

ORBIT TYPES OF THE COMPACT LIE GROUP E_7 IN THE COMPLEX FREUDENTHAL VECTOR SPACE \mathfrak{B}^C

By

Takashi MIYASAKA and Ichiro YOKOTA

1. Introduction

Let \mathfrak{J} be the exceptional Jordan algebra over R and \mathfrak{J}^C its complexification. Then the simply connected compact exceptional Lie group F_4 acts on \mathfrak{J} and F_4 has three orbit types which are

$$F_4/F_4, \quad F_4/Spin(9), \quad F_4/Spin(8).$$

Similarly the simply connected compact exceptional Lie group E_6 acts on \mathfrak{J}^C and E_6 has five orbit types which are

$$E_6/E_6, \quad E_6/F_4, \quad E_6/Spin(10), \quad E_6/Spin(9), \quad E_6/Spin(8)$$

([5]). In this paper, we determine the orbit types of the simply connected compact exceptional Lie group E_7 in the complex Freudenthal vector space \mathfrak{B}^C . As a result, E_7 has seven orbit types which are

$$E_7/E_7, \quad E_7/E_6, \quad E_7/F_4, \quad E_7/Spin(11), \quad E_7/Spin(10), \\ E_7/Spin(9), \quad E_7/Spin(8).$$

2. Preliminaries

Let \mathfrak{C} be the division Cayley algebra and $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X = X\}$ the exceptional Jordan algebra with the Jordan multiplication $X \circ Y$, the inner product (X, Y) and the Freudenthal multiplication $X \times Y$. Let \mathfrak{J}^C be the complexification of \mathfrak{J} with the Hermitian inner product $\langle X, Y \rangle$. (The definitions of $X \circ Y$, (X, Y) , $X \times Y$ and $\langle X, Y \rangle$ are found in [2]). Moreover, let $\mathfrak{B}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$ be the Freudenthal C -vector space with the Hermitian inner product $\langle P, Q \rangle$. For $P, Q \in \mathfrak{B}^C$, we can define a C -linear mapping $P \times Q$ of \mathfrak{B}^C .

(The definitions of $\langle P, Q \rangle$ and $P \times Q$ are found in [2]). The complex conjugation in the complexified spaces \mathfrak{C}^C , \mathfrak{J}^C or \mathfrak{B}^C is denoted by τ . Now, the simply connected compact exceptional Lie groups F_4, E_6 and E_7 are defined by

$$\begin{aligned} F_4 &= \{\alpha \in \text{Iso}_R(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}, \\ E_6 &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \tau\alpha\tau(X \times Y) = \alpha X \times \alpha Y, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}, \\ E_7 &= \{\alpha \in \text{Iso}_C(\mathfrak{B}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{B}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \alpha(\tau\lambda) = (\tau\lambda)\alpha\} \end{aligned}$$

(where λ is the C -linear transformation of \mathfrak{B}^C defined by $\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi)$), respectively. Then we have a natural inclusion $F_4 \subset E_6 \subset E_7$, that is,

$$\begin{aligned} E_6 &= \{\alpha \in E_7 \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\} \subset E_7, \\ F_4 &= \{\alpha \in E_6 \mid \alpha E = E\} \subset E_6 \subset E_7, \end{aligned}$$

where E is the 3×3 unit matrix. The groups F_4, E_6 and E_7 have the following subgroups

$$\begin{aligned} Spin(8) &= \{\alpha \in F_4 \mid \alpha E_k = E_k, k = 1, 2, 3\} \subset F_4 \subset E_6 \subset E_7, \\ Spin(9) &= \{\alpha \in F_4 \mid \alpha E_1 = E_1\} \subset F_4 \subset E_6 \subset E_7, \\ Spin(10) &= \{\alpha \in E_6 \mid \alpha E_1 = E_1\} \subset E_6 \subset E_7, \\ Spin(11) &= \{\alpha \in E_7 \mid \alpha(E_1, 0, 1, 0) = (E_1, 0, 1, 0)\} \subset E_7, \end{aligned}$$

where E_k is the usual notation in \mathfrak{J}^C , e.g. $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ([2]).

3. Orbit Types of F_4 in \mathfrak{J} and E_6 in \mathfrak{J}^C

We shall review the results of orbit types of F_4 in \mathfrak{J} and E_6 in \mathfrak{J}^C .

LEMMA 1 ([1]). *Any element $X \in \mathfrak{J}$ can be transformed to a diagonal form by some $\alpha \in F_4$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in \mathbf{R}, \quad (\text{which is briefly written by } (\xi_1, \xi_2, \xi_3)).$$

The order of ξ_1, ξ_2, ξ_3 can be arbitrarily exchanged under the action of F_4 .

THEOREM 2 ([5]). *The orbit types of the group F_4 in \mathfrak{J} are as follows.*

- (1) *The orbit through (ξ, ξ, ξ) is F_4/F_4 .*
- (2) *The orbit through (ξ_1, ξ, ξ) (where $\xi_1 \neq \xi$) is $F_4/Spin(9)$.*
- (3) *The orbit through (ξ_1, ξ_2, ξ_3) (where ξ_1, ξ_2, ξ_3 are distinct) is $F_4/Spin(8)$.*

LEMMA 3 ([3]). *Any element $X \in \mathfrak{J}^C$ can be transformed to a diagonal form by some $\alpha \in E_6$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in C, \quad (\text{which is briefly written by } (\xi_1, \xi_2, \xi_3)).$$

The order of ξ_1, ξ_2, ξ_3 can be arbitrarily exchanged under the action of E_6 .

THEOREM 4 ([5]). *The orbit types of the group E_6 in \mathfrak{J}^C are as follows.*

- (1) *The orbit through $(0,0,0)$ is E_6/E_6 .*
- (2) *The orbit through (ξ_1, ξ_2, ξ_3) (where $|\xi_1| = |\xi_2| = |\xi_3| \neq 0$) is E_6/F_4 .*
- (3) *The orbit through $(\xi, 0, 0)$ (where $\xi \neq 0$) is $E_6/Spin(10)$.*
- (4) *The orbit through (ξ_1, ξ_2, ξ_3) (where $|\xi_1| \neq |\xi_2| = |\xi_3| \neq 0$) is $E_6/Spin(9)$.*
- (5) *The orbit through (ξ_1, ξ_2, ξ_3) (where $|\xi_1|, |\xi_2|, |\xi_3|$ are distinct) is $E_6/Spin(8)$.*

4. Orbit Types of E_7 in \mathfrak{P}^C

LEMMA 5 ([2]). *Any element $P \in \mathfrak{P}^C$ can be transformed to the following diagonal form by some $\alpha \in E_7$:*

$$\alpha P = \left(\begin{pmatrix} ar_1 & 0 & 0 \\ 0 & ar_2 & 0 \\ 0 & 0 & ar_3 \end{pmatrix}, \begin{pmatrix} br_1 & 0 & 0 \\ 0 & br_2 & 0 \\ 0 & 0 & br_3 \end{pmatrix}, ar, br \right), \quad \begin{array}{l} r_k, r \in \mathbf{R}, 0 \leq r_k, 0 \leq r, \\ a, b \in C, |a|^2 + |b|^2 = 1. \end{array}$$

Moreover, any element $P \in \mathfrak{P}^C$ can be transformed to the following diagonal form by some $\varphi(A)\alpha \in \varphi(SU(2))E_7$:

$$\varphi(A)\alpha P = \left(\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r, 0 \right), \quad r_k, r \in \mathbf{R}, 0 \leq r_k, 0 \leq r,$$

(which is briefly written by $(r_1, r_2, r_3; r)$), where $\varphi(A) \in \varphi(SU(2)) \subset E_8$ and commutes with any element $\alpha \in E_7$. The order of r_1, r_2, r_3, r can be arbitrarily exchanged under the action of E_7 . (As for the definitions of the groups E_8 and $\varphi(SU(2))$, see

[2]). The action of $\varphi(A)$, $A \in SU(2)$, on \mathfrak{B}^C is given by

$$\begin{aligned}\varphi(A)P &= \varphi\left(\begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix}\right)(X, Y, \xi, \eta) \\ &= (aX + \tau(bY), aY - \tau(bX), a\xi + \tau(b\eta), a\eta - \tau(b\xi)).\end{aligned}$$

THEOREM 6. The group E_7 has the following seven orbit types in \mathfrak{B}^C :

$$\begin{aligned}E_7/E_7, \quad E_7/E_6, \quad E_7/F_4, \quad E_7/Spin(11), \quad E_7/Spin(10), \\ E_7/Spin(9), \quad E_7/Spin(8).\end{aligned}$$

More details,

- (1) The orbit through $(0, 0, 0; 0)$ is E_7/E_7 .
- (2) The orbit through $(0, 0, 0; 1)$ or $(1, 1, 1; 1)$ is E_7/E_6 .
- (3) The orbit through $(1, 1, 1; 0)$ or $(1, 1, 1; r)$ (where $0 < r, 1 \neq r$) is E_7/F_4 .
- (4) The orbit through $(1, 0, 0; 1)$ or $(1, r, r; 1)$ (where $0 < r, 1 \neq r$) is $E_7/Spin(11)$.
- (5) The orbit through $(1, 0, 0; r)$ (where $0 < r, 1 \neq r$) is $E_7/Spin(10)$.
- (6) The orbit through $(1, 1, r; 0)$ or $(1, 1, r; s)$ (where $0 < r, 0 < s$ and $1, r, s$ are distinct) is $E_7/Spin(9)$.
- (7) The orbit through $(1, r, s; 0)$ or $(1, r, s; t)$ (where r, s, t are positive and $1, r, s, t$ are distinct) is $E_7/Spin(8)$.

PROOF. From Lemma 5, the representatives of orbit types (up to a constant) can be given by the following.

$$\begin{aligned}(0, 0, 0; 0), \quad (0, 0, 0; 1), \quad (0, 0, 1; 1), \quad (0, 0, 1; r), \\ (0, 1, 1; 1), \quad (0, 1, 1; r), \quad (0, 1, r; s), \quad (1, 1, 1; 1), \\ (1, 1, 1; r), \quad (1, 1, r; r), \quad (1, 1, r; s), \quad (1, r, s; t)\end{aligned}$$

where r, s, t are positive, $0, 1, r, s, t$ are distinct and the order of $0, 1, r, s, t$ can be arbitrarily exchanged.

(1) The isotropy subgroup $(E_7)_{(0,0,0;0)}$ is obviously E_7 . Therefore the orbit through $(0, 0, 0; 0)$ is E_7/E_7 .

(2) The isotropy subgroup $(E_7)_{(0,0,0;1)}$ is E_6 . Therefore the orbit through $(0, 0, 0; 1)$ is E_7/E_6 .

(2') The isotropy subgroup $(E_7)_{(1,1,1;1)}$ is conjugate to E_6 in E_7 . In fact, we know that the following realization of the homogeneous space $E_7/E_6 : E_7/E_6 = \{P \in \mathfrak{B}^C \mid P \times P = 0, \langle P, P \rangle = 1\} = \mathfrak{M}$ ([4]). Since $1/2\sqrt{2}(E, E, 1, 1)$ and $(0, 0, 1, 0) \in \mathfrak{M}$, there exists $\delta \in E_7$ such that

$$\delta\left(\frac{1}{2\sqrt{2}}(E, E, 1, 1)\right) = (0, 0, 1, 0).$$

Hence the isotropy subgroup $(E_7)_{(E, E, 1, 1)}$ is conjugate to the isotropy subgroup $(E_7)_{(0, 0, 1, 0)}$ is $E_7 : (E_7)_{(E, E, 1, 1)} \sim (E_7)_{(0, 0, 1, 0)}$. On the other hand, since

$$\varphi\left(\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right)(E, 0, 1, 0) = \frac{1}{\sqrt{2}}(E, E, 1, 1),$$

we have $(E_7)_{(E, 0, 1, 0)} = (E_7)_{(E, E, 1, 1)} \sim (E_7)_{(0, 0, 1, 0)} = E_6$. Therefore the orbit through $(1, 1, 1; 1)$ is E_7/E_6 .

(3) The isotropy subgroup $(E_7)_{(1, 1, 1; 0)}$ is F_4 . In fact, for $\alpha \in E_7$ and $P \in \mathfrak{P}^C$, we have $\alpha(\tau\lambda((P \times P)P)) = \tau\lambda(\alpha((P \times P)P)) = \tau\lambda(\alpha(P \times P)\alpha^{-1}\alpha P) = \tau\lambda((\alpha P \times \alpha P)\alpha P)$. Now, let $P = (1, 1, 1; 0)$. Since $\tau\lambda((P \times P)P) = 3/2(0, 0, 0; 1)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$. Hence $\alpha \in E_6$, so together with $\alpha E = E$, we have $\alpha \in F_4$. Therefore the orbit through $(1, 1, 1; 0)$ is E_7/F_4 .

(3') The isotropy subgroup $(E_7)_{(1, 1, 1; r)}$ is F_4 . In fact, let $P = (1, 1, 1; r)$. Since $\tau\lambda((P \times P)P) = 3/2(r, r, r; 1)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r, r, r; 1) = (r, r, r; 1) \cdots$ (ii). Take (i)–(ii), then we have $\alpha(1 - r, 1 - r, 1 - r; r - 1) = (1 - r, 1 - r, 1 - r; r - 1)$. Since $1 - r \neq 0$, we have $\alpha(1, 1, 1; -1) = (1, 1, 1; -1)$. Together with $\alpha P = P$, we have $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$ and $\alpha(1, 1, 1; 0) = (1, 1, 1; 0)$. Hence $\alpha \in E_6$ and hence $\alpha \in F_4$. Therefore the orbit through $(1, 1, 1; r)$ is E_7/F_4 .

(4) The isotropy subgroup $(E_7)_{(1, 0, 0; 1)}$ is $Spin(11)$. Therefore the orbit through $(1, 0, 0; 1)$ is $E_7/Spin(11)$.

(4') The isotropy subgroup $(E_7)_{(1, r, r; 1)}$ is $Spin(11)$. In fact, let $P = (1, r, r; 1)$. Since $\tau\lambda((P \times P)P) = 3/2(r^2, r, r; r^2)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r^2, r, r; r^2) = (r^2, r, r; r^2) \cdots$ (ii). Take (i)–(ii), then we have $\alpha(1 - r^2, 0, 0; 1 - r^2) = (1 - r^2, 0, 0; 1 - r^2)$. Since $1 - r^2 \neq 0$, we have $\alpha(1, 0, 0; 1) = (1, 0, 0; 1)$. Hence $\alpha \in Spin(11)$. Therefore the orbit through $(1, r, r; 1)$ is $E_7/Spin(11)$.

(5) The isotropy subgroup $(E_7)_{(1, 0, 0; r)}$ is $Spin(10)$. In fact, for $\alpha \in E_7$ and $P \in \mathfrak{P}^C$, we have $\alpha((P \times P)\tau\lambda P) = (\alpha(P \times P)\alpha^{-1})\alpha(\tau\lambda P) = (\alpha P \times \alpha P)\tau\lambda(\alpha P)$. Now, let $P = (1, 0, 0; r)$. Since $(P \times P)\tau\lambda P = -1/2(r^2, 0, 0; r)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r^2, 0, 0; r) = (r^2, 0, 0; r) \cdots$ (ii). Take (i)–(ii), then we have $\alpha(1 - r^2, 0, 0; 0) = (1 - r^2, 0, 0; 0)$. Since $1 - r^2 \neq 0$, we have $\alpha(1, 0, 0; 0) = (1, 0, 0; 0) \cdots$ (iii). Take (i)–(iii), then $\alpha(0, 0, 0; r) = (0, 0, 0; r)$, that is, $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$. Hence $\alpha \in E_6$ and $\alpha E_1 = E_1$. Thus $\alpha \in Spin(10)$. Therefore the orbit through $(1, 0, 0; r)$ is $E_7/Spin(10)$.

(6) The isotropy subgroup $(E_7)_{(1,1,r,0)}$ is $Spin(9)$. In fact, let $P = (1, 1, r; 0)$. Since $\tau\lambda((P \times P)P) = 3/2(0, 0, 0; r)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$. Hence $\alpha \in E_6$, so together with $\alpha P = P$, we have $\alpha \in Spin(9)$ (Theorem 4.(4)). Therefore the orbit through $(1, 1, r; 0)$ is $E_7/Spin(9)$.

(6') The isotropy subgroup $(E_7)_{(1,1,r;s)}$ is $Spin(9)$. In fact, let $P = (1, 1, r; s)$. Since $\tau\lambda((P \times P)P) = 3/2(rs, rs, s; r)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(rs, rs, s; r) = (rs, rs, s; r) \cdots$ (ii). Take (i) $\times r -$ (ii) $\times s$, then we have $\alpha(r(1-s^2), r(1-s^2), r^2-s^2; 0) = (r(1-s^2), r(1-s^2), r^2-s^2; 0)$. Since $r(1-s^2)$, r^2-s^2 are non-zero and $r(1-s^2) \neq r^2-s^2$, from (6) we have $\alpha \in Spin(9)$. Therefore the orbit through $(1, 1, r; s)$ is $E_7/Spin(9)$.

(7) The isotropy subgroup $(E_7)_{(1,r,s;0)}$ is $Spin(8)$. In fact, let $P = (1, r, s; 0)$. Since $\tau\lambda((P \times P)P) = 3/2(0, 0, 0; rs)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$. Hence $\alpha \in E_6$, so together with $\alpha P = P$, we have $\alpha \in Spin(8)$ (Theorem 4.(5)). Therefore the orbit through $(1, r, s; 0)$ is $E_7/Spin(8)$.

(7') The isotropy subgroup $(E_7)_{(1,r,s;t)}$ is $Spin(8)$. In fact, let $P = (1, r, s; t)$. Since $\tau\lambda((P \times P)P) = 3/2(rst, st, rt; rs)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(rst, st, rt; rs) = (rst, st, rt; rs) \cdots$ (ii). Take (i) $\times rs -$ (ii) $\times t$, then we have $\alpha(rs(1-t^2), s(r^2-t^2), r(s^2-t^2); 0) = (rs(1-t^2), s(r^2-t^2), r(s^2-t^2); 0)$. Since $rs(1-t^2)$, $s(r^2-t^2)$ and $r(s^2-t^2)$ are non-zero and distinct, from (7) we have $\alpha \in Spin(8)$. Therefore the orbit through $(1, r, s; t)$ is $E_7/Spin(8)$.

References

- [1] H. Freudenthal, Oktaven, Ausnahmegruppen und Oktavengeometrie, Math. Inst. Rijksuniv. te Utrecht, 1951.
- [2] T. Miyasaka, O. Yasukura and I. Yokota, Diagonalization of an element P of \mathfrak{B}^C by the compact Lie group E_7 , Tsukuba J. Math., to appear.
- [3] I. Yokota, Simply connected compact simple Lie group $E_{6(-78)}$ of type E_6 and its involutive automorphisms, J. Math., Kyoto Univ., **20** (1980), 447-473.
- [4] I. Yokota, Realization of involutive automorphisms σ of exceptional Lie groups G , part II, $G = E_7$, Tsukuba J. Math., **14** (1990), 379-404.
- [5] I. Yokota, Orbit types of the compact Lie group E_6 in the complex exceptional Jordan algebra \mathfrak{J}^C , Inter. Symp. on nonassociative algebras and related topics, Hiroshima, Japan, World Scientific, 1990, 353-359.

Takashi Miyasaka
 Misuzugaoka High School
 Matsumoto 390-8602, Japan

Ichiro Yokota
 339-5, Okada-Matsuoka
 Matsumoto 390-0312, Japan