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TRIANGULAR MATRIX ALGEBRAS OVER QUASI-HEREDITARY ALGEBRAS

By

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Abstract. Let A and B be quasi-hereditary algebras and M an $A-B$-bimodule. Let Λ be the triangular matrix algebra of A and B with M. The quasi-heredity of the triangular matrix algebra Λ is proved under a suitable condition on the bimodule M. Furthermore the category of Δ-good Λ-modules and the characteristic module of Λ are described by using the corresponding ones of A and B.

1. Introduction

Let R be a commutative artin ring and A an artin algebra over R. If R is a field k, then A is a finite dimensional k-algebra. We will consider finitely generated left A-modules, maps between A-modules will be written on the right hand of the argument, thus the composition of maps $f : M_1 \to M_2$, $g : M_2 \to M_3$ will be denoted by fg. The category of all A-modules will be denoted by A-mod. All subcategories considered will be full and closed under isomorphisms.

Given a class Θ of A-modules, we denote by $\mathscr{F}(\Theta)$ the full subcategory of all A-modules which have a Θ-filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to one object in Θ for $1 \leq i \leq t$. The modules in $\mathscr{F}(\Theta)$ are called Θ-good modules, and the category $\mathscr{F}(\Theta)$ is called the Θ-good module category.

Let $E(i)$, $i \in E$ be a complete list of simple A-modules, where $E = \{1, \ldots, n\}$ is a natural ordered set. For any $i \in E$, let $P(i)$ be the projective cover of $E(i)$ and

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denote by $\Delta(i)$ the maximal factor module of $P(i)$ with composition factors of the form $E(j)$ with $j \leq i$. Dually, let $Q(i)$ be the injective hull of $E(i)$ and by $\nabla(i)$ the maximal submodule of $Q(i)$ with composition factors of the form $E(j)$ with $j \leq i$. Let Δ (respectively, ∇) be the full subcategory consisting of all $\Delta(i)$, $1 \leq i \leq n$, (respectively, all $\nabla(i)$, $1 \leq i \leq n$). The modules in Δ are called standard modules and ones in ∇ are called costandard modules.

The algebra A, or better, the pair (A, E) is called a quasi-hereditary algebra if ${}_AA$ belongs to $\mathscr{F}(\Delta)$ and $\mathrm{End}_A(\Delta(i))$ is a division ring, for any $1 \leq i \leq n$.

From now on, we will assume that A is quasi-hereditary. It was proved in [4] that $\mathscr{F}(\Delta)$ and $\mathscr{F}(\nabla)$ are functorially finite in A-mod, i.e. they are both covariantly finite and contravariantly finite in A-mod. A full subcategory $\mathscr{T}$ of A-mod is called contravariantly finite in A-mod provided that for any A-module M, there is a module M_1 in $\mathscr{T}$ with a morphism $f : M_1 \to M$ such that the restriction of $\mathrm{Hom}(-, f)$ to $\mathscr{T}$ is surjective. Such a morphism f is called a right $\mathscr{T}$-approximation of M. A right $\mathscr{T}$-approximation $f : M_1 \to M$ of M is called minimal if the restriction of f to any non-zero direct summand of M_1 is non-zero. The covariantly finiteness of $\mathscr{T}$, a left $\mathscr{T}$-approximation of M and the minimal left $\mathscr{T}$-approximation of M can be defined dually, we omit them and refer to [4]. The category $\mathscr{F}(\Delta)$ admits the following description [4]

$$\begin{aligned}\mathscr{F}(\Delta) &= \{X \in A\text{-mod} \mid \mathrm{Ext}^1(X, \nabla) = 0\}\\ &= \{X \in A\text{-mod} \mid \mathrm{Ext}^i(X, T) = 0 \text{ for all } i \geq 1\}.\end{aligned}$$

Dually, one has that

$$\begin{aligned}\mathscr{F}(\nabla) &= \{X \in A\text{-mod} \mid \mathrm{Ext}^1(\Delta, X) = 0\}\\ &= \{Y \in A\text{-mod} \mid \mathrm{Ext}^i(T, Y) = 0 \text{ for all } i \geq 1\}.\end{aligned}$$

It was also proved in [4] that there is a unique basic module ${}_AT$ such that $\mathrm{add}({}_AT) = \mathscr{F}(\Delta) \cap \mathscr{F}(\nabla)$. Such ${}_AT$ is a generalized tilting and cotilting A-module, which is called the characteristic module of A. The endomorphism ring of ${}_AT$ is again a quasi-hereditary algebra with respect to the opposite ordering E^{op} of E, which is called Ringel dual of A.

Now we recall from [5, 2.5] the notion of a subspace category. Let $\mathscr{K}$ be a Krull-Schmidt category over a field k, and $|-| : \mathscr{K} \to k$-mod an additive functor. We call the pair $(\mathscr{K}, |-|)$ a vectorspace category and denote by $\check{\mathscr{U}}(\mathscr{K}, |-|)$, called subspace category of $(\mathscr{K}, |-|)$, the category of all triples $V = (V_0, V_w, \gamma_V)$, where V_0 belongs to $\mathscr{K}$, V_ω belongs to k-mod and $\gamma_V : V_\omega \to |V_0|$ is a k-linear map. A morphism from V to V' by definition is a pair (f_0, f_ω), where $f_0 : V_0 \to V_0'$ and $f_\omega : V_\omega \to V_\omega'$ such that $\gamma_V |f_0| = f_\omega \gamma_{V'}$.

If $\mathscr{K}$ is finite, i.e. $\mathscr{K}$ has, up to isomorphisms, only finitely many indecomposable objects, then there exists an injective realization of $\mathscr{K}$, namely, there are a finite dimensional k-algebra A and a left A-module M such that we can identify $\mathscr{K}$ with A-Inj., the category of finitely generated injective left A-modules, $|-|$ with the restriction of $\mathrm{Hom}_A(M, -)$ to A-Inj., Thus $\check{\mathscr{U}}(\mathscr{K}, |-|)$ is a full subcategory of $\check{\mathscr{U}}(A\text{-mod}, \mathrm{Hom}_A(M, -))$, the later is equivalent to Λ-mod, where

$$\Lambda = A[M] = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$$

is the one-point extension of A by M, and any triple (V_0, V_ω, γ) in $\check{\mathscr{U}}(\mathscr{K}, |-|)$ corresponds to the left Λ-module $\begin{pmatrix} V_0 \\ V_\omega \end{pmatrix}$; the operation of $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ on it is given by the map $\bar{\gamma}_V : M \otimes_k V_\omega \to V_0$ adjoint to γ_V [5, 1].

If $\mathscr{K}$ is a directed vectorspace category, i.e. there are no cycles between indecomposable objects in $\mathscr{K}$, it was proved in [2] that (Λ, E) is a quasi-hereditary algebra with standard modules $E(1), E(2), \ldots, E(n)$, $P(n+1)$, and $\check{\mathscr{U}}(\mathscr{K}, |-|)$ is equivalent to the category of ∇-good modules over Λ, where $P(n+1)$ is the indecomposable projective Λ-module corresponding to the extension vertex.

Let Λ be the one-point extension of A by M. In contrasting to the ordering on simple Λ-modules above, we fix an ordering ${}_AE$ on simple A-modules and let ${}_AE = \{0\} \cup_A E$ such that $E(0)$ is the simple Λ-module corresponding to the extension vertex. It was proved in [3] that if $(A, {}_AE)$ is a quasi-hereditary algebra and M belongs to $\mathscr{F}({}_A\Delta)$, then $(\Lambda, {}_\Lambda E)$ is a quasi-hereditary algebra and $\check{\mathscr{U}}(\mathscr{F}({}_A\Delta), \mathrm{Hom}_A(M, -)) \approx \mathscr{F}({}_\Lambda\Delta)$.

In the study of a quasi-hereditary algebra A, instead of the complete module category, one is mainly interested in the category $\mathscr{F}(\Delta)$, or the category $\mathscr{F}(\nabla)$. In this paper, we study Δ-good (or ∇-good) module categories and characteristic modules of a one-point extension algebra, and of a triangular matrix algebra.

This paper is organized as follows: in Section 2 our algebras are finite dimensional over field k. We consider the one-point extension Λ of A by an arbitrary left A-module M. We prove that for an ordering ${}_AE$ on simple A-modules, if $(A, {}_AE)$ is a quasi-hereditary algebra and M is a left A-module, then $(\Lambda, {}_\Lambda E)$ is a quasi-hereditary algebra, where ${}_\Lambda E = {}_AE \cup \{n+1\}$ such that

$E(n+1)$ is the simple Λ-module corresponding to the extension vertex. We describe the category of ∇-good modules over Λ by using the notion of a subspace category and describe the characteristic module of Λ, these results generalize the main results in [2]; in Section 3, all algebras are artin algebras over a commutative artin ring R. We prove the quasi-heredity of the triangular matrix algebras of quasi-hereditary algebras A and B by a bimodule ${}_AM_B$ under a suitable condition on the bimodule M. Moreover, we describe the good module category over this quasi-hereditary triangular matrix algebra and the characteristic module of it. We note that if R is a field k, A is a finite dimensional k-algebra and B is k, then this triangular matrix algebra becomes one-point extension of A by M, but the ordering on the simple modules of the one-point extension considered in this section is different from that of the one-point extension considered in Section 2.

2. One-Point Extensions

Thoughout this section, any algebra means a finite dimensional one over a fixed field k. Let $(A, {}_AE)$ be a quasi-hereditary algebra, M an arbitrary left A-module, and $\Lambda = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$ the one-point extension. Let ${}_\Lambda E = {}_AE \cup \{n+1\}$ such that $E(n+1)$ is the simple module corresponding to the extension vertex,

THEOREM 2.1. *Let $(A, {}_AE)$ be a quasi-hereditary algebra and M a left A-module. Let Λ be the one-point extension of A by M and ${}_\Lambda E$ the ordering on simple Λ-modules as above. Then $(\Lambda, {}_\Lambda E)$ is a quasi-hereditary algebra, and $\mathscr{F}({}_\Lambda\nabla) = \check{\mathscr{U}}(\mathscr{F}({}_A\nabla), \mathrm{Hom}_A(M, -))$.*

PROOF. Let $E(1), \dots, E(n)$ be the simple A-modules. Thus there is a complete set of orthogonal primitive idempotents $\{e_1, \dots, e_n\}$ of A. Let $e_{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\{e_1, \dots, e_n, e_{n+1}\}$ is a complete set of orthogonal primitive idempotents of Λ.

It is easy to see that the costandard Λ-modules are as follows:

$$ {}_\Lambda\nabla(i) = \begin{pmatrix} {}_A\nabla(i) & 0 \\ 0 & 0 \end{pmatrix}, \quad 1 \le i \le n. $$

$$ {}_\Lambda\nabla(n+1) = {}_\Lambda Q(n+1) = E(n+1). $$

We have that $\mathrm{End}_\Lambda({}_\Lambda\nabla(i))$ is a division ring and ${}_\Lambda Q(i) \in \mathscr{F}({}_\Lambda\nabla)$ for any $1 \le i \le n+1$. Then $(\Lambda, {}_\Lambda E)$ is a quasi-hereditary algebra with costandard modules ${}_\Lambda\nabla(i) = ({}_A\nabla(i), 0, 0)$ for all $1 \le i \le n$ and ${}_\Lambda\nabla(n+1) = {}_\Lambda E(n+1) = (0, k, 0)$.

Since the subspace category $\check{\mathscr{U}}(\mathscr{F}({}_A\nabla), \mathrm{Hom}(M, -))$ is a full subcategory of Λ-mod which is closed under extensions and for any $i, {}_\Lambda\nabla(i)$ is in $\check{\mathscr{U}}(\mathscr{F}({}_A\nabla), \mathrm{Hom}(M, -))$, we have that $\mathscr{F}({}_\Lambda\nabla) \subseteq \check{\mathscr{U}}(\mathscr{F}({}_A\nabla), \mathrm{Hom}(M, -))$. For any object $(V_0, V_\omega, \gamma_V)$ in $\check{\mathscr{U}}(\mathscr{F}({}_A\nabla), \mathrm{Hom}(M, -))$, we have an exact sequence:

$$0 \to (V_0, 0, 0) \to (V_0, V_\omega, \gamma_V) \to (0, V_\omega, 0) \to 0,$$

where V_0 is in $\mathscr{F}({}_A\nabla)$, hence $(V_0, 0, 0)$ is in $\mathscr{F}({}_\Lambda\nabla)$. We know that $(0, V_\omega, 0)$ is in $\mathscr{F}({}_\Lambda\nabla(n+1))$ from the fact ${}_\Lambda\nabla(n+1) = (0, k, 0)$. Then $(V_0, V_\omega, \gamma_V)$ is in $\mathscr{F}({}_\Lambda\nabla)$. Therefore $\mathscr{F}({}_\Lambda\nabla) = \check{\mathscr{U}}(\mathscr{F}({}_A\nabla), \mathrm{Hom}(M, -))$. The proof is finished.

Let $(\Lambda, {}_\Lambda E)$ be the quasi-hereditary algebra in Theorem 2.1. Let $f : M \to P(n+1)$ be the injection such that $\mathrm{coker} f$ is the simple projective $E(n+1)$ (the existence of f is from the fact that M is the radical of $P_\Lambda(n+1)$). Let $f_0 : M \to M_0$ be the minimal left $\mathscr{F}({}_\Lambda\nabla)$-approximation of M. Thus by [4], we have that the following exact sequence:

$$0 \to M \xrightarrow{f_0} M_0 \to N_0 \to 0, \quad \text{where } N_0 \in \mathscr{F}({}_\Lambda\Delta).$$

Then we have a commutative diagram which is the pull-out diagram of morphisms f and f_0.

$$\begin{array}{ccc} M & \longrightarrow & P(n+1) \\ \downarrow & & \downarrow \\ M_0 & \longrightarrow & G. \end{array}$$

Let T_0 be an indecomposable direct summand of G having a composition factor as $E(n+1)$. We have that

THEOREM 2.2. *Let A, M and Λ be the same as in Theorem 2.1. and ${}_A T$ the characteristic module of A. Let ${}_\Lambda T = {}_A T \oplus T_0$. Then ${}_\Lambda T$ is the characteristic module of the quasi-hereditary algebra $(\Lambda, {}_\Lambda E)$.*

PROOF. We have the exact sequence: $0 \to M \to M_0 \to N_0 \to 0$ with $N_0 \in \mathscr{F}({}_\Lambda\Delta)$, and a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r, "f_0"] \arrow[d, "f_0"'] & P(n+1) \arrow[r] \arrow[d] & E(n+1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_0 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & E(n+1) \arrow[r] & 0, \\
 & N_0 \arrow[r, equal] \arrow[d] & N_0 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the rows and the columns are exact sequences. Since $\mathscr{F}({}_\Lambda\nabla)$ and $\mathscr{F}({}_\Lambda\Delta)$ are closed under extensions, we have that G is in $\mathscr{F}({}_\Lambda\Delta)$ and in $\mathscr{F}({}_\Lambda\nabla)$. From the constructions of standard (or costandard) Λ-modules, we have that ${}_AT \in \mathscr{F}({}_\Lambda\Delta) \cap \mathscr{F}({}_\Lambda\nabla)$. Since T_0 has a composition factor as $E(n+1)$ and T_0 is not the direct summand of ${}_AT$, we have that ${}_\Lambda T$ is the direct sum of $n+1$ non-isomorphic indecomposable modules belonging to $\mathscr{F}({}_\Lambda\Delta) \cap \mathscr{F}({}_\Lambda\nabla)$. Thus it is the characteristic module of the quasi-hereditary algebra $(\Lambda, {}_\Lambda E)$. The proof is finished.

EXAMPLE. Let A be the algebra given by

$$\begin{tikzcd}
\overset{2}{\circ} \arrow[r, shift left, "\alpha"] & \overset{1}{\circ} \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

with relation $\beta\alpha = 0$. Then A is a quasi-hereditary algebra with standard modules ${}_A\Delta(1) = E(1)$, ${}_A\Delta(2) = \begin{matrix} E(2) \\ E(1) \end{matrix}$. The characteristic module of A is $T = E(1) \oplus \begin{matrix} E(1) \\ E(2) \\ E(1) \end{matrix}$.

Let Λ be the one-point extension of A by $M = E(2)$. Then Λ is the algebra given by

$$\begin{tikzcd}
\overset{3}{\circ} \arrow[r, "\gamma"] & \overset{2}{\circ} \arrow[r, shift left, "\alpha"] & \overset{1}{\circ} \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

with relations $\beta\alpha = \alpha\gamma = 0$. Then Λ is a quasi-hereditary algebra with standard modules ${}_\Lambda\Delta(1) = E(1)$, ${}_\Lambda\Delta(2) = \begin{matrix} E(2) \\ E(1) \end{matrix}$, ${}_\Lambda\Delta(3) = \begin{matrix} E(3) \\ E(2) \end{matrix}$. Its characteristic module ${}_\Lambda T = E(1) \oplus \begin{matrix} E(1) \\ E(2) \\ E(1) \end{matrix} \oplus \begin{matrix} E(1), E(3) \\ E(2) \end{matrix}$, where $\begin{matrix} E(1), E(3) \\ E(2) \end{matrix}$ is determined as follows,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E(2) \arrow[r] \arrow[d] & \begin{matrix} E(3) \\ E(2) \end{matrix} \arrow[r] \arrow[d] & E(3) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \begin{matrix} E(1) \\ E(2) \end{matrix} \arrow[r] \arrow[d] & \begin{matrix} E(1), E(3) \\ E(2) \end{matrix} \arrow[r] \arrow[d] & E(3) \arrow[r] & 0 \\
0 \arrow[r] & E(1) \arrow[r, equal] \arrow[d] & E(1) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

REMARK. The Ringel dual of the quasi-hereditary algebras in Theorem 2.1. is neither a one-point extension of algebras, nor a one-point coextension of algebras in general. For example, the Ringel dual of Λ in the example above is the algebra given by:

$$\begin{tikzcd}
\underset{\circ}{1} \arrow[r, "\alpha"] & \underset{\circ}{3} \arrow[r, "\beta"] & \underset{\circ}{2} \arrow[ll, bend left, "\gamma"]
\end{tikzcd}$$

with relation $\gamma\beta = 0$.

3. Triangular Matrix Algebras over Quasi-Hereditary Algebras

Throughtout this section, we assume that A and B are artin R-algebras, where R is a commutative artin ring. Let

$$\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

be the triangular matrix algebra, where M is an $A-B$-bimodule such that Λ is an artin R-algebra. It is well known that any Λ-module N can be identified with a triple (X, Y, f), where X is an A-module, Y a B-module, and $f : M \otimes_B Y \to X$ an A-module morphism [1].

THEOREM 3.1. *Let $(A, {}_AE)$ and $(B, {}_BE)$ be quasi-hereditary algebras and ${}_\Lambda E = ({}_BE, {}_AE)$. If ${}_AM$ is in $\mathscr{F}({}_A\Delta)$, then $(\Lambda, {}_\Lambda E)$ is a quasi-hereditary algebra. Moreover, $\mathscr{F}({}_\Lambda\Delta) = \{(X, Y, f) \mid X \in \mathscr{F}({}_A\Delta),\ Y \in \mathscr{F}({}_B\Delta)\}$.*

PROOF. Let $(A, {}_AE)$ and $(B, {}_BE)$ be quasi-hereditary algebras and ${}_\Lambda E = ({}_BE, {}_AE)$ the ordering on simple Λ-modules. An easy calculation shows that $(\Lambda, {}_\Lambda E)$ is a quasi-hereditary algebra with standard modules

$$
{}_\Lambda\Delta(1) = \begin{pmatrix} 0 & 0 \\ 0 & {}_B\Delta(1) \end{pmatrix},
$$

$$\cdots \quad \cdots$$

$$
{}_\Lambda\Delta(m) = \begin{pmatrix} 0 & 0 \\ 0 & {}_B\Delta(m) \end{pmatrix},
$$

$$
{}_\Lambda\Delta(m+1) = \begin{pmatrix} {}_A\Delta(1) & 0 \\ 0 & 0 \end{pmatrix},
$$

$$\cdots \quad \cdots$$

$$
{}_\Lambda\Delta(m+n) = \begin{pmatrix} {}_A\Delta(n) & 0 \\ 0 & 0 \end{pmatrix}.
$$

We now prove the second assertion. Let $\mathscr{T}$ be the subcategory of Λ-mod consisting of all triples (X, Y, f) with X is from $\mathscr{F}({}_A\Delta)$ and Y is from $\mathscr{F}({}_B\Delta)$. For any triple (X, Y, f) in $\mathscr{T}$, we have an exact sequence:

$$
0 \to (X, 0, 0) \to (X, Y, f) \to (0, Y, 0) \to 0,
$$

where $(X, 0, 0)$ and $(0, Y, 0)$ are in $\mathscr{F}({}_\Lambda\Delta)$. Thus (X, Y, f) is in $\mathscr{F}({}_\Lambda\Delta)$ since $\mathscr{F}({}_\Lambda\Delta)$ is closed under extensions in Λ-mod. Therefore $\mathscr{T} \subseteq \mathscr{F}({}_\Lambda\Delta)$.

By the construction of standard Λ-modules, we have that all standard Λ-modules ${}_\Lambda\Delta(i)$ are in $\mathscr{T}$, where $1 \le i \le m+n$. By identifying an A-module X with a triple $(X, 0, 0)$, and a B-module Y with a triple $(0, Y, 0)$, we can consider both A-mod and B-mod as subcatgories of Λ-mod, namely, we identify A-mod with subcategory (A-mod, 0, 0), and B-mod with subcategory $(0, B\text{-mod}, 0)$. Then $\mathrm{Ext}^1_\Lambda(A\text{-mod}, B\text{-mod}) = 0$, $\mathscr{F}({}_A\Delta)$ and $\mathscr{F}({}_B\Delta)$ are closed under extensions in Λ-mod. We know from [4] that $\mathscr{F}({}_B\Delta) \int \mathscr{F}({}_A\Delta) := \{N \in \Lambda\text{-mod} \mid \text{there is an exact sequence } 0 \to X \to N \to Y \to 0, \text{ with } X \in \mathscr{F}({}_A\Delta),\ Y \in \mathscr{F}({}_B\Delta)\}$ is closed under extensions in Λ-mod. Then $\mathscr{T} = \mathscr{F}(_B\Delta) \int \mathscr{F}(_A\Delta)$ is a subcategory closed under extensions in Λ-mod. For any Δ-good Λ-module N, we have N is in $\mathscr{T}$ since N has a $_\Lambda\Delta$-filtration and all $_\Lambda\Delta(i)$ are in $\mathscr{T}$. Therefore

$$\mathscr{F}(_\Lambda\Delta) = \mathscr{T} = \{(X, Y, f) \mid X \in \mathscr{F}(_A\Delta), Y \in \mathscr{F}(_B\Delta)\}.$$

The proof is finished.

We keep all notation in Theorem 3.1. in the following. We will describe the characteristic module of Λ.

Let $\underline{e} = (e_1, \ldots, e_n)$ be a complete set of orthogonal primitive idempotents of A corresponding to the ordered index set $_AE$ of simple A-modules, $\underline{f} = (f_1, \ldots, f_m)$ a complete set of orthogonal primitive idempotents of B corresponding to the ordered index set $_BE$ of simple B-modules. Thus $(\underline{f}, \underline{e}) = (f_1, \ldots, f_m, e_1, \ldots, e_n)$ is a complete set of orthogonal primitive idempotents of Λ corresponding to the ordered index set $_\Lambda E = (_BE, {_AE})$ of simple Λ-modules. We have a chain of ideals of Λ:

$$\Lambda = J_0 \supset J_1 \supset \cdots \supset J_{m-1} \supset J_m \supset J_{m+1} \supset \cdots \supset J_{m+n-1} \supset J_{m+n} = 0,$$

where

$$J_0 = \begin{pmatrix} A & R \\ 0 & B \end{pmatrix},$$

$$J_1 = \begin{pmatrix} A & R \\ 0 & B(f_2 + \cdots + f_m)B \end{pmatrix},$$

$$\cdots \quad \cdots,$$

$$J_{m-1} = \begin{pmatrix} A & R \\ 0 & Bf_mB \end{pmatrix},$$

$$J_m = \begin{pmatrix} A & R \\ 0 & 0 \end{pmatrix},$$

$$J_{m+1} = \begin{pmatrix} A(e_2 + \cdots + e_n)A & A(e_2 + \cdots + e_n)R \\ 0 & 0 \end{pmatrix},$$

$$\cdots \quad \cdots,$$

$$J_{m+n} = \begin{pmatrix} Ae_nA & Ae_nR \\ 0 & 0 \end{pmatrix},$$

$$J_{m+n} = 0.$$

For each i in $\{1,2,\dots,m+n\}$, let Λ_i be the quotient of Λ by J_i. Then all Λ_i are quasi-hereditary algebras, whose standard modules are ${}_\Lambda\Delta(1),\dots,{}_\Lambda\Delta(i)$. In particular, for any $i \geq m+1$, ${}_\Lambda\Delta(i)$ is an Λ_i-module. We assume that the injective Λ_i-hull of ${}_\Lambda\Delta(i)$ is ${}_{\Lambda_i}Q'(i)$. We have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N_i' \arrow[r, equal] \arrow[d] & N_i' \arrow[d] & \\
0 \arrow[r] & \Delta(i) \arrow[r] \arrow[d, equal] & \Omega(i) \arrow[r] \arrow[d] & N_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta(i) \arrow[r] & {}_{\Lambda_i}Q'(i) \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $N_i \to M_i$ is the minimal right $\mathscr{F}({}_{\Lambda_i}\Delta)$-approximation of M_i. Then we have N_i' is in $\mathscr{F}({}_{\Lambda_i}\nabla)$ by [4]. Therefore $\Omega(i) \in \mathscr{F}({}_{\Lambda_i}\Delta) \cap \mathscr{F}({}_{\Lambda_i}\nabla)$, since $\mathscr{F}({}_{\Lambda_i}\Delta)$ and $\mathscr{F}({}_{\Lambda_i}\nabla)$ are closed under extensions in Λ_i-mod, ${}_{\Lambda_i}Q'(i)$ and N_i' are in $\mathscr{F}({}_{\Lambda_i}\nabla)$, while $\Delta(i)$ and N_i are in $\mathscr{F}({}_{\Lambda_i}\Delta)$. Let $\overline{T}(i)$ be an indecomposable direct summand, which has a composition factor as $E(i)$, of $\Omega(i)$. Then we have that $\overline{T}(m+1)$, $\overline{T}(m+2),\dots,\overline{T}(m+n)$ are non-isomorphic indecomposable modules.

THEOREM 3.2. *Let A, B, ${}_AM_B$, and Λ be the same as in Theorem 3.1. and ${}_BT$ the characteristic module of B. Then ${}_BT \oplus (\bigoplus_{j=1}^n \overline{T}(m+j))$ is the characteristic module of Λ.*

PROOF. By Theorem 3.1., we have that $\mathscr{F}({}_\Lambda\Delta) = \{(X,Y,f) \mid X \in \mathscr{F}({}_A\Delta), Y \in \mathscr{F}_B\Delta\}$, and ${}_BT \in \mathscr{F}({}_B\Delta) \subseteq \mathscr{F}({}_\Lambda\Delta)$. Let $0 \to {}_BT \to (M,N,g) \to (X,Y,f) \to 0$ be an exact sequence with $(X,Y,f) \in \mathscr{F}({}_\Lambda\Delta)$. Then $0 \to {}_BT \to N \to Y \to 0$ is an exact sequence with $Y \in \mathscr{F}({}_B\Delta)$. Since ${}_BT$ is the characteristic module of B, the exact sequence above splits, and $N \cong {}_BT \oplus Y$. It implies that the exact sequence $0 \to {}_BT \to (M,N,g) \to (X,Y,f) \to 0$ splits. We have that $\mathrm{Ext}^1_\Lambda(\mathscr{F}({}_\Lambda\Delta), {}_BT) = 0$, and ${}_BT \in \mathscr{F}({}_\Lambda\Delta) \cap \mathscr{F}({}_\Lambda\nabla)$. Let ${}_\Lambda T$ be the characteristic module of Λ with a decomposition of indecomposable direct summands ${}_\Lambda T = {}_\Lambda T(1) \oplus \cdots \oplus {}_\Lambda T(m) \oplus {}_\Lambda T(m+1) \oplus \cdots \oplus {}_\Lambda T(m+n)$. Then ${}_\Lambda T(1) \oplus \cdots \oplus {}_\Lambda T(m)$ is the characteristic module of quasi-hereditary algebra Λ_m. It follows that the characteristic module of B is isomorphic to ${}_\Lambda T(1) \oplus \cdots \oplus {}_\Lambda T(m)$ from the fact that Λ_m is isomorphic to B. By the construction of $\bar{T}(i)$, the modules ${}_B T \oplus \bar{T}(m+1)$, and ${}_\Lambda T(1) \oplus \cdots \oplus {}_\Lambda T(m) \oplus {}_\Lambda T(m+1)$ are the characteristic module of Λ_{m+1}, thus $\bar{T}(m+1) \cong {}_\Lambda T(m+1)$. We can get that $\bar{T}(m+j)$ is isomorphic to $T(m+j)$ for each $1 \leq j \leq n$ by an easy induction on j. The proof is finished.

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