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REMARKS ON SPACES WITH SPECIAL TYPE OF k-NETWORKS

By

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Abstract: We negatively answer the following questions posed by Y. Ikeda and Y. Tanaka. (1) Does every closed image of a space X with a star-countable k-network have a star-countable k-network, or a point-countable k-network? (2) Is every space X with a locally countable k-network a σ-space, or a space in which every closed subset is a G_σ-set?

1. Introduction

All spaces we consider here are completely regular Hausdorff and all maps are continuous and onto. A collection of subsets of a space is said to be *star-countable* (resp. *point-countable*) if each element (resp. single point) meets only countably many members. Obviously a star-countable collection is point-countable. A collection $\mathscr{P}$ of subsets of a space X is called a *k-network* if whenever K is a compact subset of an open set U, there exists a finite subset $\mathscr{P}'$ of $\mathscr{P}$ such that $K \subset \bigcup \mathscr{P}' \subset U$. If we replace "compact" by "single point", then $\mathscr{P}$ is called a *network*. A space with a σ-locally finite network is called a *σ-space*.

Concerning spaces with special type of k-networks, Y. Ikeda and Y. Tanaka posed the following questions in [7], see also [10] and [12].

QUESTIONS. (1) Does every closed image of a space X with a star-countable k-network have a star-countable k-network, or a point-countable k-network?

(2) Is every space X with a locally countable k-network a σ-space, or a space in which every closed subset is a G_σ-set?

The question (1) has a positive answer under some conditions.

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THEOREM 1 [10]. Let $f: X \to Y$ be a closed map such that X has a point-countable k-network. If one of the following properties holds, then Y has a point-countable k-network.

(a) X is a k-space,

(b) Each point of X is a G_σ-set,

(c) X is a normal, isocompact space,

(d) For each $y \in Y$ $\partial f^{-1}(y)$ is Lindelöf, where $\partial f^{-1}(y)$ is the boundary of $f^{-1}(y)$.

The question (2) has a positive answer if X is a k-space, in fact a k-space with a locally countable k-network is the topological sum of $\aleph_0$-spaces, see [7] (cf. [9]).

In this paper we give counterexamples for the questions and slightly generalize the case (d) of Theorem 1.

2. Counterexamples

In this paper we endow ω_1 with the discrete topology. For a subset A of a discrete space D we put $A^* = Cl_{\beta D}A - A$, where βD is the Stone-Čech compactification of D.

For convenience, we call a space X a *CF-space* if every compact subset of X is finite. If X is a CF-space, then the collection $\{\{x\} : x \in X\}$ is obviously a start-countable k-network of X.

Recall that the one-point compactification of ω_1 does not have any point-countable k-network. In fact, a compact space with a point-countable k-network is metrizable, see Theorem 3.1 in [3].

Hence the following example shows that the first question has a negative answer.

EXAMPLE 1. There exists a closed map f from a CF-space X onto the one-point compactification of ω_1.

PROOF. A point z of a space Z is called a *weak P-point* if $z \notin \bar{E}$ for any countable $E \subset Z - \{z\}$. It is known that ω^* contains weak P-points [8]. Hence we can see that the set $P = \{p \in \omega_1^* : p \text{ is a weak P-point in } \omega_1^*\}$ is dense in ω_1^*.

We set $X = \omega_1 \cup P$, the subspace of $\beta\omega_1$. It is easy to check that X is a CF-space, because a compact space in which every point is a weak P-point is finite, and every convergent sequence of $\beta\omega_1$ is finite. Let Y be the space obtained by collapsing the closed set P to one point, and let f the canonical map from X onto Y. Then f is a closed map, and since the closure of an infinite subset of ω_1 intersects with P, Y is the one-point compactification of ω_1. $\square$

A collection $\mathscr{P}$ of subsets of a space X is called a *cs-network* if whenever σ is a sequence converging to a point x such that $\sigma \cup \{x\} \subset U$ with U open in X, then there exists a $P \in \mathscr{P}$ such that $x \in P \subset U$ and σ is eventually in P.

It is known that every space is the perfect image of an extremally disconnected space, see [13], where a space is *extremally disconnected* if the closure of an open set is open. Since every convergent sequence of an extremally disconnected space is finite, every space is the perfect image of a space with a point-countable cs-network. So, it is natural to ask whether every space is the closed image of a space with a point-countable k-network.

The author does not know if it is true. But, at least, the following holds.

PROPOSITION. Every space is the quotient image of a CF-space.

PROOF. Let Z be a space. As noted above, Z is the perfect image of an extremally disconnected space Y. For each point $y \in Y$, let Y_y be the space obtained by isolating all points of Y but y. Then Y is canonically the quotient image of the topological sum $X = \bigoplus\{Y_y : y \in Y\}$. If K is an infinite compact subset of Y_y, then it contains a non-trivial convergent sequence to y. Hence Y must have a non-trivial convergent sequence. This is a contradiction. Thus X is a CF-space, and Z is the quotient image of X. $\square$

If X is a locally countable CF-space, then the collection $\{\{x\} : x \in X\}$ is obviously a locally countable k-network of X.

A space is *countably metacompact* if every countable open cover has a point-finite open refinement. It is not difficult to check that a space X is countable metacompact iff whenever $\{C_n\}$ is a decreasing sequence of closed sets of X with empty intersection, there exist open sets $U_n \supset C_n$ with $\bigcap\{U_n : n \in \omega\} = \emptyset$.

Recall the diagram below:

σ-space $\rightarrow$ perfect (every closed set is a G_σ-set) $\rightarrow$ countably metacompact

Hence the following example shows that the second question is also negative.

Example 2. There exists a locally countable CF-space X which is not countably metacompact.

Proof. Let D be a set of cardinality 2^{ω}. Let $\{P_{\alpha} : \alpha < 2^{\omega}\}$ be an almost disjoint family of countable infinite subsets of D such that for every uncountable $P \subset D$ there exists some α with $P_{\alpha} \subset P$. Such a family exists, for example see [1, Example 4.2]. For each α, let $\{P_{\alpha n} : n \in \omega\}$ be a disjoint family of infinite subsets of P_{α}. We set $\mathscr{P} = \{P_{\alpha n} : \alpha < 2^{\omega}, n \in \omega\}$. We endow D with the discrete topology. For each α, n, pick a point $p_{\alpha n} \in P^{*}_{\alpha n}$.

We set $X = D \cup \{p_{\alpha n} : \alpha < 2^{\omega}, n \in \omega\}$, the subspace of βD. Since $\mathscr{P}$ is almost disjoint, $X - D$ is a closed discrete subset of X. X is obviously a locally countable CF-space.

We see that X is not countably metacompact. For each $n \in \omega$, let $C_n = \{p_{\alpha k} : \alpha < 2^{\omega}, k \geq n\}$. Each C_n is closed in X and $\bigcap\{C_n : n \in \omega\} = \emptyset$. Assume that there exist open sets $U_n \supset C_n$ with $\bigcap\{U_n : n \in \omega\} = \emptyset$. Since D is uncountable, there exists $n \in \omega$ such that $D - U_n$ is uncountable. Then there exists some α with $P_{\alpha} \subset D - U_n$. Hence the closure of $D - U_n$ must contain $p_{\alpha n} \in C_n$. This is a contradiction. Thus X is not countably metacompact. □

3. A generalization

In this section we slightly generalize the case (d) in Theorem 1.

A subset S of a space X is *z-embedded* in X if every zero-set of S is the restriction to S of some zero-set of X. A map $f : X \to Y$ is *compact-covering* if every compact subset of Y is the image of a compact subset of X. For realcompact spaces, see [5].

Lemma 1. Let $f : X \to Y$ be a closed map. Then (1) and (2) below hold.

(1) If Y is realcompact and for each $y \in Y f^{-1}(y)$ is realcompact, z-embedded in X, then X is realcompact. [2, Theorem 3.9]

(2) If X is realcompact, then f is compact-covering. [4, Theorem 3.4]

Corollary. Let $f : X \to Y$ be a closed map. If for each $y \in Y$ $\partial f^{-1}(y)$ is realcompact, z-embedded in X, then f is compact-covering.

Proof. Let K be a compact subset of Y. For each $y \in K$, choose any $x_y \in f^{-1}(y)$. We set:

$$A_y = \begin{cases} \partial f^{-1}(y) & \text{if } \partial f^{-1}(y) \neq \emptyset \\ \{x_y\} & \text{if } \partial f^{-1}(y) = \emptyset \end{cases}$$

Then the set $A = \cup\{A_y : y \in K\}$ is closed in X, hence the restricted map $g = f|A : A \to K$ is a closed map. By lemma 1 (1), A is realcompact. By Lemma 1 (2), g is compact-covering. So there exists a compact set $K' \subset A$ with $f(K') = K$. □

Let $\mathscr{P}$ be a collection of subsets of a space $X, \mathscr{P}$ is called a *wcs*-network* of X if whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is an open set of X with $\{x\} \cup \{x_n\} \subset U$, there exists a $P \in \mathscr{P}$ such that $P \subset U$ and P contains some subsequence of $\{x_n\}$.

LEMMA 2 [11, Proposition 1.2.(1)]. Let $\mathscr{P}$ be a point-countable cover of X. Then $\mathscr{P}$ is a k-network of X iff $\mathscr{P}$ is a wcs*-network of X and each compact subset of X is sequentially compact.

A Lindelöf space is realcompact [5, 8.2], and every Lindelöf subspace of a space X is z-embedded in X [6, 5.3]. Hence the following theorem generalizes the case (d) of Theorem 1.

THEOREM 2. Let $f : X \to Y$ be a closed map such that for each $y \in Y$ $\partial f^{-1}(y)$ is realcompact, z-embedded in X. If X has a point-countable k-network, then so does Y.

PROOF. The idea of the proof is due to [10].

Let K be a compact subset of Y. By the corollary above, there exists a compact set K' of X with $f(K') = K$. As noted in the second section, a compact space with a point-countable k-network is metrizable, so K' is metrizable. Therefore K is metrizable, in particular sequentially compact.

By Lemma 2 we have only to construct a point-countable wcs*-network of Y.

Let $\mathscr{P}$ be a point-countable k-network of X. For each $y \in Y$ choose any $x_y \in f^{-1}(y)$. We set $A = \{x_y : y \in Y\}$ and $\mathscr{P}' = \{f(P \cap A) : P \in \mathscr{P}\}$. Obviously $\mathscr{P}'$ is point-countable. We see that $\mathscr{P}'$ is a wcs*-network of Y. Let $\{y_n : n \in \omega\}$ be a sequence converging to a point $y \in Y$, and U be an open set of Y with $K \subset U$, where $K = \{y\} \cup \{y_n : n \in \omega\}$. Since the set $J = \partial f^{-1}(y) \cup \{x_n : n \in \omega\}$, where $x_n = x_{y_n}$, is closed in X, the restricted map $g = f|J : J \to K$ is a closed map. By Lemma 1 (1), J is realcompact. By Lemma 1 (2), g is compact-covering. Hence there exists a compact set $J' \subset J$ such that $g(J') = K$. Note that $\{x_n : n \in \omega\} \subset J' \subset f^{-1}(U)$. Since $\mathscr{P}$ is a k-network of X, there exists a $P \in \mathscr{P}$ such that

$P \subset f^{-1}(U)$ and $P \cap \{x_n : n \in \omega\}$ is infinite. The set $f(P \cap A)$ is a desired one. Thus $\mathscr{P}'$ is a wcs*-network of Y. $\square$

Lemma 1 (2) and the same idea as the proof of Theorem 2 lead to the following theorem.

THEOREM 3. Let $f : X \to Y$ be a closed map such that X is realcompact. If X has a point-countable k-network, then so does Y.

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