$$(3.9) \quad \begin{aligned} \nabla_{\lambda} R_{\mu\nu} &= -2(n+2)a^2tf'^4(1+at^2f'^2)\bar{z}_{\lambda}\bar{z}_{\mu}z_{\nu} \\ &\quad - (n+2)af'^2(1+at^2f'^2)(\bar{z}_{\lambda}\delta_{\mu\nu} + \bar{z}_{\mu}\delta_{\lambda\nu}) \\ &= -(n+2)af'^2(1+at^2f'^2)(2atf'^2\bar{z}_{\lambda}\bar{z}_{\mu}z_{\nu} + \bar{z}_{\lambda}\delta_{\mu\nu} + \bar{z}_{\mu}\delta_{\lambda\nu}). \end{aligned}$$

Now we shall calculate the right hand side of $(\#')$ for our metrics. Substituting (2.10) and (2.13) in (2.14), we have

$$\Delta = -atf',$$

from which

$$(3.10) \quad r = 2(n+1)(n+2)\Delta = -2(n+1)(n+2)atf'.$$

Differentiating (3.10) by z_λ , and taking account of (2.13) we have

$$(3.11) \quad \begin{aligned} r_\lambda &= -2(n+1)(n+2)a(\bar{z}_\lambda f' + tf''\bar{z}_\lambda) \\ &= -2(n+1)(n+2)af'(1 + at^2 f'^2)\bar{z}_\lambda. \end{aligned}$$

Substituting (3.11) and (2.17), we can see that the right hand side of $(\#')$ coincides to that of (3.9).

Thus we conclude that the metrics given by $(\#\#)$ satisfy $(\#)$.

References

- [1] Tachibana, S. and Liu, R. C., Notes on Kählerian metrics with vanishing Bochner curvature tensor, *Kodai Math. Sem. Rep.* **22** (1970), 313–321.
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