

# ORBIT TYPES OF THE COMPACT LIE GROUP $E_7$ IN THE COMPLEX FREUDENTHAL VECTOR SPACE $\mathfrak{P}^C$

By

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## 1. Introduction

Let  $\mathfrak{J}$  be the exceptional Jordan algebra over  $R$  and  $\mathfrak{J}^C$  its complexification. Then the simply connected compact exceptional Lie group  $F_4$  acts on  $\mathfrak{J}$  and  $F_4$  has three orbit types which are

$$F_4/F_4, \quad F_4/Spin(9), \quad F_4/Spin(8).$$

Similarly the simply connected compact exceptional Lie group  $E_6$  acts on  $\mathfrak{J}^C$  and  $E_6$  has five orbit types which are

$$E_6/E_6, \quad E_6/F_4, \quad E_6/Spin(10), \quad E_6/Spin(9), \quad E_6/Spin(8)$$

([5]). In this paper, we determine the orbit types of the simply connected compact exceptional Lie group  $E_7$  in the complex Freudenthal vector space  $\mathfrak{P}^C$ . As a result,  $E_7$  has seven orbit types which are

$$E_7/E_7, \quad E_7/E_6, \quad E_7/F_4, \quad E_7/Spin(11), \quad E_7/Spin(10), \\ E_7/Spin(9), \quad E_7/Spin(8).$$

## 2. Preliminaries

Let  $\mathfrak{C}$  be the division Cayley algebra and  $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X = X^{\circ}\}$  the exceptional Jordan algebra with the Jordan multiplication  $X \circ Y$ , the inner product  $(X, Y)$  and the Freudenthal multiplication  $X \times Y$ . Let  $\mathfrak{J}^C$  be the complexification of  $\mathfrak{J}$  with the Hermitian inner product  $\langle X, Y \rangle$ . (The definitions of  $X \circ Y$ ,  $(X, Y)$ ,  $X \times Y$  and  $\langle X, Y \rangle$  are found in [2]). Moreover, let  $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$  be the Freudenthal  $C$ -vector space with the Hermitian inner product  $\langle P, Q \rangle$ . For  $P, Q \in \mathfrak{P}^C$ , we can define a  $C$ -linear mapping  $P \times Q$  of  $\mathfrak{P}^C$ .

(The definitions of  $\langle P, Q \rangle$  and  $P \times Q$  are found in [2]). The complex conjugation in the complexified spaces  $\mathfrak{C}^C$ ,  $\mathfrak{J}^C$  or  $\mathfrak{P}^C$  is denoted by  $\tau$ . Now, the simply connected compact exceptional Lie groups  $F_4, E_6$  and  $E_7$  are defined by

$$\begin{aligned} F_4 &= \{\alpha \in \text{Iso}_R(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}, \\ E_6 &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \tau\alpha\tau(X \times Y) = \alpha X \times \alpha Y, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}, \\ E_7 &= \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \alpha(\tau\lambda) = (\tau\lambda)\alpha\} \end{aligned}$$

(where  $\lambda$  is the  $C$ -linear transformation of  $\mathfrak{P}^C$  defined by  $\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi)$ ), respectively. Then we have a natural inclusion  $F_4 \subset E_6 \subset E_7$ , that is,

$$\begin{aligned} E_6 &= \{\alpha \in E_7 \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\} \subset E_7, \\ F_4 &= \{\alpha \in E_6 \mid \alpha E = E\} \subset E_6 \subset E_7, \end{aligned}$$

where  $E$  is the  $3 \times 3$  unit matrix. The groups  $F_4, E_6$  and  $E_7$  have the following subgroups

$$\begin{aligned} Spin(8) &= \{\alpha \in F_4 \mid \alpha E_k = E_k, k = 1, 2, 3\} \subset F_4 \subset E_6 \subset E_7, \\ Spin(9) &= \{\alpha \in F_4 \mid \alpha E_1 = E_1\} \subset F_4 \subset E_6 \subset E_7, \\ Spin(10) &= \{\alpha \in E_6 \mid \alpha E_1 = E_1\} \subset E_6 \subset E_7, \\ Spin(11) &= \{\alpha \in E_7 \mid \alpha(E_1, 0, 1, 0) = (E_1, 0, 1, 0)\} \subset E_7, \end{aligned}$$

where  $E_k$  is the usual notation in  $\mathfrak{J}^C$ , e.g.  $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  ([2]).

### 3. Orbit Types of $F_4$ in $\mathfrak{J}$ and $E_6$ in $\mathfrak{J}^C$

We shall review the results of orbit types of  $F_4$  in  $\mathfrak{J}$  and  $E_6$  in  $\mathfrak{J}^C$ .

LEMMA 1 ([1]). *Any element  $X \in \mathfrak{J}$  can be transformed to a diagonal form by some  $\alpha \in F_4$ :*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in \mathbf{R}, \quad (\text{which is briefly written by } (\xi_1, \xi_2, \xi_3)).$$

The order of  $\xi_1, \xi_2, \xi_3$  can be arbitrarily exchanged under the action of  $F_4$ .

THEOREM 2 ([5]). *The orbit types of the group  $F_4$  in  $\mathfrak{J}$  are as follows.*

- (1) *The orbit through  $(\xi, \xi, \xi)$  is  $F_4/F_4$ .*
- (2) *The orbit through  $(\xi_1, \xi, \xi)$  (where  $\xi_1 \neq \xi$ ) is  $F_4/Spin(9)$ .*
- (3) *The orbit through  $(\xi_1, \xi_2, \xi_3)$  (where  $\xi_1, \xi_2, \xi_3$  are distinct) is  $F_4/Spin(8)$ .*

LEMMA 3 ([3]). *Any element  $X \in \mathfrak{J}^C$  can be transformed to a diagonal form by some  $\alpha \in E_6$ :*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in C, \quad (\text{which is briefly written by } (\xi_1, \xi_2, \xi_3)).$$

*The order of  $\xi_1, \xi_2, \xi_3$  can be arbitrarily exchanged under the action of  $E_6$ .*

THEOREM 4 ([5]). *The orbit types of the group  $E_6$  in  $\mathfrak{J}^C$  are as follows.*

- (1) *The orbit through  $(0, 0, 0)$  is  $E_6/E_6$ .*
- (2) *The orbit through  $(\xi_1, \xi_2, \xi_3)$  (where  $|\xi_1| = |\xi_2| = |\xi_3| \neq 0$ ) is  $E_6/F_4$ .*
- (3) *The orbit through  $(\xi, 0, 0)$  (where  $\xi \neq 0$ ) is  $E_6/Spin(10)$ .*
- (4) *The orbit through  $(\xi_1, \xi_2, \xi_3)$  (where  $|\xi_1| \neq |\xi_2| = |\xi_3| \neq 0$ ) is  $E_6/Spin(9)$ .*
- (5) *The orbit through  $(\xi_1, \xi_2, \xi_3)$  (where  $|\xi_1|, |\xi_2|, |\xi_3|$  are distinct) is  $E_6/Spin(8)$ .*

#### 4. Orbit Types of $E_7$ in $\mathfrak{P}^C$

LEMMA 5 ([2]). *Any element  $P \in \mathfrak{P}^C$  can be transformed to the following diagonal form by some  $\alpha \in E_7$ :*

$$\alpha P = \left( \begin{pmatrix} ar_1 & 0 & 0 \\ 0 & ar_2 & 0 \\ 0 & 0 & ar_3 \end{pmatrix}, \begin{pmatrix} br_1 & 0 & 0 \\ 0 & br_2 & 0 \\ 0 & 0 & br_3 \end{pmatrix}, ar, br \right), \quad \begin{array}{l} r_k, r \in \mathbf{R}, 0 \leq r_k, 0 \leq r, \\ a, b \in C, |a|^2 + |b|^2 = 1. \end{array}$$

*Moreover, any element  $P \in \mathfrak{P}^C$  can be transformed to the following diagonal form by some  $\varphi(A)\alpha \in \varphi(SU(2))E_7$ :*

$$\varphi(A)\alpha P = \left( \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r, 0 \right), \quad r_k, r \in \mathbf{R}, 0 \leq r_k, 0 \leq r,$$

*(which is briefly written by  $(r_1, r_2, r_3; r)$ ), where  $\varphi(A) \in \varphi(SU(2)) \subset E_8$  and commutes with any element  $\alpha \in E_7$ . The order of  $r_1, r_2, r_3, r$  can be arbitrarily exchanged under the action of  $E_7$ . (As for the definitions of the groups  $E_8$  and  $\varphi(SU(2))$ , see*

[2]). The action of  $\varphi(A), A \in SU(2)$ , on  $\mathfrak{P}^C$  is given by

$$\begin{aligned}\varphi(A)P &= \varphi\left(\begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix}\right)(X, Y, \xi, \eta) \\ &= (aX + \tau(bY), aY - \tau(bX), a\xi + \tau(b\eta), a\eta - \tau(b\xi)).\end{aligned}$$

**THEOREM 6.** The group  $E_7$  has the following seven orbit types in  $\mathfrak{P}^C$ :

$$\begin{aligned}E_7/E_7, \quad E_7/E_6, \quad E_7/F_4, \quad E_7/Spin(11), \quad E_7/Spin(10), \\ E_7/Spin(9), \quad E_7/Spin(8).\end{aligned}$$

*More details,*

- (1) The orbit through  $(0, 0, 0; 0)$  is  $E_7/E_7$ .
- (2) The orbit through  $(0, 0, 0; 1)$  or  $(1, 1, 1; 1)$  is  $E_7/E_6$ .
- (3) The orbit through  $(1, 1, 1; 0)$  or  $(1, 1, 1; r)$  (where  $0 < r, 1 \neq r$ ) is  $E_7/F_4$ .
- (4) The orbit through  $(1, 0, 0; 1)$  or  $(1, r, r; 1)$  (where  $0 < r, 1 \neq r$ ) is  $E_7/Spin(11)$ .
- (5) The orbit through  $(1, 0, 0; r)$  (where  $0 < r, 1 \neq r$ ) is  $E_7/Spin(10)$ .
- (6) The orbit through  $(1, 1, r; 0)$  or  $(1, 1, r; s)$  (where  $0 < r, 0 < s$  and  $1, r, s$  are distinct) is  $E_7/Spin(9)$ .
- (7) The orbit through  $(1, r, s; 0)$  or  $(1, r, s; t)$  (where  $r, s, t$  are positive and  $1, r, s, t$  are distinct) is  $E_7/Spin(8)$ .

**PROOF.** From Lemma 5, the representatives of orbit types (up to a constant) can be given by the following.

$$\begin{aligned}(0, 0, 0; 0), \quad (0, 0, 0; 1), \quad (0, 0, 1; 1), \quad (0, 0, 1; r), \\ (0, 1, 1; 1), \quad (0, 1, 1; r), \quad (0, 1, r; s), \quad (1, 1, 1; 1), \\ (1, 1, 1; r), \quad (1, 1, r; r), \quad (1, 1, r; s), \quad (1, r, s; t)\end{aligned}$$

where  $r, s, t$  are positive,  $0, 1, r, s, t$  are distinct and the order of  $0, 1, r, s, t$  can be arbitrarily exchanged.

(1) The isotropy subgroup  $(E_7)_{(0,0,0;0)}$  is obviously  $E_7$ . Therefore the orbit through  $(0, 0, 0; 0)$  is  $E_7/E_7$ .

(2) The isotropy subgroup  $(E_7)_{(0,0,0;1)}$  is  $E_6$ . Therefore the orbit through  $(0, 0, 0; 1)$  is  $E_7/E_6$ .

(2') The isotropy subgroup  $(E_7)_{(1,1,1;1)}$  is conjugate to  $E_6$  in  $E_7$ . In fact, we know that the following realization of the homogeneous space  $E_7/E_6 : E_7/E_6 = \{P \in \mathfrak{P}^C \mid P \times P = 0, \langle P, P \rangle = 1\} = \mathfrak{M}$  ([4]). Since  $1/2\sqrt{2}(E, E, 1, 1)$  and  $(0, 0, 1, 0) \in \mathfrak{M}$ , there exists  $\delta \in E_7$  such that

$$\delta\left(\frac{1}{2\sqrt{2}}(E, E, 1, 1)\right) = (0, 0, 1, 0).$$

Hence the isotropy subgroup  $(E_7)_{(E, E, 1, 1)}$  is conjugate to the isotropy subgroup  $(E_7)_{(0, 0, 1, 0)}$  is  $E_7 : (E_7)_{(E, E, 1, 1)} \sim (E_7)_{(0, 0, 1, 0)}$ . On the other hand, since

$$\varphi\left(\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right)(E, 0, 1, 0) = \frac{1}{\sqrt{2}}(E, E, 1, 1),$$

we have  $(E_7)_{(E, 0, 1, 0)} = (E_7)_{(E, E, 1, 1)} \sim (E_7)_{(0, 0, 1, 0)} = E_6$ . Therefore the orbit through  $(1, 1, 1; 1)$  is  $E_7/E_6$ .

(3) The isotropy subgroup  $(E_7)_{(1, 1, 1; 0)}$  is  $F_4$ . In fact, for  $\alpha \in E_7$  and  $P \in \mathfrak{P}^C$ , we have  $\alpha(\tau\lambda((P \times P)P)) = \tau\lambda(\alpha((P \times P)P)) = \tau\lambda(\alpha(P \times P)\alpha^{-1}\alpha P) = \tau\lambda((\alpha P \times \alpha P)\alpha P)$ . Now, let  $P = (1, 1, 1; 0)$ . Since  $\tau\lambda((P \times P)P) = 3/2(0, 0, 0; 1)$ , if  $\alpha \in E_7$  satisfies  $\alpha P = P$ , then  $\alpha$  also satisfies  $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$ . Hence  $\alpha \in E_6$ , so together with  $\alpha E = E$ , we have  $\alpha \in F_4$ . Therefore the orbit through  $(1, 1, 1; 0)$  is  $E_7/F_4$ .

(3') The isotropy subgroup  $(E_7)_{(1, 1, 1; r)}$  is  $F_4$ . In fact, let  $P = (1, 1, 1; r)$ . Since  $\tau\lambda((P \times P)P) = 3/2(r, r, r; 1)$ , if  $\alpha \in E_7$  satisfies  $\alpha P = P \cdots$  (i), then  $\alpha$  also satisfies  $\alpha(r, r, r; 1) = (r, r, r; 1) \cdots$  (ii). Take (i)–(ii), then we have  $\alpha(1 - r, 1 - r, 1 - r; r - 1) = (1 - r, 1 - r, 1 - r; r - 1)$ . Since  $1 - r \neq 0$ , we have  $\alpha(1, 1, 1; -1) = (1, 1, 1; -1)$ . Together with  $\alpha P = P$ , we have  $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$  and  $\alpha(1, 1, 1; 0) = (1, 1, 1; 0)$ . Hence  $\alpha \in E_6$  and hence  $\alpha \in F_4$ . Therefore the orbit through  $(1, 1, 1; r)$  is  $E_7/F_4$ .

(4) The isotropy subgroup  $(E_7)_{(1, 0, 0; 1)}$  is  $Spin(11)$ . Therefore the orbit through  $(1, 0, 0; 1)$  is  $E_7/Spin(11)$ .

(4') The isotropy subgroup  $(E_7)_{(1, r, r; 1)}$  is  $Spin(11)$ . In fact, let  $P = (1, r, r; 1)$ . Since  $\tau\lambda((P \times P)P) = 3/2(r^2, r, r; r^2)$ , if  $\alpha \in E_7$  satisfies  $\alpha P = P \cdots$  (i), then  $\alpha$  also satisfies  $\alpha(r^2, r, r; r^2) = (r^2, r, r; r^2) \cdots$  (ii). Take (i)–(ii), then we have  $\alpha(1 - r^2, 0, 0; 1 - r^2) = (1 - r^2, 0, 0; 1 - r^2)$ . Since  $1 - r^2 \neq 0$ , we have  $\alpha(1, 0, 0; 1) = (1, 0, 0; 1)$ . Hence  $\alpha \in Spin(11)$ . Therefore the orbit through  $(1, r, r; 1)$  is  $E_7/Spin(11)$ .

(5) The isotropy subgroup  $(E_7)_{(1, 0, 0; r)}$  is  $Spin(10)$ . In fact, for  $\alpha \in E_7$  and  $P \in \mathfrak{P}^C$ , we have  $\alpha((P \times P)\tau\lambda P) = (\alpha(P \times P)\alpha^{-1})\alpha(\tau\lambda P) = (\alpha P \times \alpha P)\tau\lambda(\alpha P)$ . Now, let  $P = (1, 0, 0; r)$ . Since  $(P \times P)\tau\lambda P = -1/2(r^2, 0, 0; r)$ , if  $\alpha \in E_7$  satisfies  $\alpha P = P \cdots$  (i), then  $\alpha$  also satisfies  $\alpha(r^2, 0, 0; r) = (r^2, 0, 0; r) \cdots$  (ii). Take (i)–(ii), then we have  $\alpha(1 - r^2, 0, 0; 0) = (1 - r^2, 0, 0; 0)$ . Since  $1 - r^2 \neq 0$ , we have  $\alpha(1, 0, 0; 0) = (1, 0, 0; 0) \cdots$  (iii). Take (i)–(iii), then  $\alpha(0, 0, 0; r) = (0, 0, 0; r)$ , that is,  $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$ . Hence  $\alpha \in E_6$  and  $\alpha E_1 = E_1$ . Thus  $\alpha \in Spin(10)$ . Therefore the orbit through  $(1, 0, 0; r)$  is  $E_7/Spin(10)$ .

(6) The isotropy subgroup  $(E_7)_{(1,1,r;0)}$  is  $Spin(9)$ . In fact, let  $P = (1, 1, r; 0)$ . Since  $\tau\lambda((P \times P)P) = 3/2(0, 0, 0; r)$ , if  $\alpha \in E_7$  satisfies  $\alpha P = P$ , then  $\alpha$  also satisfies  $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$ . Hence  $\alpha \in E_6$ , so together with  $\alpha P = P$ , we have  $\alpha \in Spin(9)$  (Theorem 4.(4)). Therefore the orbit through  $(1, 1, r; 0)$  is  $E_7/Spin(9)$ .

(6') The isotropy subgroup  $(E_7)_{(1,1,r;s)}$  is  $Spin(9)$ . In fact, let  $P = (1, 1, r; s)$ . Since  $\tau\lambda((P \times P)P) = 3/2(rs, rs, s; r)$ , if  $\alpha \in E_7$  satisfies  $\alpha P = P \cdots (i)$ , then  $\alpha$  also satisfies  $\alpha(rs, rs, s; r) = (rs, rs, s; r) \cdots (ii)$ . Take  $(i) \times r - (ii) \times s$ , then we have  $\alpha(r(1-s^2), r(1-s^2), r^2-s^2; 0) = (r(1-s^2), r(1-s^2), r^2-s^2; 0)$ . Since  $r(1-s^2)$ ,  $r^2-s^2$  are non-zero and  $r(1-s^2) \neq r^2-s^2$ , from (6) we have  $\alpha \in Spin(9)$ . Therefore the orbit through  $(1, 1, r; s)$  is  $E_7/Spin(9)$ .

(7) The isotropy subgroup  $(E_7)_{(1,r,s;0)}$  is  $Spin(8)$ . In fact, let  $P = (1, r, s; 0)$ . Since  $\tau\lambda((P \times P)P) = 3/2(0, 0, 0; rs)$ , if  $\alpha \in E_7$  satisfies  $\alpha P = P$ , then  $\alpha$  also satisfies  $\alpha(0, 0, 0; 1) = (0, 0, 0; 1)$ . Hence  $\alpha \in E_6$ , so together with  $\alpha P = P$ , we have  $\alpha \in Spin(8)$  (Theorem 4.(5)). Therefore the orbit through  $(1, r, s; 0)$  is  $E_7/Spin(8)$ .

(7') The isotropy subgroup  $(E_7)_{(1,r,s;t)}$  is  $Spin(8)$ . In fact, let  $P = (1, r, s; t)$ . Since  $\tau\lambda((P \times P)P) = 3/2(rst, st, rt; rs)$ , if  $\alpha \in E_7$  satisfies  $\alpha P = P \cdots (i)$ , then  $\alpha$  also satisfies  $\alpha(rst, st, rt; rs) = (rst, st, rt; rs) \cdots (ii)$ . Take  $(i) \times rs - (ii) \times t$ , then we have  $\alpha(rs(1-t^2), s(r^2-t^2), r(s^2-t^2); 0) = (rs(1-t^2), s(r^2-t^2), r(s^2-t^2); 0)$ . Since  $rs(1-t^2)$ ,  $s(r^2-t^2)$  and  $r(s^2-t^2)$  are non-zero and distinct, from (7) we have  $\alpha \in Spin(8)$ . Therefore the orbit through  $(1, r, s; t)$  is  $E_7/Spin(8)$ .

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