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# NOTE ON MACAULAY SEMIGROUPS

By

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Almost all of ideal theory of a commutative ring $R$ concerns properties of ideals of $R$ with respect to the multiplication "$\times$" on $R$. Abandoning the addition "$+$" on $R$ we extract the multiplication on $R$. Then we have the idea of the algebraic system $S$ of a semigroup. We denote the operation on $S$ by addition. $S$ is called a grading monoid. Concretely, a submonoid $S$ of a torsion-free abelian (additive) group is called a grading monoid (or a $g$-monoid). Many terms in commutative ring theory are defined analogously for $S$. For example, a non-empty subset $I$ of $S$ is called an ideal of $S$ if $S+I \subset I$. Let $I$ be an ideal of $S$ with $I \subsetneq S$. If $s_1+s_2 \in I$ (for $s_1, s_2 \in S$) implies $s_1 \in I$ or $s_2 \in I$, then $I$ is called a prime ideal of $S$. If there exists an element $s \in S$ such that $I = S+s$, then $I$ is called a principal ideal of $S$. The group $q(S) = \{s_1 - s_2 \mid s_1, s_2 \in S\}$ is called the quotient group of $S$. A subsemigroup of $q(S)$ containing $S$ is called an oversemigroup of $S$. Let $\Gamma$ be a totally ordered abelian (additive) group. A mapping $v$ of a torsion-free abelian group $G$ onto $\Gamma$ is called a valuation on $G$ if $v(x+y) = v(x)+v(y)$ for all $x, y \in G$. Then $v$ is called a $\Gamma$-valued valuation on $G$. The subsemigroup $\{x \in G \mid v(x) \geq 0\}$ of $G$ is called the valuation semigroup of $G$ associated with $v$. A $\boldsymbol{Z}$-valued valuation is called a discrete valuation of rank 1. The valuation semigroup associated with a discrete valuation of rank 1 is called a discrete valuation semigroup of rank 1. An element $x$ of an extension semigroup $T$ of $S$ is called integral over $S$ if $nx \in S$ for some $n \in N$. Let $\bar{S}$ be the set of all integral elements of $q(S)$ over $S$. Then $\bar{S}$ is an oversemigroup of $S$, and is called the integral closure of S. If $\bar{S} = S$, then $S$ is called an integrally closed semigroup (or a normal semigroup). An ideal $I$ of $S$ is called a cancellation ideal of $S$ if $I+J_1 = I+J_2$ (for ideals $J_1, J_2$ of $S$) implies $J_1 = J_2$. The maximum number $n$ so that there exists a chain $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n (\subsetneq S)$ of prime ideals of $S$ is called the dimension of $S$. Many propositions for commutative rings are known to hold for $S$. The author conjectures that almost all propositions of multi-

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plicative ideal theory of $R$ hold for $S$. It is usually expected that ideal theory of $S$ is simpler than that of $R$. Therefore investigating the ideal theory of $S$ may be an auxiliary means for investigating that of $R$. Of course we may say that the ideal theory of $S$ has its proper significance (cf. [M2])). For an example, Anderson-Anderson [AA] posed a question: Is every cancellation ideal of a quasi-local domain principal? The answer to this question is open. However, every cancellation ideal of $S$ is principal ([M1]). For another example, let $\sum'(D)$ (resp. $\sum'(S)$) be the set of all semistar-operations on $D$ (resp. $S$). Assume that $S$ is integrally closed and of dimension $n$. Then $S$ is a valuation semigroup if and only if $n+1 \leq |\sum'(S)| \leq 2n+1$ ([OMS]). If a similar property holds for $D$ is open. The aim of this paper is to show that almost all the propositions in Chapter 3 of Kaplansky's Commutative Rings [K] hold for $g$-monoids.

Since this paper is a semigroup version of commutative ring theory, a $g$-monoid is denoted by $R$. Let $A$ be a non-empty set. Assume that, for every $r \in R$ and $a \in A$, there is defined $r + a \in A$ such that, for every $r_1, r_2 \in R$ and $a \in A$, we have $(r_1 + r_2) + a = r_1 + (r_2 + a)$ and $0 + a = a$. Then $A$ is called an $R$-module. Let $A$ be an $R$-module and $r \in R$. If $r + a_1 = r + a_2$ (for $a_1, a_2 \in A$) implies $a_1 = a_2$, then $r$ is called a non-zerodivisor on $A$. If $r$ is not a non-zerodivisor, then $r$ is called a zerodivisor on $A$. The set of zerodivisors on $A$ is denoted by $Z(A)$. Let $B$ be a submodule of an $R$-module $A$, and $r \in R$. If $r + a \in B$ (for $a \in A$) implies $a \in B$, then $r$ is called a non-zerodivisor on $A$ modulo $B$. A non-zerodivisor on $A$ modulo $B$ is also called a non-zerodivisor on $A/B$. If $r$ is not a non-zerodivisor on $A/B$, then $r$ is called a zerodivisor. The set of zerodivisors on $A/B$ is denoted by $Z(A/B)$. If $\{x_1, \dots, x_n\}$ is a finite subset of $R$, then the ideal $\bigcup_{i=1}^n (R + x_i)$ of $R$ is denoted by $(x_1, \dots, x_n)$. The ordered sequence of elements $x_1, \dots, x_n$ of $R$ is called a regular sequence on $A$, if $(x_1, \dots, x_n) + A \subsetneq A$ and if $x_1 \notin Z(A)$, $x_2 \notin Z(A/((x_1) + A)), \dots, x_n \notin Z(A/((x_1, \dots, x_{n-1}) + A))$.

THEOREM 1. *Let $A$ be an $R$-module, and let $x$, $y$ be a regular sequence on $A$. Then* $x \notin Z(A/(y + A))$.

PROOF. Assume that $x + a = y + a_1$ (for $a, a_1 \in A$). Since $y \notin Z(A/(x + A))$, we have $a_1 \in x + A$. Since $x \notin Z(A)$, we have $a \in y + A$, and hence $x \notin Z(A/(y + A))$.

Let $A$ be an $R$-module. If $Z(A) = \varnothing$, then $A$ is called torsion-free. Theorem 1 implies the following, THEOREM 2. *Let $A$ be a torsion-free $R$-module, and $x_1, \dots, x_n$ a regular sequence on $A$. Then the sequence obtained by interchanging $x_i$ and $x_{i+1}$ is a regular sequence on $A$ if and only if $x_{i+1} \notin Z(A/((x_1, \dots, x_{i-1}) + A))$.*

Let $A$ be an $R$-module. If $r_1 + a = r_2 + a$ (for $r_1, r_2 \in R$ and $a \in A$) implies $r_1 = r_2$, then $A$ is called cancellative. If every ideal of a $g$-monoid $R$ is finitely generated, then $R$ is called a Noetherian semigroup.

LEMMA 3 ([M3, Proposition 1]). *Let $R$ be a Noetherian semigroup, and $A$ a finitely generated $R$-module. Then $A$ satisfies the ascending chain condition on submodules.*

THEOREM 4. *Let $R$ be a Noetherian semigroup, and $A$ a finitely generated torsion-free cancellative $R$-module. Let $x_1, \dots, x_n$ be a regular sequence on $A$. Then any permutation of the $x$'s is a regular sequence on $A$.*

PROOF. Set $S = ((x_1, \dots, x_{n-2}) + A : x_n)_A$. By Theorem 2, it suffices to show that $S \subset (x_1, \dots, x_{n-2}) + A$. Suppose the contrary. Take $s$ in $S$ with $s \notin (x_1, \dots, x_{n-2}) + A$. Since $x_n \notin Z(A/((x_1, \dots, x_{n-1}) + A))$, we have $s \in (x_1, \dots, x_{n-1}) + A$, and hence $s = x_{n-1} + a$ for some $a \in A$. It follows that $x_n + a \in (x_1, \dots, x_{n-2}) + A$, and hence $a \in S$. Then we have $S = x_{n-1} + S$; a contradiction to Lemma 3.

THEOREM 5. *Let $A$ be an $R$-module, and $x_1, \dots, x_n$ a regular sequence on $A$. Then $(x_1), (x_1, x_2), \dots, (x_1, \dots, x_n)$ form a properly ascending chain.*

Let $A$ be an $R$-module, and $I$ an ideal of $R$. Let $x_1, \dots, x_n$ be a regular sequence in $I$ on $A$. If $x_1, \dots, x_n, x$ is not a regular sequence on $A$ for each $x \in I$, then $x_1, \dots, x_n$ is called a maximal regular sequence in $I$ on $A$.

REMARK. Let $R$ be a Noetherian semigroup. Then two maximal regular sequences on $R$ need not have the same length.

For example, let $Z_0$ be the monoid of non-negative integers and let $R = Z_0 \oplus Z_0$. Set $p = (1, 0), q = (0, 1)$ and $x = (1, 1)$. Then $p, q$ is a maximal regular sequence on $R$. Also, $x$ is a maximal regular sequence on $R$.

Let $A$ be an $R$-module, and $I$ an ideal of $R$. Then the maximum of lengths of all regular sequences in $I$ on $A$ is called the grade of $I$ on $A$, and is denoted by $G(I, A)$.

THEOREM 6. *Let $A$ be an $R$-module, and $I$ an ideal of $R$ with $I + A \subsetneq A$. Let $x_1, \dots, x_n$ be a maximal regular sequence in $I$ on $A$. Then there exists a prime ideal $P$ such that $x_1, \dots, x_n$ is a maximal regular sequence in $P$ on $A$.*

PROOF. Set $P = Z(A/((x_1, \dots, x_n) + A))$. Then $P$ is a prime ideal containing $I$, and $x_1, \dots, x_n$ is a maximal regular sequence in $P$ on $A$.

LEMMA 7. *Let $R$ be a Noetherian semigroup. Then there exists only a finite number of prime ideals of $R$.*

PROOF. Let $x_1, \dots, x_n$ be the set of all irreducible elements of $R$ any two of which are not associated. Let $P$ be a prime ideal of $R$. Then $P$ is generated by a subset of $\{x_1, \dots, x_n\}$.

THEOREM 8. *Let $R$ be a Noetherian semigroup, $A$ a finitely generated torsion-free cancellative $R$-module, and $J$ a $k$-generated ideal of $R$ with $J + A \subsetneq A$. Then $G(J, A) \leq k$.*

PROOF. Let $J = (x_1, \dots, x_k)$. Suppose that there exists a regular sequence $y_1, \dots, y_{k+1}$ in $J$ on $A$. By Theorem 4, we may assume that $y_1 = x_1 + r_1$, $y_2 = x_1 + r_2$ for $r_1, r_2 \in R$. Choose $a \in A - (J + A)$. Then we have $r_1 + a \notin y_1 + A$ and $y_2 + (r_1 + a) \in y_1 + A$. Hence $y_2 \in Z(A/(y_1 + A))$; a contradiction.

Let $A$ be an $R$-module. If any two maximal regular sequences in $I$ on $A$ have the same length for every ideal $I$ with $I + A \subsetneq A$, then $A$ is said to satisfy property (*). If $A$ satisfies property (*), we say also that $(R, A)$ satisfies property (*).

THEOREM 9. *In Theorem 8, let $J = (x_1, \dots, x_k)$, and assume that $G(J, A) = k$ and $(R, A)$ satisfies property (*). Then $x_1, \dots, x_k$ is a maximal regular sequence in $J$ on $A$.*

PROOF. Assume that $x_{i_1} \notin Z(A), x_{i_2} \notin Z(A/((x_{i_1}) + A)), \dots, x_{i_h} \notin Z(A/((x_{i_1}, \dots, x_{i_{h-1}}) + A))$ for $1 < h < k$. Then we have $J \not\subset Z(A/((x_{i_1}, \dots, x_{i_h}) + A))$. Hence there exists $i_{h+1}$ such that $x_{i_{h+1}} \notin Z(A/((x_{i_1}, \dots, x_{i_h}) + A))$. Thus $x_{i_1}, \dots, x_{i_k}$ is a regular sequence on $A$. Then Theorem 4 completes the proof.

THEOREM 10. *Let $R$ be a Noetherian semigroup, and $A$ a finitely generated torsion-free cancellative $R$-module which satisfies property (*). Let $I$ be an ideal of $R$, and $x \in R$ with $J = (I, x) \subsetneq R$. Then $G(J, A) \leq 1 + G(I, A)$.*

PROOF. Let $x_1, \ldots, x_m$ be a maximal regular sequence in $I$ on $A$. If $J \subset Z(A/((x_1, \ldots, x_m) + A))$, we have $G(J, A) \leq 1 + G(I, A)$. Assume that $J \not\subset Z(A/((x_1, \ldots, x_m) + A))$. Then $x_1, \ldots, x_m, x$ is a regular sequence in $J$ on $A$. Suppose that $J \not\subset Z(A/((x_1, \ldots, x_m, x) + A))$. Then there exists $y \in I$ such that $x_1, \ldots, x_m$, $x, y$ is a regular sequence in $J$ on $A$. Then $x_1, \ldots, x_m, y$ is a regular sequence in $I$ on $A$ by Theorem 4; a contradiction.

THEOREM 11. *Let $R$ be a Noetherian semigroup, and $A$ a finitely generated torsion-free cancellative $R$-module which satisfies property (*). Let $I$ be an ideal of $R$ contained in a maximal ideal $M$. Assume that $G(I, A) < G(M, A)$. Then there exists a prime ideal $P$ of $R$ such that $G(P, A) = 1 + G(I, A)$.*

PROOF. Let $x_1, \ldots, x_k$ be a maximal regular sequence in $I$ on $A$, and set $I_0 = (x_1, \ldots, x_k)$. We may take $x \in M - Z(A/(I_0 + A))$. By Theorem 10, we have $G((I, x), A) = k + 1$. Then $P = Z(A/((I, x) + A))$ is a desired prime ideal.

THEOREM 12. *Let $R$ be a Noetherian semigroup, and $A$ a finitely generated torsion-free cancellative $R$-module with property (*). Let $I = (x_1, \ldots, x_n)$ be a proper ideal of $R$. Then $G(I, A) = n$ if and only if $x_1, \ldots, x_n$ is a regular sequence on $A$.*

PROOF. The necessity: Let $J = (x_1, \ldots, x_{n-1})$, and assume that the assertion holds for $x_1, \ldots, x_{n-1}$. We have $G(J, A) = n - 1$ by Theorem 10, and hence $x_1, \ldots, x_{n-1}$ is a regular sequence on $A$. Since $G(I, A) = n$, we have $I \not\subset Z(A/((x_1, \ldots, x_{n-1}) + A))$. It follows that $x_1, \ldots, x_n$ is a regular sequence on $A$.

Let $P$ be a prime ideal of $R$. Then the maximum number $n$ so that there exists a chain $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n = P$ of prime ideals of $R$ is called the height of $P$, and is denoted by $\mathrm{ht}(P)$. For an ideal $I$ of $R$, the minimum of $\mathrm{ht}(P)$, $P$ ranging over the prime ideals containing $I$ is called the height of $I$, and is denoted by $\mathrm{ht}(I)$.

THEOREM 13. *Let $I$ be an ideal of $R$, and $x_1, \ldots, x_n$ a regular sequence in $I$ on $R$. Then $n \leq ht(I)$.*

PROOF. Assume that the assertion holds for $n-1$. We may assume that $P = I$ is a prime ideal. Set $T = \{kx_n + s \mid k \geq 0, s \in R - P\}$. Suppose that $T \cap (x_1, \dots, x_{n-1}) \neq \emptyset$, say $kx_n + s \in (x_1, \dots, x_{n-1})$. By the choice of $s$ and $k_n$, we have $k \geq 1$ and $(k-1)x_n + s \in (x_1, \dots, x_{n-1})$. Thus $x_n + s \in (x_1, \dots, x_{n-1})$; a contradiction. Hence $T \cap (x_1, \dots, x_{n-1}) = \emptyset$. Then there exists a prime ideal $Q$ such that $Q \supset (x_1, \dots, x_{n-1})$ and $Q \cap T = \emptyset$. By the assumption, we have $ht(Q) \geq n-1$, and hence $ht(P) \geq n$.

Let $R$ be a Noetherian semigroup with maximal ideal $M$. If $G(M,R) = dim(R)$, then $R$ is called a Macaulay semigroup. Let $A$ be an $R$-module, and $S$ an additive system in $R$. If, for $a_1, a_2 \in A$ and $s_1, s_2 \in S$, we have $a_1 + s_2 + s = a_2 + s_1 + s$ for some $s \in S$, we define $a_1 - s_1 = a_2 - s_2$. Thus $A_S = \{a - s \mid a \in A, s \in S\}$ is an $R_S$-module. If $P$ is a prime ideal of $R$, then $A_{R-P}$ is denoted by $A_P$.

THEOREM 14. *Let $A$ be an $R$-module, and $x_1, \dots, x_n$ a regular sequence on $A$. Let $S$ be an additive system in $R$ such that $(x_1, \dots, x_n) + A_S \subsetneqq A_S$. Then $x_1, \dots, x_n$ is a regular sequence in $R_S$ on $A_S$.*

Theorem 14 implies the following,

THEOREM 15. *Let $P$ be a prime ideal of $R$, and $I$ an ideal contained in $P$. Then $G(I,P) \leq G(I_P, R_P)$.*

THEOREM 16. *Let $R$ be a Macaulay semigroup such that $(R,R)$ satisfies property (*). Then we have $G(I,R) = ht(I)$ for every ideal $I$ of $R$.*

PROOF. Suppose the contrary. Let $P$ be a maximal member in the set of all ideals $I$ with $G(I,R) < ht(I)$. Then $P$ is a prime ideal by Theorem 6. By Theorem 11, there exists a prime ideal $Q$ containing $P$ such that $G(Q,R) = 1 + G(P,R)$. Then $G(Q,R) < ht(P) + 1 < ht(Q) + 1$, and hence $G(Q,R) < ht(Q)$; a contradiction.

Let $P$ be a prime ideal of $R$. Then the minimum of $n+1$ such that there exists a saturated chain of prime ideals $P \supsetneqq P_1 \supsetneqq \cdots \supsetneqq P_n$ of $R$ is called the little height of $P$, and is denoted by $lh(P)$. If $R$ satisfies the following conditions (1) and (2), we say that $R$ satisfies the saturated chain condition:

(1) $lh(P) = ht(P)$ for every prime ideal $P$ of $R$.

(2) For all prime ideals $P$, $Q$ with $P \supsetneqq Q$, any two saturated chains from $P$ to $Q$ have the same length.

THEOREM 17. *Let $R$ be a Noetherian semigroup such that $(R_P, R_P)$ satisfies property (*) for every prime ideal $P$. Then we have $G(P, R) \leq lh(P)$ for every prime ideal $P$.*

PROOF. We may assume that $M = P$ is a maximal ideal of $R$. Let $lh(M) = m$, and assume that the assertion holds for $m-1$. There exists a prime ideal $Q$ with $lh(Q) = m-1$. Then $G(Q + R_Q, R_Q) \leq m-1$ by the assumption, and hence $G(Q, R) \leq m-1$. Then $G(M, R) = 1 + G(Q, R)$ by Theorem 11, and hence $G(M, R) \leq m$.

THEOREM 18. *Let $R$ be a Macaulay semigroup such that $(R, R)$ satisfies property (*). Then $R_S$ is a Macaulay semigroup for every additive system $S$ of $R$.*

PROOF. There exists a prime ideal $P$ of $R$ such that $N = P + R_S$ is a maximal ideal of $R_S$. Then we have $G(P, R) = ht(P)$ by Theorem 16. It follows that $ht(N) \leq G(N, R_S)$ by Theorem 15, and $G(N, R_S) \leq ht(N)$ by Theorem 13.

LEMMA 19 ([M3, Theorem 1]). *Let $R$ be a Noetherian semigroup, and $x$ a nonunit of $R$. If $P$ is a minimal prime ideal over $(x)$, then $ht(P) = 1$.*

Let $S$ be a $g$-monoid and $R$ a submonoid of $S$. If $x$ is an element of $S$, then the submonoid $R + Z_0 s$ of $S$ is denoted by $R[s]$. Let $X$ be an indeterminate over $R$. Then the $g$-monoid $R + Z_0 X$ is denoted by $R[X]$, and is called the polynomial semigroup of $X$ over $R$.

LEMMA 20 ([TM]). (1) *Assume that $R$ satisfies the ascending chain condition on radical ideals, and let $I$ be an ideal. Then there exists only a finite number of prime ideals minimal over $I$.*

(2) *Assume that $R$ satisfies the ascending chain condition on radical ideals. If $R$ has an infinite number of prime ideals of height* 1, *then their intersection is empty.*

(3) *Let $R$ be a g-monoid with $G = q(R)$, and let $u \in R$. Then every prime ideal of $R$ contains $u$ if and only if $G = R[-u]$.*

If $q(R)$ is generated by one element over R as a monoid, that is, if $q(R)$ is of the form $R[x]$ for some $x \in q(R)$, then $R$ is called a $G$-semigroup. Lemmas 7 and 20 imply the following,

THEOREM 21. *Any Noetherian semigroup is a G-semigroup.*

LEMMA 22 ([TM]). (1) *Let $P$ be a prime ideal of $R$ of height* 1. *Then $P+R[X]$ is a prime ideal of $R[X]$ of height* 1.

(2) *Let $P$ be a prime ideal of $R$ with $n=ht(P)$. Let $Q$ be a prime ideal of $R[X]$ properly containing $P+R[X]$ and with $P=Q\cap R$. Then $ht(P+R[X])=n$ and $ht(Q)=n+1$.*

QUESTION. If $R$ is Noetherian, what are conditions for $(R[X],R[X])$ to satisfy property (*)?

THEOREM 23. *Let $R$ be a Noetherian semigroup such that $(R[X],R[X])$ satisfies property* (*). *Then $R$ is a Macaulay semigroup if and only If $R[X]$ is a Macaulay semigroup.*

PROOF. Let $M$ be a maximal ideal of $R$ with $n=ht(M)$. The necessity: Let $x_1,\dots,x_n$ be a regular sequence in $M$ on $R$, Then $x_1,\dots,x_n,X$ is a regular sequence in $R[X]$ on $R[X]$. Hence $R[X]$ is a Macaulay semigroup by Theorem 13 and Lemma 22. The sufficiency: Let $X,f_1,\dots,f_n$ be a regular sequence in $R[X]$ on $R[X]$. Set $f_i=a_i+k_iX$ for each $i$ (for $a_i\in R$ and $k_i\geq 0$). Then we have $k_i=0$ for each $i$. It follows that $a_1,\dots,a_n$ is a regular sequence on $R$.

LEMMA 24 ([M3, Theorem 1]). *Let $R$ be a Noetherian semigroup, and $I$ an $n$-generated proper ideal. Let $P$ be a prime ideal minimal over $I$. Then $ht(P)\leq n$.*

Let $P$ be a prime ideal, and $I$ an ideal contained in $P$. The maximum of $n$ so that there exists a chain of prime ideals $P\supsetneqq P_1\supsetneqq\cdots\supsetneqq P_n\supset I$ is called the height of $P/I$, and is denoted by $ht(P/I)$.

THEOREM 25. *Let $R$ be a Noetherian semigroup, $P$ a prime ideal, and $I$ an $n$-generated ideal contained in $P$. Then $ht(P)\leq n+ht(P/I)$.*

PROOF. Let $I=(a_1,\dots,a_n)$ with $a_i\neq a_j$ (for $i\neq j$), and set $ht(P/I)=k$. For any prime ideal $Q$, the cardinality of $Q\cap\{a_1,\dots,a_n\}$ is called the capacity of $Q$, and is denoted by $c(Q)$. Let $ht(P)=l+1$, and let $P\supsetneqq P_1\supsetneqq\cdots\supsetneqq P_l$ be a chain of prime ideals. Then $\sum_1^l c(P_i)$ is called the capacity of the chain. We will show $l+1\leq k+n$ by the induction on the number $\sum_1^l c(P_i)$. Thus, if $\sum_1^l c(P_i)=nl$, then $l\leq k$. Hence $l+1\leq k+n$. Assume that $0=c(P_l)=\cdots=c(P_\alpha)<c(P_{\alpha-1})$. The case that $ht(P_\alpha)=1$: Since $ht(P_{\alpha-1})=2$, $P_{\alpha-1}$ is not a prime ideal minimal over $(a_1)$. Hence there exists a prime ideal $Q$ with $a_1\in Q$ such that $P_{l-1}\supsetneqq Q$.

Then the capacity of the chain $P \supsetneqq P_1 \supsetneqq \cdots \supsetneqq P_{l-1} \supsetneqq Q$ is larger than that of the chain $P \supsetneqq P_1 \supsetneqq \cdots \supsetneqq P_{l-1} \supsetneqq P_l$. By the induction hypothesis, we have $l+1 \leq k+n$. The case that $ht(P_\alpha) > 1$: If $P_{\alpha+1}$ is a prime ideal minimal over $(P_{\alpha-1}, a_1)$, then $P_\alpha$ is a prime ideal minimal over $P_{\alpha-1}$; a contradiction. Hence $P_{a+1}$ is not a prime ideal minimal over $(P_{\alpha-1}, a_1)$. Then there exists a prime ideal $Q$ such that $P_{\alpha+1} \supsetneqq Q \supsetneqq P_{\alpha-1}$. Then the capacity of the chain $P \supsetneqq P_1 \supsetneqq \cdots \supsetneqq P_{\alpha+1} \supsetneqq Q \supsetneqq P_{\alpha-1} \supsetneqq \cdots \supsetneqq P_l$ is larger than that of the chain $P \supsetneqq P_1 \supsetneqq \cdots \supsetneqq P_{\alpha+1} \supsetneqq P_\alpha \supsetneqq P_{\alpha-1} \supsetneqq \cdots \supsetneqq P_l$. By the induction hypothesis, we have $l+1 \geq k+n$.

THEOREM 26. *Let $R$ be a Noetherian semigroup, $P$ a prime ideal, and $x \in P$. Then $ht(P/(x)) = ht(P) - 1$.*

By Theorem 26, we have the following,

THEOREM 27. *Let $R$ be a Noetherian semigroup, $M$ a maximal ideal of $R$, and $x \in M$. Let $k = ht(M/(x))$, and $x_1, \ldots, x_k$ be elements of $M$ such that $x_1 \notin Z(R/(x))$, $x_2 \notin Z(R/(x, x_1)), \ldots, x_{k-1} \notin Z(R/(x, x_1, \ldots, x_{k-1}))$. Then $R$ is a Macaulay semigroup.*

LEMMA 28 ([TM]). *Let $R$ be a Noetherian semigroup with maximal ideal $M$, and $A$ a finitely generated $R$-module. Assume that $A = (a_1, \ldots, a_r, M + A)$. Then $M = (a_1, \ldots, a_r)$.*

Let $R$ be a Noetherian semigroup with maximal ideal $M$. If $M$ is generated by a finite subset $\{a_1, \ldots, a_n\}$ of $R$, and If $M$ is not generated by any proper subset of $\{a_1, \ldots, a_n\}$, then $\{a_1, \ldots, a_n\}$ is called a minimal generators of $M$. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$ be two set of minimal generators of $M$. Then each $a_i$ is contained in the ideal $(b_{m(i)})$ for some $m(i)$, and each $b_j$ is contained in some $(a_{n(i)})$. Then it follows that $i = n(m(i))$ and $j = m(n(j))$ for all $i$ and $j$. Hence $n = m$. The cardinality of a minimal generators of $M$ is called the V-dimension of $R$, and is denoted by $V(R)$.

THEOREM 29. *Let $R$ be a Noetherian semigroup with maximal ideal $M$, and let $x \in M - 2M$. Let $r$ be the minimum number so that there exist $x_1, \ldots, x_r$ with $(x, x_1, \ldots, x_r) = M$. Then $r = V(R) - 1$.*

For a Noetherian semigroup $R$, we have $V(R) \geq dim(R)$ by Lemma 24. A Noetherian semigroup $R$ is called a regular semigroup if $V(R) = dim(R)$.

THEOREM 30. *Let $R$ be a Noetherian semigroup with maximal ideal $M$. Assume that $M$ is generated by a regular sequence $a_1, \dots, a_k$ on $R$. Then $k = dim(R) = V(R)$, and $R$ is a regular semigroup.*

PROOF. We have $k \leq G(M, R) \leq ht(M)$ by Theorem 13. Also we have $ht(M) \leq V(R) \leq k$ by Lemma 24.

THEOREM 31. *Let $R$ be a regular semigroup with maximal ideal $M$, and $x \in M - 2M$. Put $ht(M/(x)) = k$. Then there exist $x_1, \dots, x_k$ such that $M = (x, x_1, \dots, x_k)$.*

PROOF. We have $dim(R) = k + 1$ by Theorem 26. By Theorem 29, there exist $x_1, \dots, x_k$ such that $M = (x, x_1, \dots, x_k)$.

THEOREM 32. *Let $R$ be a Noetherian semigroup with maximal ideal $M$, and let $x \in M - 2M$. Put $ht(M/(x)) = k$. Assume that there exist elements $x_1, \dots, x_k$ such that $M = (x, x_1, \dots, x_k)$. Then $R$ is a regular semigroup.*

PROOF. Because $ht(M) = k + 1$ by Theorem 26.

THEOREM 33. *Let $R$ be a regular semigroup of dimension $n$ with maximal ideal $M$. Let $M = (x_1, \dots, x_n)$. Then $x_i \notin 2M$ for each $i$, and $x_1, \dots, x_n$ is a regular sequence on $R$.*

PROOF. By Lemma 28, we have $x_i \notin 2M$. It follows that $x_1, \dots, x_n$ is a complete representatives of irreducible elements of $R$. Suppose that $x_k \in Z(R/(x_1, \dots, x_{k-1}))$. There exists $y \in M - (x_1, \dots, x_{k-1})$ such that $x_k + y \in (x_1, \dots, x_{k-1})$. Let $P$ be a prime ideal minimal over $(x_1, \dots, x_{k-1})$. Then there exists $l \geq k$ such that $x_l \in P$. There exist irreducible elements $a_1, \dots, a_{n-k}$ of $R$ such that $M = (P, a_1, \dots, a_{n-k})$. Then we have $ht(M) \leq ht(P) + n - k \leq k - 1 + n - k = n - 1$, namely $ht(M) \leq n - 1$; a contradiction.

Theorem 33 implies the following,

THEOREM 34. *Any regular semigroup is a Macaulay semigroup.*

THEOREM 35. *Let $R$ be a Noetherian semigroup such that $R_P$ is regular for every prime ideal $P$ of $R$. Then $R[X]_Q$ is regular for every prime ideal $Q$ of $R[X]$.*

PROOF. Set $R[X] = T$. We may assume that $R$ is a regular semigroup with maximal ideal $M$, and $R \cap N = M$ for a prime ideal $N$ of $T$. Then $M$ is generated by a regular sequence $a_1, \dots, a_n$. If $M + R[X] = N$, then our assertion holds. If $N \supsetneqq M + R[X]$, then $N = (a_1, \dots, a_n, X)$, and $a_1, \dots, a_n, X$ is a regular sequence on $R[X]$. Theorem 30 completes the proof.

THEOREM 36. *Let $R$ be a Macaulay semigroup such that $(R, R)$ satisfies property (*). Let $I$ be a proper ideal of height $n$, which can be generated by $n$ elements $x_1, \dots, x_n$. Then $P = Z(R/I)$ is a prime ideal of $R$, has height $n$ and a minimal prime over $I$.*

PROOF. We see that $P$ is a prime ideal of $R$ ([TM]). Theorem 16 implies that $G(I, R) = n$. Theorem 9 implies that $x_1, \dots, x_n$ is a maximal regular sequence in $I$ on $R$. Theorem 6 implies that $x_1, \dots, x_n$ is a maximal regular sequence in $P$ on $R$. Then Theorem 16 implies that $ht(P) = n$.

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