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ON COVERINGS OF MODULES

By

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Abstract. Let R be a ring, and let τ be a torsion theory for R-mod. We give a necessary condition for every R-module to have a τ-torsionfree cover; this necessary condition is close to the known sufficient condition. Then we present a method for computing τ-torsionfree covers of modules that can be embedded in Q_τ-modules, where Q_τ is the quotient ring for τ.

In this paper, we let R be a ring, and we let τ be an hereditary torsion theory of left R-modules with torsion class $\mathscr{T}$, torsionfree class $\mathscr{F}$, filter of left ideals $\mathscr{L}$, and quotient ring Q_τ. For a module M, we let $\tau(M)$ denote the largest submodule of M that is in $\mathscr{T}$ and $Q_\tau(M)$ be the localization of M. For the basic definitions and results on torsion theories, the reader may consult [7].

After the characterization of projective covers by Bass [2], Enochs [4] found the existence of torsionfree covers of modules for the usual torsion theory over an integral domain. A concrete method for constructing these covers was obtained by Banaschewski [1]. The concept of a torsionfree cover was extended to modules over associative rings by Teply [13]: given an hereditary torsion theory τ and a module M, an epimorphism $\theta : F \to M$ is called a τ-torsionfree cover if

(1) F is τ-torsionfree,

(2) for any homomorphism $h : F' \to M$ with F' τ-torsionfree, there is a homomorphism $g : F' \to F$ such that $h = \theta g$, and

(3) $\ker\theta$ contains no nonzero τ-pure submodule of F.

General results about the existence and uniqueness of τ-torsionfree covers was obtained in [13], [8] and [14]; τ-torsionfree covers exist when τ has finite type (i.e., when the filter $\mathscr{L}$ for τ has a cofinal subset of finitely generated left ideals.) The extension of Banaschewski's construction only works when τ is a perfect torsion theory. Since the existence proof calls for forming an infinite direct sum of τ-

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injective modules and factoring out by a module obtained from Zorn's Lemma, no general method for realistic computation of τ-torsionfree covers is known. Several researchers have studied this problem and found particular cases (mostly when R is commutative) in which constructions can be given for the τ-torsionfree cover; for example, see [3], [10], [11], and [12]. Important problems in this area are

(1) to give a precise characterization of the torsion theories τ for which every module has a τ-torsionfree cover, and

(2) to find a construction for the τ-torsionfree cover when τ is not perfect.

If every module has a τ-torsionfree cover, it is trivial to show that R must be τ-torsionfree. But no other necessary conditions for every module to have a τ-torsionfree cover have been published. In this paper, we present a necessary condition for every module to have a τ-torsionfree cover; this necessary condition is close to the sufficient condition given in [14]. Then we present a method for computing the τ-torsionfree cover of a module N that embeds in a Q_τ-module M, where M has a Q_τ-projective cover.

We need one definition before we present our necessary condition in Theorem 1.

A τ-torsionfree module M is called *τ-exact* if every τ-torsionfree homomorphic image of M is τ-injective.

REMARKS. (1) The localization functor $Q_\tau(\_)$ for τ is an exact functor if and only if every τ-torsionfree τ-injective module is τ-exact. This observation is immediate from [7, Proposition 44.1, (1) $\Leftrightarrow$ (3)].

(2) Any τ-injective τ-cocritical module is τ-exact, as the only τ-torsionfree homomorphic images of such a module M are 0 and M.

(3) If E is τ-exact and E' is a τ-pure submodule of E, then E' and E/E' are τ-exact.

PROOF. It is clear from the definition that E/E' is τ-exact; so we show that E' is τ-exact. Let K be τ-pure in E'; we need to show that E'/K is τ-injective. Since E'/K and E/E' are τ-torsionfree, so is E/K; the τ-exactness of E implies that E/K is τ-injective. Since E'/K is τ-pure in E/K, then E'/K is τ-injective by [7, Proposition 8.4].

(4) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence and if E' and E'' are τ-exact, then E is also τ-exact.

PROOF. Since E' and E'' are τ-exact, it follows from [7, Prop. 8.2] that E is τ-injective. We let N be a τ-pure submodule of E and show that E/N is τ-

injective. Since

$$E'/(E' \cap N) \cong (E' + N)/N \subseteq E/N \in \mathscr{F},$$

then $(E' + N)/N$ is τ-injective by the τ-exactness of E'. Thus

$$(E/N)/((E' + N)/N) \cong E/(E' + N) \in \mathscr{F}.$$

From the τ-exactness of $E'' \cong E/E'$ and the induced epimorphism $E/E' \to E/(E' + N)$, it now follows that $E/(E' + N)$ is τ-injective. Now the exact sequence

$$0 \to (E' + N)/N \to E/N \to E/(E' + N) \to 0$$

and [7, Prop. 8.2] imply that E/N is τ-injective, as desired.

THEOREM 1. *If every R-module has a τ-torsionfree cover, then any directed union of τ-exact submodules of a module is τ-injective.*

PROOF. Let M be the directed union of τ-exact submodules M_α $(\alpha \in A)$ of a given module. Let $\theta : F \to E_\tau(M)/M$ be a τ-torsionfree cover. For each $\alpha \in A$, let $\rho_\alpha : E_\tau(M)/M_\alpha \to E_\tau(M)/M$ be the natural epimorphism. By the directedness of the M_α's, M is τ-torsionfree, and hence each $E_\tau(M)/M_\alpha$ is τ-torsionfree. Consequently, there exist homomorphisms $g_\alpha : E_\tau(M)/M_\alpha \to F$ such that $\theta g_\alpha = \rho_\alpha$. If $\ker g_\beta \neq M/M_\beta$ for some $\beta \in A$, choose an M_γ such that $(M_\gamma + M_\beta)/M_\beta$ is not contained in $\ker g_\beta$. Then $g_\beta((M_\gamma + M_\beta)/M_\beta)$ is a τ-injective submodule of $\ker \theta$, which contradicts the definition of a τ-torsionfree cover. Therefore, we must have $\ker g_\alpha = M/M_\alpha$ for each $\alpha \in A$, and hence $E_\tau(M)/M \cong img_\alpha$ is τ-torsionfree, which forces M to be τ-injective.

REMARK. The known sufficient condition for every R-module to have a τ-torsionfree cover is equivalent to the condition, the directed union of τ-torsionfree τ-injective submodules of a given module is τ-injective. (See [14, Theorem] and [7, Proposition 42.9].) This latter condition is close to the necessary condition obtained in Theorem 1.

COROLLARY 2. (T. Cheatham, personal letter). *If every module has a τ-torsionfree cover, then any direct sum of τ-cocritical τ-injective modules is τ-injective.*

Now we turn our attention toward computing τ-torsionfree covers of modules. We recall that the ring homomorphism $R \to Q_\tau$ is a flat epimorphism if $Q_\tau \otimes_R Q_\tau \cong Q_\tau$ and Q_τ is flat as a right R-module.

PROPOSITION 3. *Let $i : R \to Q_\tau$ be a flat epimorphism of rings. If $\theta : F \to M$ is a τ-torsionfree cover of a Q_τ-module M, then F is a Q_τ-module.*

PROOF. Consider the diagram

$$\begin{tikzcd} F \arrow[r, "i\otimes 1"] \arrow[d, "\theta"'] & Q_\tau \otimes_R F \arrow[d, "1\otimes\theta"] \\ M & Q_\tau \otimes_R M \arrow[l, "\mu"'] \end{tikzcd}$$

where μ is the multiplication map. Since $R \to Q_\tau$ is a flat epimorphism, μ is an isomorphism, and hence $Q_\tau \otimes_R F \in \mathscr{F}$. Since $\theta : F \to M$ is a τ-torsionfree cover, there exists $g : Q_\tau \otimes_R F \to F$ such that $\theta_g = \mu(1 \otimes \theta)$. Therefore, $\theta g(i \otimes 1) = \theta$. By the uniqueness of τ-torsionfree covers, $g(i \otimes 1)$ must be an automorphism α of F. Hence $Q_\tau \otimes_R F = (i \otimes 1)F \oplus \ker \alpha^{-1} g$. Since $R \to Q_\tau$ is a flat epimorphism, the canonical map $Q_\tau \otimes_R F \to Q_\tau(F)$ is a monomorphism, and hence F is essential in $Q_\tau \otimes F$. It follows $\ker \alpha^{-1} g = 0$. Therefore, $i \otimes 1 : F \to Q_\tau \otimes F$ is an isomorphism, so that F is a Q_τ-module via $qf = q \otimes f$.

Next we make an elementary observation that is useful in computing some τ-torsionfree covers of Q_τ-modules.

PROPOSITION 4. *Let M be a Q_τ-module. If $\Phi : P \to M$ is a Q_τ-projective cover of M and if $\theta : F \to M$ is a τ-torsionfree cover of M, then there is a R-homomorphism $g : P \to F$ such that $\theta g = \Phi$ and $F = img + \ker \theta$.*

PROOF. Since P is τ-torsionfree, the definition of a τ-torsionfree cover gives the existence of $g : P \to F$ with the desired properties.

REMARKS. (1) If g is an epimorphism, then $F \cong P/\ker g$ and the homomorphism $\bar{\Phi} : P/\ker g \to M$ induced by Φ is a τ-torsionfree cover of M.

(2) If $R \to Q_\tau$ is a flat epimorphism, then Propositions 3 and 4 show that F is a Q_τ-module and the homomorphism $g : P \to img$ is a Q_τ-projective cover.

We can now give our method for computing the τ-torsionfree cover of a R-submodule N of a Q_τ-module M such that M has a Q_τ-projective cover. For example, we can apply our method when τ has finite type (so that every R-module has a τ-torsionfree cover) and Q_τ is a left perfect ring (so that every Q_τ-

module has a projective cover). We also note that if τ is not a perfect torsion theory, then there are nonzero τ-torsion modules that are R-submodules of Q_τ-modules.

The method for computing the τ-torsionfree cover of a given R-module N consists of the following steps:

(1) Embed N into a Q_τ-module M.

(2) Find the Q_τ-projective cover $\Phi : P \to M$ of M.

(3) By Proposition 4, there is a homomorphism $g : P \to F$ with $\theta g = \Phi$, where $\theta : F \to M$ is the τ-torsionfree cover. Using the properties of a τ-torsionfree cover, compute $\ker g$. Since $img \cong P/\ker g$ and $F = img + \ker\theta$, then F must be very close to $P/\ker g$.

(4) Using the structure of P, M and img, we determine F; the map $\bar{\Phi} : P/\ker g \to M$ induced by Φ can be used to find the map $\theta : F \to M$ for the τ-torsionfree cover of M.

(5) Then the τ-torsionfree cover for N will be either the restriction of θ to $\theta^{-1}(N)$,

$$\theta : \theta^{-1}(N) \to N,$$

or else an induced map of some easily found factor of $\theta^{-1}(N)$,

$$\bar{\theta} : \theta^{-1}(N)/K \to N.$$

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