TSUKUBA J. MATH.
Vol. 26 No. 1 (2002), 119–132

ON CHEN INVARIANT OF CR-SUBMANIFOLDS IN A COMPLEX HYPERBOLIC SPACE

By

Tooru SASAHARA

1. Introduction

One of the very basic problems in submanifold theory is to find relations between extrinsic and intrinsic invariants of submanifolds. Many famous results in differential geometry, such as isoperimetric inequality, Chern-Lashof's theorem among others, are regarded as results in this respect.

Recently, Bang-Yen Chen has introduced new type of Riemannian curvature invariants and obtained sharp inequalities involving these invariants and the square mean curvature for arbitrary submanifolds in real and complex space forms ([5], [6]). Roughly speaking, an isometric immersion of a Riemannian manifold into a space form satisfying an equality case of the inequalities is an immersion which produces the least possible amount of tension from the ambient space at each point of the submanifold. It is natural and interesting to investigate such submanifolds, from both geometric and physical point of views.

Let M be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1, \dots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined to be $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$. Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by $\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta)$, $1 \leq \alpha, \beta \leq r$.

For an integer $k \geq 0$, denote by $\mathscr{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1, \dots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathscr{S}(n)$ the set of k-tuples with $k \geq 0$ for a fixed n.

For each k-tuple $(n_1, \dots, n_k) \in \mathscr{S}(n)$, Chen's curvature invariant $\delta(n_1, \dots, n_k)$ introduced in [5, 6] are given by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{1.1}$$

Received September 13, 2000.
Revised January 22, 2001.

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j, \ j = 1, \ldots, k$.

In [6] Chen proved that, for every n-dimensional submanifold M^n in a real space form $R^m(\varepsilon)$ of constant sectional curvature ε, the invariant $\delta(n_1, \ldots, n_k)$ and the square mean curvature H^2 of M^n satisfy the following sharp inequality:

$$\delta(n_1, \ldots, n_k) \le c(n_1, \ldots, n_k)H^2 + b(n_1, \ldots, n_k)\varepsilon, \tag{1.2}$$

where $c(n_1, \ldots, n_k)$ and $b(n_1, \ldots, n_k)$ are positive constants defined by

$$c(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)}, \tag{1.3}$$

$$b(n_1, \ldots, n_k) = \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\right). \tag{1.4}$$

It was proved in [6] that the same inequality holds for totally real submanifolds in a complex space form of constant holomorphic sectional curvature 4ε, too.

Let M be a submanifold in a Kaehler manifold $\tilde{M}$. A subspace $V \subset T_pM$ is called *totally real* if $JV \subset T_p^{\perp}M$, where $T_p^{\perp}M$ denote the normal space of M at p. M is called *totally real* if each tangent space of M is totally real. A submanifold M of $\tilde{M}$ is called a CR-submanifold if there exists on M a differentiable holomorphic distribution $\mathscr{H}$ such that its orthogonal complement $\mathscr{H}^{\perp} \subset TM$ is a totally real distribution ([2]).

For a $(2n + p)$-dimensional CR-submanifolds with $2n$-dimensional maximal holomorphic tangent subspace (i.e, $\dim \mathscr{H}^{\perp} = p$) in a complex hyperbolic space $\boldsymbol{CH}^m(-4)$ of constant holomorphic sectional curvature -4, we have the following sharp inequality:

$$\delta(n_1, \ldots, n_k) \le c(n_1, \ldots, n_k)H^2 - b(n_1, \ldots, n_k) - 3n + \frac{3}{2}\sum_{i=1}^{k} n_i. \tag{1.5}$$

Let M be a real $2n$-dimensional Kaehler manifold. For a k-tuple $(2n_1, \ldots, 2n_k) \in \mathscr{S}(2n)$, Chen has also introduced the *complex δ-invariants* $\delta^c(2n_1, \ldots, 2n_k)$ by

$$\delta^c(2n_1, \ldots, 2n_k) = \tau - \inf\{\tau(L_1^c) + \cdots + \tau(L_k^c)\},$$

where $L_1^c, \ldots, L_k^c$ run over all k mutually orthogonal complex subspaces of T_pM, $p \in M$, with dimensions $2n_1, \ldots, 2n_k$, respectively.

For $\delta_n^c := \delta^c(2, \dots, 2)$ (2 appears n times) of a $2n$-dimensional Kaehler submanifold in the complex Euclidean space, we have the following result from [5].

$$\delta_n^c \le 0. \tag{1.6}$$

In [9] the author investigated CR-submanifolds with $\dim \mathscr{H}^\perp = 1$ in a complex hyperbolic space satisfying the equality case of (1.5) with $\delta(2, \dots, 2)$ (2 appears k times) and established the explicit representation of such submanifolds in an anti-de Sitter space-time via Hopf's fibration, in terms of Kaehler submanifolds of the complex Euclidean $(m-1)$-space which satisfying the equality case of (1.6). This result is a generalization of Chen and Vrancken's result with $n = 1$ and $k = 1$ ([7]).

CR-submanifolds we constructed in [9] have the following properties:

(1) *The shape operator A_η with respect to the unit vector field $\eta \in \mathscr{H}^\perp$ has two constant principal curvatures.*

(2) *The mean curvature vector field is parallel.*

A submanifold is said to be *linearly full* in $\boldsymbol{CH}^m(-4)$ if it does not lie in any totally geodesic complex hypersurface of $\boldsymbol{CH}^m(-4)$.

The purpose of this paper is to determine linearly full CR-submanifolds with $\dim \mathscr{H}^\perp = 1$ in $\boldsymbol{CH}^m(-4)$ $(m > n+1)$ satisfying the equality case of (1.5) with a general k-tuple $(n_1, \dots, n_k)$, under the condition that the shape operator A_η with respect to $\eta \in \mathscr{H}^\perp$ has constant principal curvatures.

Acknowledgment

The author would like to express his hearty gratitude to Professor B. Y. Chen for many valuable advices.

2. Preliminaries

For a submanifold M^n of a complex space form $\tilde{M}^n(4\varepsilon)$, we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connection on M and $\tilde{M}^n(4\varepsilon)$, respectively. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

where h, A and D are the second fundamental form, the shape operator and the normal connection. Denote by R and $\tilde{R}$ the Riemann curvature tensors of M^n and $\tilde{M}^n(4\varepsilon)$. Then the equations of Gauss, Codazzi and Ricci are given respectively by

$$\langle R(X,Y)Z,W\rangle = \langle A_{h(Y,Z)}X,W\rangle - \langle A_{h(X,Z)}Y,W\rangle$$
$$+\varepsilon\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\}, \quad (2.3)$$

$$(\tilde{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z), \quad (2.4)$$

$$R^D(X,Y;\xi,\eta) = \tilde{R}(X,Y;\xi,\eta) + \langle [A_\xi, A_\eta](X), Y\rangle, \quad (2.5)$$

where X,Y,Z,W (respectively, η and ξ) are vector tangent (respectively, normal) to M, $R^D(X,Y) = [D_X, D_Y] - D_{[X,Y]}$, and $\overline{\nabla}h$ is defined by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (2.6)$$

Let M be a CR-submanifold. Denote by $T^{\perp}M = J\mathscr{H}^{\perp} \oplus v$ the orthogonal decomposition of the normal bundle, where $\mathscr{H}^{\perp}$ is the totally real distribution and v a complex subbundle of $T^{\perp}M$. We have from [4]

$$A_{JZ}W = A_{JW}Z, \quad A_{J\xi}X = -A_\xi JX, \quad (2.7)$$

for vector fields Z,W in $\mathscr{H}^{\perp}$, ξ in v, U in TM and vector field X in the holomorphic distribution $\mathscr{H}$.

3. Main Results

Consider the complex number $(m+1)$-space $\boldsymbol{C}_1^{m+1}$ endowed with the pseudo-Euclidean metric g_0 given by (for the details, cf. [8]) $g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^m dz_j d\bar{z}_j$, where $\bar{z}_k$ denotes the complex conjugate of z_k. On $\boldsymbol{C}_1^{m+1}$ we define $(z,w) = -z_0\overline{w}_0 + \sum_{k=1}^m z_k \overline{w}_k$. Put $H_1^{2m+1}(-1) = \{z = (z_0.z_1.\dots.z_m) \in \boldsymbol{C}_1^{m+1} : (z,z) = -1\}$. Then $H_1^{2m+1}(-1)$ is a real hypersurface of $\boldsymbol{C}_1^{m+1}$ whose tangent space at $z \in H_1^{2m+1}(-1)$ is given by $T_z H_1^{2m+1}(-1) = \{w \in \boldsymbol{C}_1^{m+1} : \mathrm{Re}(z,w) = 0\}$. It is known that $H_1^{2m+1}(-1)$ together with the induced metric g is a pseudo-Riemannian manifold of constant sectional curvature -1, which is known as an anti-de Sitter space time.

We put $H_1^1 = \{\lambda \in \boldsymbol{C} : \lambda\bar{\lambda} = 1\}$. Then we have an H_1^1-action on $H_1^{2m+1}(-1)$ given by $z \mapsto \lambda z$. At each point z in $H_1^{2m+1}(-1)$, the vector iz is tangent to the flow of the action. Since $(\,,\,)$ is Hermitian, we have $(iz,iz) = -1$. Note that the orbit is given by $x(t) = e^{it}z$ and $dx(t)/dt = ix(t)$. Thus the orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2m+1}(-1)/\sim$, under the identification induced from the action, is the complex hyperbolic space $\boldsymbol{C}H^m(-4)$ with constant holomorphic sectional curvature -4. The almost complex structure J on $\boldsymbol{C}H^m(-4)$ is induced from the canonical almost complex structure J on $\boldsymbol{C}_1^{m+1}$, the multiplication by i, via the totally geodesic fibration: $\pi : H_1^{2m+1}(-1) \to \boldsymbol{C}H^m(-4)$.

The main result is the following.

Theorem 1. *Let M be a linearly full $(2n+1)$-dimensional CR-submanifolds with* $\dim \mathscr{H}^{\perp} = 1$ *in $\boldsymbol{C}H^m(-4)$ $(m > n+1)$ satisfying the equality case of* (1.5). *Then the shape operator A_η with respect to the unit vector field $\eta \in \mathscr{H}^{\perp}$ has constant principal curvatures if and only if up to rigid motions, the immersion is the composition $\pi \circ z$, where z is locally, one of the following.*

(1) *$z : \hat{M} = \boldsymbol{R}^2 \times U \to \boldsymbol{C}_1^{m+1}$ is given by*

$$z(u, t, w_1, \dots, w_n) = e^{it}\left(-1 - \frac{1}{2}|\Psi|^2 + iu, -\frac{1}{2}|\Psi|^2 + iu, \Psi\right), \tag{3.1}$$

where U is a domain of $\boldsymbol{C}^n$ and $\Psi : U \to \boldsymbol{C}^{m-1}$ is a holomorphic isometric immersion in $\boldsymbol{C}^{m-1}$ satisfying the equality case of (1.6).

(2) *$z : \boldsymbol{R}^{2n+2} \supset U \to \boldsymbol{C}_1^{m+1}$ is given by*

$$\begin{aligned} &z(s, t, x_1, x_2, \dots, y_1, y_2) \\ &\quad = \left(g(x, y)e^{-(1-\alpha^2)is}, \frac{\alpha\sqrt{(1-\alpha^2)}}{1-\alpha^2}e^{((1-\alpha^2)/\alpha)it}, \phi(x, y)e^{-(1-\alpha^2)is}\right), \end{aligned} \tag{3.2}$$

where $\alpha = \sqrt{k/(2n-k)}$, $-|g|^2 + |\phi|^2 = -1/(1-\alpha^2)$ and $z_1 = (g(x, y)e^{-(1-\alpha^2)is}, 0, \phi(x, y)e^{-(1-\alpha^2)is})$ is a CR-submanifold with pseudo-Riemannian metric in $\boldsymbol{C}_1^m$ which satisfies the following conditions:

There exists an orthonormal basis $\{E_1, \dots, E_{2n}, \tilde{E}_{2n+1}\}$ such that $E_{2l} = iE_{2l-1}$ $(l = 1, \dots, n)$, $\tilde{E}_{2n+1} = (1/\sqrt{1-\alpha^2})\partial/\partial s$ and the second fundamental form $\tilde{h}$ takes the following form.

$$\tilde{h}(E_{2r-1}, E_{2r-1}) = \sqrt{1-\alpha^2}i\tilde{E}_{2n+1} + \phi_r\tilde{\xi}_r, \tag{3.3}$$

$$\tilde{h}(E_{2r}, E_{2r}) = \sqrt{1-\alpha^2}i\tilde{E}_{2n+1} - \phi_r\tilde{\xi}_r, \tag{3.4}$$

$$\tilde{h}(E_{2r-1}, E_{2r}) = i\phi\tilde{\xi}_r, \quad \tilde{h}(X_i, X_j) = \tilde{h}(X_i, \tilde{E}_{2n+1}) = 0, \quad (i \neq j) \tag{3.5}$$

$$\tilde{h}(\tilde{E}_{2n+1}, \tilde{E}_{2n+1}) = -\sqrt{1-\alpha^2}i\tilde{E}_{2n+1} \tag{3.6}$$

where $X_j \in \tilde{L}_j := Span\{E_{n_1+\cdots+n_{j-1}+1}, \dots, E_{n_1+\cdots+n_j}\}$ $(j = 1, \dots, n)$, $n_1 = \cdots = n_n = 2n/k$, ϕ_r are functions and $\tilde{\xi}_r$ are unit normal vector fields perpendicular to $i\tilde{E}_{2n+1}$.

Another purpose is to prove the following general property.

THEOREM 2. *Let M be a $(2n+p)$-dimensional CR-submanifolds with* $\dim \mathscr{H}^{\perp} = p$ *in $\boldsymbol{C}H^m(-4)$ satisfying the equality case of* (1.5). *Then M has parallel mean curvature vector fields $\vec{H}$ i.e., $D\vec{H} = 0$ and, moreover, M is foliated by geodesics or circles of $\boldsymbol{C}H^m(-4)$. In particular, if $m > n+1$, $p = 1$ and M is linearly full, then M is non-minimal and foliated by circles of $\boldsymbol{C}H^m(-4)$. If $p > 1$, then M is minimal and foliated by geodesics of $\boldsymbol{C}H^m(-4)$.*

4. Proof of Theorem 1

For a subspace $L \in T_pM$ of dimension r we put $\Psi(L) = \sum_{i<j} \langle Jv_i, v_j \rangle^2$, $1 \le i, j \le r$, where $\{v_1, \dots, v_r\}$ is an orthonormal basis of L.

We have the following general inequalities from [6].

LEMMA 3. *Let $x : M \to \boldsymbol{C}H^m(-4)$ be a $(2n+p)$-dimensional CR-submanifold with* $\dim \mathscr{H}^{\perp} = p$. *Then*

$$\delta(n_1, \dots, n_k) \le c(n_1, \dots, n_k)H^2 - b(n_1, \dots, n_k) - 3n + \frac{3}{2}\sum_{i=1}^{k} n_i \tag{4.1}$$

Equality sign of (4.1) *holds for some $(n_1, \dots, n_k) \in \mathscr{S}(2n+p)$ if and only if, there exists an orthonormal basis $e_1, \dots, e_{2m}$ at p, such that*

(a) *$L_j := Span\{e_{n_1+\cdots+n_{j-1}+1}, \dots, e_{n_1+\cdots+n_j}\}$ satisfy $\Psi(L_j) = n_j/2$,*

(b) *the shape operators of M in $\boldsymbol{C}H^m(-4)$ at p take the following forms*:

$$A_r = \begin{pmatrix} A_1^r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & A_k^r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mu_r & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \mu_r \end{pmatrix},$$

$$r = 2n+2, \dots, 2m,$$

where $A_r := A_{e_r}$ and each A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$\mathrm{trace}(A_1^r) = \cdots = \mathrm{trace}(A_k^r) = \mu_r. \tag{4.2}$$

In the rest of this paper we shall assume that M is a $(2n+1)$-dimensional CR-submanifolds with $\dim \mathscr{H}^{\perp} = 1$ in a complex hyperbolic space satisfying the equality case of (4.1). Under the hypothesis, we have $\vec{H} \in J\mathscr{D}^{\perp}$ in the same way as [9]. Let $\{e_1, \dots, e_{2m}\}$ be an orthonormal frame field on M mentioned in

Lemma 3 such that e_{2n+2} is parallel to the mean curvature vector field and $\{e_1, \dots, e_{2n+1}\}$ diagonalize the shape operator A_{2n+2}. Without loss of generality we may assume that $Je_{2n+1} = e_{2n+2}$. In the same way as [9] we have Je_{2n+1} is a parallel normal vector field, i.e., $D(Je_{2n+1}) = 0$. Then we get

$$2\langle PX, Y\rangle + 2\langle A_{2n+2}PA_{2n+2}X, Y\rangle = (X\mu_{2n+2})\langle e_{2n+1}, Y\rangle$$
$$- (Y\mu_{2n+2})\langle e_{2n+1}, X\rangle + \mu_{2n+2}\langle PA_{2n+2}X, Y\rangle - \mu_{2n+2}\langle PA_{2n+2}Y, X\rangle, \quad (4.3)$$

where PX is tangential components of JX.

From now on we shall assume that all principal curvatures of A_{2n+2} are constant. Then we have the following lemmas.

Lemma 4. *Let $\{e_1, \dots, e_{2n}\}$ be an orthonormal frame field of $\mathscr{H}$ with $A_{2n+2}e_i = \lambda_i e_i$ as above. Then for any $i \in \{1, \dots, 2n\}$ we get,*

$$\sum_{j=1,\, \lambda_j \neq \lambda_i}^{2n} \left(\frac{-1 + \lambda_i\lambda_j}{\lambda_i - \lambda_j}(1 + 2\langle Pe_i, e_j\rangle)^2 + \frac{1}{\lambda_i - \lambda_j}\sum_{2n+3}^{m}(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) \right) = 0. \quad (4.4)$$

where $h_{ij}^r = \langle A_r e_i, e_j\rangle$.

Proof. The proof is given in the same way as the proof of lemma 2 in [3]. □

Lemma 5. *If $\mu_{2n+2}^2 = 4$, then A_{2n+2} has exactly two distinct principal curvatures.*

Proof. Let $\sigma(\mathscr{H})$ be the spectrum of $A_{2n+2}|\mathscr{H}$. For $\lambda \in \sigma(\mathscr{H})$ we denote by T_λ the sub-bundle of $\mathscr{H}$ formed by the eigenspaces corresponding to the eigenvalue λ. By using (4.3) for $\lambda \in \sigma(\mathscr{H})$, $X \in T_\lambda$ we have

$$(2\lambda - \mu_{2n+2})A_{2n+2}PX = (-2 + \lambda\mu_{2n+2})PX. \quad (4.5)$$

Assume that there exists $\lambda \in \sigma(\mathscr{H})$ with $\lambda \neq \alpha/2$. We obtain from (4.5) that $A_{2n+2}PX = (\alpha/2)PX$ for $X \in T_\lambda$. Hence $\alpha/2$ is an eigenvalue. Let E_j be the eigenvectors corresponding to $\lambda_j \neq \alpha/2$.

By the way, we have $\tilde{R}(X, Y; Je_{2n+1}, \xi) = R^D(X, Y; Je_{2n+1}, \xi) = 0$ for any $\xi \in \nu$ by virtue of $D(Je_{2n+2}) = 0$. Hence, Ricci equation implies

$$[A_{2n+2}, A_\xi] = 0. \quad (4.6)$$

It follows from (2.7) and (4.6) that $\langle A_r E_j, E_j\rangle\langle A_r X, X\rangle - \langle A_r E_j, X\rangle^2 = 0$, for eigenvector $X \in T_{\alpha/2}$. Hence we have

$$\sum_{j=1,\,\lambda_j\neq\alpha/2}^{2n}\frac{-1+(\alpha/2)\lambda_j}{\alpha/2-\lambda_j}(1+2\langle PX,E_j\rangle^2)=-\frac{\alpha}{2}\sum_{j=1,\,\lambda_j\neq\alpha/2}^{2n}(1+2\langle PX,E_j\rangle^2)\neq 0, \tag{4.7}$$

which contradicts (4.4). Therefore we obtain that $\sigma(\mathscr{H})=\{\alpha/2\}$. □

LEMMA 6. *If $\mu_{2n+2}^2\neq 4$, A_{2n+2} has at most three distinct principal curvatures.*

PROOF. If $\sharp\sigma(\mathscr{H})\geq 2$, then we have the following orthogonal decomposition:

$$\mathscr{H}=T_{\alpha_1}\oplus JT_{\alpha_1}\oplus\cdots T_{\alpha_s}\oplus JT_{\alpha_s}\oplus T_\lambda\oplus T_{\mu_{2n+2}-\lambda}, \tag{4.8}$$

where $\lambda=\left(\mu_{n+2}+\sqrt{\mu_{n+2}^2-4}/2\right)$, T_λ and $T_{\mu_{2n+2}-\lambda}$ are J-invariant, and $\lambda\neq\alpha_j$ from (4.5). We may assume that we can choose the eigenvalue $\beta\in\sigma(\mathscr{H})$ with $\beta>0$ and that there are no further eigenvalues between β and $1/\beta$. Hence, for all eigenvalues $\gamma\in\sigma(\mathscr{H})$, we have

$$\frac{-1+\beta\gamma}{\beta-\gamma}\leq 0. \tag{4.9}$$

On the other hand by virtue of (2.7) and (4.6), we have $\langle A_re_i,e_j\rangle=0$ for $r\geq 2n+2$ and $(e_i,e_j)\notin T_\lambda\oplus T_{\mu_{2n+2}-\lambda}\times T_\lambda\oplus T_{\mu_{2n+2}-\lambda}$. Hence we obtain

$$\sum_{j=1,\,\lambda_j\neq\alpha_l}^{2n}\sum_{r=2n+3}^{m}\frac{1}{\alpha_l-\lambda_j}(\langle A_rX,X\rangle\langle A_re_j,e_j\rangle-\langle A_re_j,X\rangle^2)=0 \tag{4.10}$$

for each eigenvector X corresponding to α_l $(l=1,\dots,s)$. Further, for each eigenvector Y corresponding to λ, we have

$$\begin{aligned}&\sum_{j=1,\,\lambda_j\neq\lambda}^{2n}\sum_{r=2n+3}^{m}\frac{1}{\lambda-\lambda_j}(\langle A_rY,Y\rangle\langle A_re_j,e_j\rangle-\langle A_rY,e_j\rangle^2)\\&=\sum_{r=2n+3}^{m}\left(\frac{1}{2\lambda-\mu_{2n+2}}\langle A_rY,Y\rangle\sum_{j=1,\,\lambda_j\neq\lambda}^{t}\langle A_r\tilde{E}_j,\tilde{E}_j\rangle\right)=0,\end{aligned} \tag{4.11}$$

where $\tilde{E}_j$ are eigenvectors corresponding to $\mu_{2n+2}-\lambda$ and $t=\dim T_{\mu_{2n+2}-\lambda}$. It follows from (4.4), (4.9), (4.10) and (4.11) that $-1+\beta\gamma=0$. Hence we obtain that $T_\lambda=\phi$, $T_{\mu_{2n+2}-\lambda}=\phi$ and $s=2$. Therefore $\sharp\sigma(\mathscr{H})=2$. □

LEMMA 7. *If $m>n+1$ and M is linearly full, then $n_1=\cdots=n_k$, $n_1+\cdots+n_k=2n$ and, moreover, with respect to some suitable orthonormal frame field*

$\{e_1, \dots, e_{2m}\}$, *the second fundamental form of* M *in* $\boldsymbol{C}H^m(-4)$ *satisfies*

$$h(e_{2r-1}, e_{2r-1}) = \sqrt{\frac{k}{2n-k}} Je_{2n+1} + \phi_r \xi_r, \tag{4.12}$$

$$h(e_{2r}, e_{2r}) = \sqrt{\frac{k}{2n-k}} Je_{2n+1} - \phi_r \xi_r, \tag{4.13}$$

$$h(e_{2r-1}, e_{2r}) = \phi_r J\xi_r, \quad h(e_{2n+1}, e_{2n+1}) = \frac{2n}{\sqrt{k(2n-k)}} Je_{2n+1} \tag{4.14}$$

$$h(f_i, f_j) = h(f_i, e_{2n+1}) = 0 \quad (i \neq j), \tag{4.15}$$

where $r = 1, \dots, n, \phi_r$ *are functions,* $\xi_r \in \nu$ *and* $f_j \in L_j := Span\{e_{n_1+\cdots+n_{j-1}+1}, \dots, e_{n_1+\cdots+n_j}\}$.

Proof. If $n_1 + \cdots + n_k < 2n$, then we have $A_{2n+2}e_{2n-1} = \mu_{2n+2}e_{2n-1}$, $A_{2n+2}Je_{2n-1} = \mu_{2n+2}Je_{2n-1}$. We obtain from (4.3) that $2 + 2\mu_{2n+2}^2 = 2\mu_{2n+2}^2$. It is a contradiction. Therefore $n_1 + \cdots + n_k = 2n$.

Suppose that A_{2n+2} has exactly three distinct principal curvatures, μ_{2n+2}, $\left(\mu_{n+2} + \sqrt{\mu_{n+2}^2 - 4}\right)/2$ and $\left(\mu_{n+2} - \sqrt{\mu_{n+2}^2 - 4}\right)/2$. If each submatrix A_{2n+2}^i has only one eigenvalue λ_i, then we have $n_i\lambda_i = n_i/\sqrt{n_i - 1} = \mu_{2n+2}$ for any i. Hence $\sharp\sigma(\mathscr{H}) = 1$. But it is a contradiction. If there exists a submatrix A_{2n+2}^j which has two eigenvalues $\left(\mu_{n+2} + \sqrt{\mu_{n+2}^2 - 4}\right)/2$ and $\left(\mu_{n+2} - \sqrt{\mu_{n+2}^2 - 4}\right)/2$ whose multiplicities are l and m $(l > m)$ respectively. Then we get $(l - m)\sqrt{\mu_{2n+2}^2 - 4} = (2 - l - m)\mu_{2n+2}$. But it does not hold, since $l, m > 2$. If $\mathscr{H} = T_{\alpha_1} \oplus JT_{\alpha_1}$, where $\alpha_1 \neq \mu_{2n+2}$, $\left(\mu_{n+2} \pm \sqrt{\mu_{n+2}^2 - 4}\right)/2$, then it follows from (2.7) and (4.6) that M is contained in a totally geodesic complex hyperbolic space $\boldsymbol{C}H^{n+1}(-4)$, since Je_{2n+1} is parallel. This is a contradiction. Therefore, A_{2n+2} has exactly two distinct eigenvalues.

Let X be the eigenvector corresponding to the second eigenvalue $\alpha \neq \mu_{2n+2}$. From (4.5) we obtain that PX is also an eigenvector corresponding to the eigenvalue $\beta = (-2 + \alpha\mu_{2n+2})/(2\alpha - \mu_{2n+2})$. Since A_{2n+2} has exactly two distinct eigenvalues, we have $\beta = \mu_{2n+2}$ or $\beta = \alpha$.

Suppose that A_{2n+2} has two distinct eigenvalues μ_{2n+2} and $(-2 + \mu_{2n+2}^2)/\mu_{2n+2}$, i.e. $\mu_{2n+2} = \beta$. Then, using (2.7) and (4.6), we obtain that M is contained in a totally geodesic complex hyperbolic space $\boldsymbol{C}H^{n+1}(-4)$. This is a contradiction.

Hence A_{2n+2} has two distinct eigenvalues μ_{2n+2} and $\alpha = \beta$. Then from (4.3) we obtain that $\alpha^2 - \mu_{2n+2}\alpha = -1$. Moreover, using (4.5), we obtain that T_α is

J-invariant. Further, let us suppose that the multiplicity of μ_{2n+2} is greater than one. Let $X \in T_{\mu_{2n+2}}$ and $\langle X, e_{2n+1}\rangle = 0$. Then it follows from (4.5) that $A_{2n+2}PX = ((-2+\mu_{2n+2}^2)/\mu_{2n+2})PX = \alpha PX$. Hence, $PX \in T_\alpha$. Therefore we obtain that $A_{2n+2}P^2X = \alpha P^2X$, i.e. $A_{2n+2}X = \alpha X$. But it is a contradiction, since $\mu_{2n+2} \neq \alpha$. Therefore, the multiplicity of μ_{2n+2} is one. Moreover $n_1 = \cdots = n_k$ by virtue of lemma 3.

Consequently, from lemma 3, replace e_{2n+1} by $-e_{2n+1}$ if necessary, we obtain that $\alpha = 1/\sqrt{n_1 - 1}$ and $\mu_{2n+2} = n_1/\sqrt{n_1 - 1}$. □

Let $\hat{M} = \pi^{-1}(M)$ denote the inverse image of M via the Hopf fibration $\pi : H_1^{2m+1} \to \boldsymbol{C}H^m(-4)$. Then $\hat{M}$ is a principal circle bundle over M with time-like totally geodesic fibers. Let $z : \hat{M} \to H_1^{2m+1}(-1) \subset \boldsymbol{C}_1^{m+1}$ denote the immersion of $\hat{M}$ in $\boldsymbol{C}_1^{m+1}$. Let $\tilde{\nabla}$ and $\hat{\nabla}$ denote the metric connections of $\boldsymbol{C}_1^{m+1}$ and $\hat{M}$, respectively. We denote by X^* the horizontal lift of a tangent vector X of $\boldsymbol{C}H^m(-4)$. Then we have (cf. [8])

$$\tilde{\nabla}_{X^*} Y^* = (\nabla_X Y)^* + (h(X, Y))^* + \langle JX, Y\rangle V + \langle X, Y\rangle z, \tag{4.16}$$

$$\tilde{\nabla}_{X^*} V = \tilde{\nabla}_V X^* = (JX)^*, \tag{4.17}$$

$$\tilde{\nabla}_V V = -z, \tag{4.18}$$

for vector fields X, Y tangent to M, where z is the position vector of $\hat{M}$ in $\boldsymbol{C}_1^{2m+1}$ and $V = iz \in T_z H_1^{2m+1}(-1)$.

Let $E_1, \dots, E_{2n+1}, \xi_r^*$ be the horizontal lifts of $e_1, \dots, e_{2n+1}, \xi_r$, respectively and let $E_{2n+2} = iz$, and let $\{\omega_i^j\}$ be connection forms of $\hat{M}$. Then, in the same way as [7, 9], from lemma 7, (4.16), (4.17) and (4.18), we obtain

$$\tilde{\nabla}_{E_{2r-1}} E_{2r-1} = \sum_{j=1}^{2n} \omega_{2r-1}^j(E_{2r-1})E_j + \alpha i E_{2n+1} + \phi_r \xi_r^* - iE_{2n+2}, \tag{4.19}$$

$$\tilde{\nabla}_{E_{2r-1}} E_{2r} = \sum_{j=1}^{2n} \omega_{2r}^j(E_{2r-1})E_j - \alpha E_{2n+1} + i\phi_r \xi_r^* + E_{2n+2}, \tag{4.20}$$

$$\tilde{\nabla}_{E_{2r}} E_{2r-1} = \sum_{j=1}^{2n} \omega_{2r-1}^j(E_{2r})E_j + \alpha E_{2n+1} + i\phi_r \xi_r^* - E_{2n+2}, \tag{4.21}$$

$$\tilde{\nabla}_{E_{2r}} E_{2r} = \sum_{j=1}^{2n} \omega_{2r}^j(E_{2r})E_j + i\alpha E_{2n+1} - \phi_r \xi_r^* - iE_{2n+2}, \tag{4.22}$$

$$\tilde{\nabla}_{E_{2r-1}} E_{2n+1} = \alpha E_{2r}, \tag{4.23}$$

$$\tilde{\nabla}_{E_{2r}} E_{2n+1} = -\alpha E_{2r-1}, \tag{4.24}$$

$$\tilde{\nabla}_{E_{2n+1}} E_{2n+1} = \frac{2n}{k} \alpha i E_{2n+1} - i E_{2n+2}, \tag{4.25}$$

$$\tilde{\nabla}_{E_{2r-1}} E_{2n+2} = \tilde{\nabla}_{E_{2n+2}} E_{2r-1} = E_{2r}, \tag{4.26}$$

$$\tilde{\nabla}_{E_{2r}} E_{2n+2} = \tilde{\nabla}_{E_{2n+2}} E_{2r} = -E_{2r-1}, \tag{4.27}$$

$$\tilde{\nabla}_{E_{2n+1}} E_{2n+2} = \tilde{\nabla}_{E_{2n+2}} E_{2n+1} = i E_{2n+1}, \tag{4.28}$$

$$\tilde{\nabla}_{E_{2n+2}} E_{2n+2} = i E_{2n+2}, \tag{4.29}$$

$$\tilde{\nabla}_{X_i} X_j \in Span\{E_1, \dots, E_{2n}\} \quad (i \neq j), \tag{4.30}$$

where $r = 1, \dots, n$, $\alpha = \sqrt{k/(2n-k)}$ and $X_j \in L_j := Span\{E_{n_1+\cdots+n_{j-1}+1}, \dots, E_{n_1+\cdots+n_j}\}$.

By using the above equations, we obtain the following lemmas.

Lemma 8. $\hat{\nabla}_{E_{2n+1}-\alpha E_{2n+2}}(E_{2n+1} - \alpha E_{2n+2}) = 0$, $\hat{\nabla}_{X'} Y' \in \mathscr{D}_1$ *for* $X', Y' \in \mathscr{D}_1$, *where* $\mathscr{D}_1 := Span\{E_1, E_2, \dots, E_{2n}, \alpha E_{2n+1} - E_{2n+2}\}$.

Lemma 9. *Let* $X' \in \mathscr{D}_1$. *Then* $\hat{\nabla}_{E_{2n+1}-\alpha E_{2n+2}} X' \in \mathscr{D}_1$, $\hat{\nabla}_{X'}(E_{2n+1} - \alpha E_{2n+2}) = 0$.

Lemma 10. $Z := E_{2n+1} - \alpha E_{2n+2}$ *is a constant vector in* $\boldsymbol{C}_1^{m+1}$ *along each integral manifold of* $\mathscr{D}_1$.

Proof. If follows from (4.23), (4.24), (4.26) and (4.27) that $\tilde{\nabla}_{E_{2r-1}}(E_{2n+1} - \alpha E_{2n+2}) = \tilde{\nabla}_{E_{2r}}(E_{2n+1} - \alpha E_{2n+2}) = 0$. Using (4.25), (4.28) and (4.29), we get $\tilde{\nabla}_{\alpha E_{2n+1}-E_{2n+2}}(E_{2n+1} - \alpha E_{2n+2}) = ((2n/k)\alpha^2 - \alpha^2 - 1) i E_{2n+1} = 0$. □

If $\alpha = 1$, we have $k = n$ and $n_1 = \cdots = n_n = 2$. In this case, $\hat{M}$ is represented by (1) of Theorem 1 by virtue of main theorem in [9].

Suppose that $\alpha \neq 1$. Then from lemma 8, 9, there exist coordinates $\{s, t, x_1, y_1, \dots, x_n, y_n\}$ such that $\partial/\partial s, \partial/\partial x_1, \dots, \partial/\partial y_n$ are tangent to integral manifolds of $\mathscr{D}_1$, $\partial/\partial s = \alpha E_{2n+1} - E_{2n+2}$ and $\partial/\partial t = E_{2n+1} - \alpha E_{2n+2}$. Then $\hat{M}$ is locally a Riemannian product $\hat{M}_1 \times \hat{M}_2$, where $\hat{M}_1$ is a integral manifold of $\mathscr{D}_1$ and $\hat{M}_2$ is a integral curve of $E_{2n+1} - \alpha E_{2n+2}$. Moreover, $z : \hat{M} \to \boldsymbol{C}_1^{m+1}$ is a product immersion. We put $Z_0 := Z|_{t=0}$.

We may assume $Z_0 = (0, \sqrt{1-\alpha^2}, 0, \dots, 0)$, up to rigid motions. In the same way as [7, 9], since (z, Z_0) is constant, we have

$$z(s, 0, x_1, y_1, \dots, x_n, y_n) = (f, c, \Psi_1, \dots, \Psi_{m-1}), \tag{4.31}$$

where c is a constant determined by the initial conditions and $f, \Psi_1, \dots, \Psi_{m-1}$ are functions.

Since $z_s + (1-\alpha^2)iz = \alpha E_{2n+1} - E_{2n+2} + (1-\alpha^2)E_{2n+2} = \alpha(E_{2n+1} - \alpha E_{2n+2}) = \alpha Z$, we have

$$\frac{\partial f}{\partial s} + (1-\alpha^2)if = 0, \quad \frac{\partial \Psi_j}{\partial s} + (1-\alpha^2)i\Psi_j = 0, \tag{4.32}$$

$$c(1-\alpha^2)i = \alpha\sqrt{1-\alpha^2}, \quad (1-\alpha^2)iz_2 = \alpha\frac{\partial z_2}{\partial t}, \tag{4.33}$$

where, z_2 is a position vector of $\hat{M}_2$ in $\boldsymbol{C}_1^{m+1}$. By solving differential equations (4.32) and (4.33), we have

$$z(s, t, x_1, \dots, y_n)$$
$$= \left(g(x_1, \dots, y_n)e^{-(1-\alpha^2)is}, \frac{\alpha\sqrt{1-\alpha^2}}{\alpha^2 - 1}ie^{((1-\alpha^2)/\alpha)it}, \phi(x_1, \dots, y_n)e^{-(1-\alpha^2)is}\right). \tag{4.34}$$

Since $(z, z) = -1$, we have

$$-|g|^2 + \frac{\alpha^2}{1-\alpha^2} + |\phi|^2 = -1. \tag{4.35}$$

We put $\tilde{E}_{2n+1} = 1/\sqrt{1-\alpha^2}(\alpha E_{2n+1} - E_{2n+2})$ and $\tilde{E}_{2n+2} = (1/\sqrt{1-\alpha^2}) \cdot (E_{2n+1} - \alpha E_{2n+2})$. It follows from (4.19)–(4.29) that the second fundamental form of $\hat{M}_1$ in $\boldsymbol{C}_1^m$ satisfies (3.3)–(3.6).

Conversely, suppose that $\hat{M}$ satisfies the conditions in (2) of Theorem 1. Let $\tilde{E}_{2n+2}$ be a unit vector field of $\hat{M}_2$. We put $E_{2n+1} = (\alpha\sqrt{1-\alpha^2}/(\alpha^2-1)) \cdot \tilde{E}_{2n+1} - (\sqrt{1-\alpha^2}/(\alpha^2-1))\tilde{E}_{2n+2}$ and $E_{2n+2} = (\sqrt{1-\alpha^2}/(\alpha^2-1))\tilde{E}_{2n+1} - (\alpha\sqrt{1-\alpha^2}/(\alpha^2-1))\tilde{E}_{2n+2}$. Then, $\{E_1, \dots, E_{2n}, E_{2n+1}, E_{2n+2}\}$ is an orthonormal basis $\hat{M}$ and the second fundamental form of $\hat{M}$ in $\boldsymbol{C}_1^{m+1}$ satisfies

$$\tilde{h}(E_{2r-1}, E_{2r-1}) = \alpha i E_{2n+1} - iE_{2n+2} + \phi_r\tilde{\xi}_r, \tag{4.36}$$

$$\tilde{h}(E_{2r}, E_{2r}) = \alpha i E_{2n+1} - iE_{2n+2} - \phi_r\tilde{\xi}_r, \tag{4.37}$$

$$\tilde{h}(E_{2r-1}, E_{2r}) = i\phi_r\tilde{\xi}_r, \quad \tilde{h}(X_i, X_j) = h(X_i, E_{2n+1}) = 0, \quad (i \neq j), \tag{4.38}$$

$$\tilde{h}(E_{2n+1}, E_{2n+1}) = \frac{2n}{k}\alpha i E_{2n+1} - iE_{2n+2}, \tag{4.39}$$

$X_i \in L_i$, ϕ_r are functions and $\tilde{\xi}_r$ are unit normal vector fields perpendicular to iE_{2n+1}, iE_{2n+2}. Therefore, we obtain that $e_1 = \pi_*(E_1), \dots, e_n = 2\pi_*(E_{2n})$, $e_{2n+1} = \pi_*(E_{2n+1})$ satisfy (4.12)–(4.15). This completes the proof of Theorem 1.

In the rest of this section we shall determine *normal CR*-submanifolds with $\dim \mathscr{H}^\perp = 1$ in a complex hyperbolic space satisfying the equality case of (1.7).

$(P, e_{2n+1}, \omega^1, g)$ defines an almost contact metric structure on (M, g), where $\omega^1(X) := \langle e_{2n+1}, X\rangle$ and g is an induced metric ([10]). M is said to be *normal* if the tensor field S defined by

$$\begin{aligned} S(X, Y) = [PX, PY] + P^2[X, Y] - P[X, PY] \\ - P[PX, Y] + 2\, d\omega_1(X, Y)e_{2n+1} \end{aligned} \tag{4.40}$$

vanishes ([1]).

Theorem 11. *Let M be a linearly full $(2n+1)$-dimensional CR-submanifolds with* $\dim \mathscr{H}^\perp = 1$ *in $\boldsymbol{C}H^m(-4)$ $(m > n+1)$ satisfying the equality case of* (15). *Then M is normal if and only if $\hat{M}$ is represented by* (1) *or* (2) *of Theorem* 1.

Proof. It is known that M is normal if and only if $PA_{2n+2} = A_{2n+2}P$ ([10]). From (4.3) we obtain that the shape operator A_{2n+2} has at most three distinct constant eigenvalues μ_{2n+2}, $\left(\mu_{n+2} + \sqrt{\mu_{n+2}^2 - 4}\right)/2$ and $\left(\mu_{n+2} - \sqrt{\mu_{n+2}^2 - 4}\right)/2$. The assertion follows immediately from Theorem 1. □

5. Proof of Theorem 2

Case 1. $p = 1$. In this case, in the same way as [9] we obtain that Je_{2n+1} is a parallel normal vector field, i.e., $D(Je_{2n+1}) = 0$. Putting $Y = e_{2n+1}$ in (4.3), we get $X\mu_{2n+1} = \omega^1(X)e_{2n+1}\mu_{2n+2}$.

Differentiating this relation covariantly and using relation $(\nabla_Y\omega^1)(X) = \langle PA_{2n+2}Y, X\rangle$, we obtain

$$\begin{aligned} Y(e_{2n+1}\mu_{2n+2})\omega^1(X) - X(e_{2n+1}\mu_{2n+2})\omega^1(Y) \\ + e_{2n+1}\mu_{2n+2}\langle(PA_{2n+2} + A_{2n+2}P)Y, X\rangle = 0. \end{aligned} \tag{5.1}$$

Putting $Y = e_{2n+1}$ in (5.1), we get $X(e_{2n+1}\mu_{2n+2}) = e_{2n+1}(e_{2n+1}\mu_{2n+2})\omega^1(X)$. Combining this and (5.2) yields $(e_{2n+1}\mu_{2n+2})\langle(PA_{2n+2} + A_{2n+2}P)Y, X\rangle = 0$.

Putting $X = \sum_{l=1}^{l=n_1/2} Je_{2l-1}$ and $Y = \sum_{l=1}^{l=n_1/2} e_{2l-1}$ in this relation, we have $e_{2n+1}\mu_{2n+1}\,\mathrm{trace}(A_1^{2n+1}) = 0$. If M is nonminimal, then $e_{2n+1}\mu_{2n+2} = 0$, since $\mathrm{trace}(A_1^{2n+1}) \neq 0$ from Lemma 3. Since $X\mu_{2n+1} = 0$ for any vector X perpendicular to e_{2n+1}, we obtain that μ_{2n+2} is constant and hence $D\vec{H} = 0$. If M is minimal, then M is is contained in a totally geodesic complex hyperbolic space $\boldsymbol{C}H^{n+1}(-4)$ by (2.7), (4.3) and (4.6).

Case 2. $p \geq 2$. In this case, by applying (2.7) we have $\mu_{2n+2}e_{2n+2} = A_{Je_{2n+1}}e_{2n+2} = A_{Je_{2n+2}}e_{2n+1} = 0$. Therefore, in this case M is minimal.

Finally, we obtain from (2.1) and (2.2) that

$$\tilde{\nabla}_{e_{2n+1}}e_{2n+1} = \mu_{2n+2}Je_{2n+1}, \quad \tilde{\nabla}_{e_{2n+1}}Je_{2n+1} = -\mu_{2n+2}e_{2n+1}. \tag{5.2}$$

Hence the integral curve of $\mathscr{H}^{\perp}$ are geodesic or circle of $\boldsymbol{C}H^m(-4)$. This completes the proof of Theorem 2. □

References

[1] A. Bejancu, Normal *CR*-submanifolds of a Kaehler manifold. An. St. Univ. Al. I. Cuza Iasi. **26** (1980), 123–132.
[2] A. Bejancu, Geometry of *CR*-submanifolds. D. Reidel Publ. Dordrecht, Holland. (1986).
[3] J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space. J. reine angrew. Math. **395** (1989), 132–141.
[4] B. Y. Chen, *CR*-submanifolds of a Kaehler manifold, I, II. J. Differential Geometry. **16** (1981), 305–322; **16** (1981), 493–509.
[5] B. Y. Chen, Strings of Riemannian invariants, inequalities, ideal immersions and their applications. Third Pacific Rim Geom. Conf. (1998), pp. 7–60 (Internat. Press, Cambridge, MA).
[6] B. Y. Chen, Some new obstructions to minimal and Lagrangian isometric immersions. Japan. J. Math. **26** (2000), 105–127.
[7] B. Y. Chen and L. Vrancken, *CR*-submanifolds of complex hyperbolic spaces satisfying a basic equality. Israel. J. Math. **110** (1999), 341–358.
[8] B. Y. Chen, G. D. Ludden and S. Montiel, Real submanifolds of a Kaehler manifold. Algebras, Groups and Geom. **1** (1984), 176–212.
[9] T. Sasahara, *CR*-submanifolds in complex hyperbolic spaces satisfying an equality of Chen. Tsukuba. J. Math. **23** (1999), 565–583.
[10] Y. Tashiro, Relations between almost complex spaces and almost contact spaces (in Japanese). Sugaku. **16** (1964), 34–61.

Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan
E-mail: t-sasa@math.sci.hokudai.ac.jp