

ON THE EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS WITH SLOWLY DECAYING INITIAL DATA

By

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1. Introduction

In this paper, we study the Cauchy problem

$$\begin{aligned} u_t &= \Delta u + |x|^\sigma u^p && \text{in } D \times (0, T), \\ \text{(P)} \quad u(x, t) &= 0 && \text{on } \partial D \times (0, T), \\ u(x, 0) &= a(x) \geq 0 && \text{in } D, \end{aligned}$$

where $\sigma \geq 0, p > 1$.

The domain D is a cone in \mathbb{R}^N , such as

$$D = \{x \in \mathbb{R}^N \setminus \{0\}; x/|x| \in \Omega\},$$

where $\Omega \subset \mathbb{S}^{N-1}$ is an open connected subset with smooth boundary. The nonnegative initial condition $a(x)$ is continuous and $\langle x \rangle^{\sigma/(p-1)} a(x)$ is bounded in \bar{D} and $a = 0$ on ∂D ($\langle x \rangle = \sqrt{1 + |x|^2}$).

Let ω_1 be the smallest Dirichlet eigenvalue for the Laplace-Beltrami operator on Ω , and γ_+ be the positive root of $\gamma(\gamma + N - 2) = \omega_1$. The following results are well known by the papers of Levine and Meier [3], [4] and Hamada [2].

- (I) If $1 < p \leq 1 + (2 + \sigma)/(N + \gamma_+)$, there is no nontrivial nonnegative global solution for any initial data.
- (II) If $p = 1 + (2 + \sigma)/(N + \gamma_+)$, there is no nontrivial nonnegative global solution for any initial data.
- (III) If $p > 1 + (2 + \sigma)/(N + \gamma_+)$, then there exist nontrivial nonnegative global solutions for sufficiently small initial data.

In the case (III), it is clear if the initial condition is not decreasing as $|x| \rightarrow \infty$, so the solution of (P) is not global in time.

In the case of $D = \mathbf{R}^N$, when $p > 1 + (2/N)$ and $\sigma = 0$, Fujita [1] proved that there exist global solutions of (P) with the initial data satisfying

$$0 \leq a(x) \leq \delta \exp(-\gamma|x|^2)$$

for any $\gamma > 0$ and some $\delta > 0$.

Our motivation of this paper is to find the critical decay rate of the initial datum between existence and nonexistence of the nonnegative global solution of (P).

Throughout this paper we assume $p > 1 + (2 + \sigma)/(N + \gamma_+)$ and $\sigma \geq 0$.

DEFINITION 1.1. For $T > 0$, $u = u(x, t)$ is called a solution of (P) in $(0, T)$, if

- (A) u is continuous in $D \times [0, T)$,
- (B) u_t, u_{x_i} and $u_{x_i x_j}$ ($i, j = 1, \dots, N$) are continuous in $D \times (0, T)$,
- (C) $\|u(t)\|_{\sigma/(p-1)}$ is finite for each $t \in [0, T)$,
- (D) u satisfies (P),

where $\|u(t)\|_{\sigma/(p-1)} := \sup_D \langle x \rangle^{\sigma/(p-1)} |u(x, t)|$.

DEFINITION 1.2. $\bar{T} := \sup\{T > 0; \|u(t)\|_{\sigma/(p-1)} \text{ is finite for } 0 \leq t < T\}$ is called the nontrivial existence time of u . If $\bar{T} = +\infty$, then u is called a global solution of (P).

The local existence theorem for (P) is stated as follows (see [2, Theorem 1.1]).

THEOREM. For any nonnegative function $a = a(x)$ in $C(\bar{D})$ satisfying $\|a\|_{\sigma/(p-1)} < \infty$ and $a = 0$ on ∂D there is a solution $u(x, t)$ of (P) in $(0, t_0)$ such that $\|u(t)\|_{\sigma/(p-1)}$ is finite in $(0, t_0)$ where $t_0 > 0$ depends only on σ, p, N and a .

Let $\{\psi_n\}_{n=1}^\infty$ be a normalized orthogonal system for the Laplace-Beltrami operator on Ω corresponding to the sequence $\{\omega_n\}$ of Dirichlet eigenvalues for this problem and we take $\psi_1 > 0$ in Ω .

We assume $M(\alpha)$ be positive constants such that

- (i) if $\alpha < (2 + \sigma)/(p - 1)$ then $M(\alpha)$ is an arbitrary positive constant,
- (ii) if $\alpha = (2 + \sigma)/(p - 1)$ then

$$M(\alpha) = \frac{2^{\gamma_+ + (3\alpha)/2} \exp(1/4) \Gamma(N/2 + \gamma_+) \int_{\Omega} \psi_1(\theta) dS_{\theta}}{(p - 1)^{1/(p-1)} \Gamma((N + \gamma_+ - \alpha)/2) \int_{\Omega_1} \psi_1^2(\theta) dS_{\theta}},$$

where $\Omega_1 \subset \Omega$ and $|\Omega_1| \neq 0$.

The main results of this paper are the following.

THEOREM 1.1. *Let $0 \leq \sigma \leq (p - 1)(N - 2)$ ($N \geq 2$) and Ω_1 be stated as above. We assume $\|a\|_{\sigma/(p-1)} < \infty$ and*

$$(1) \quad a(x) \geq M \langle x \rangle^{-\alpha} \psi_1(x/|x|) \quad \text{for } x \in \{|x| > 1 \text{ and } x/|x| \in \Omega_1\},$$

where $0 \leq \alpha \leq (2 + \sigma)/(p - 1)$ and $M > M(\alpha)$. Then the solution of (P) is not global.

THEOREM 1.2. *Let $\alpha > (2 + \sigma)/(p - 1)$. If there exists $m > 0$ such that the nontrivial initial data of (P) satisfies*

$$(2) \quad 0 \leq a(x) \leq m \langle x \rangle^{-\alpha} \psi_1(x/|x|) \quad \text{for } x \in D,$$

then there exists the unique nontrivial global solution of (P).

The paper is divided as follows. In Section 2 we will prove Theorem 1.1. In Section 3 we will show the existence of the global solution of (P) (Theorem 1.2).

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

We introduce the Green's function $G(x, y; t) = G(r, \theta, \rho, \phi; t)$ ($r = |x|$, $\theta = x/|x| \in \Omega$, $\rho = |y|$, $\phi = y/|y| \in \Omega$), for the linear heat equation in the cone. Here

$$\begin{aligned} G(x, y, t) &= G(r, \theta, \rho, \phi; t) \\ &= \frac{1}{2t} (r\rho)^{-(N-2)/2} \exp\left(-\frac{\rho^2 + r^2}{4t}\right) \sum_{n=1}^{\infty} I_{\nu_n}\left(\frac{r\rho}{2t}\right) \psi_n(\theta) \psi_n(\phi), \end{aligned}$$

where $\nu_n = \{((N - 2)/2)^2 + \omega_n\}^{1/2}$, and

$$(3) \quad I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(z/2)^{2k}}{k! \Gamma(\nu + k + 1)} \sim \begin{cases} (z/2)^\nu / \Gamma(\nu + 1) & z \rightarrow 0^+ \\ e^z / \sqrt{2\pi z} & z \rightarrow +\infty \end{cases}$$

(see Watson [5]).

Let $v = v(x, t)$ be a solution of

$$(H) \quad \begin{aligned} v_t &= \Delta v && \text{in } D \times (0, \infty), \\ v(x, t) &= 0 && \text{on } \partial D \times (0, \infty), \\ v(x, 0) &= a(x) && \text{in } D, \end{aligned}$$

where $a(x)$ is the same initial data as (P).

Then the solution of (H) has the form

$$v(x, t) = \int_D G(x, y; t) a(y) dy.$$

To prove Theorem 1.1 we need the following Lemma (See [2, Lemma 3.2]).

LEMMA 2.1. *We assume $0 \leq \sigma/(p - 1) \leq N - 2$. Let \bar{T} be the maximal existence time of u , and v be the solution of (H) with the same initial-boundary condition as u .*

Then

$$v(x, t) < ((p - 1)|x|^\sigma t)^{-1/(p-1)} \quad \text{in } D \times (0, \bar{T}).$$

PROOF OF THEOREM 1.1. We assume that there exists a global solution of (P), then from Lemma 2.1 we have

$$(p - 1)^{-1/(p-1)} > |x|^{\sigma/(p-1)} t^{1/(p-1)} v(x, t) \quad \text{in } D \times (0, \infty).$$

Integrating the above both sides over Ω with respect to $\psi_1(\theta) dS_\theta$,

$$\begin{aligned} c_1 &> \int_\Omega r^{\sigma/(p-1)} t^{1/(p-1)} v(r, \theta; t) \psi_1(\theta) dS_\theta \\ &= \int_\Omega r^{\sigma/(p-1)} t^{1/(p-1)} \int_0^\infty \int_\Omega G(r, \theta, \rho, \phi; t) a(\rho, \phi) \rho^{N-1} dS_\phi d\rho \psi_1(\theta) dS_\theta \end{aligned}$$

$$\begin{aligned}
 &= r^{\sigma/(p-1)} t^{1/(p-1)} \int_0^\infty \int_\Omega \frac{1}{2t} (r\rho)^{-(N-2)/2} \\
 &\quad \times I_{\nu_1} \left(\frac{r\rho}{2t} \right) \exp \left(-\frac{r^2 + \rho^2}{4t} \right) a(\rho, \phi) \psi_1(\phi) \rho^{N-1} dS_\phi d\rho \\
 &\geq \frac{1}{2\Gamma(\nu_1 + 1)} r^{\sigma/(p-1) - (N-2)/2} t^{1/(p-1) - 1} \exp \left(-\frac{r^2}{4t} \right) \\
 &\quad \times \int_0^\infty \int_\Omega \rho^{-(N-2)/2 + N-1} \left(\frac{r\rho}{4t} \right)^{\nu_1} \exp \left(-\frac{\rho^2}{4t} \right) a(\rho, \phi) \psi_1(\phi) dS_\phi d\rho,
 \end{aligned}$$

where $c_1 = (p - 1)^{-1/(p-1)} \int_\Omega \psi_1(\theta) dS_\theta$.

By using (1) and $\nu_1 = (N - 2)/2 + \gamma_+$, we get

$$\begin{aligned}
 c_2 &\geq 2^{-\alpha/2} M r^{\sigma/(p-1) + \gamma_+} t^{1/(p-1) - N/2 - \gamma_+} \\
 &\quad \times \exp \left(-\frac{r^2}{4t} \right) \int_1^\infty \int_{\Omega_1} \rho^{N + \gamma_+ - \alpha - 1} \exp \left(-\frac{\rho^2}{4t} \right) \psi_1^2(\phi) dS_\phi d\rho,
 \end{aligned}$$

where $c_2 = 2^{N+2\gamma_+ - 1} \Gamma(N/2 + \gamma_+) c_1$.

Let $r = t^{1/2}$ and $c_3 = c_2 2^{\alpha/2} \exp(1/4) / \int_{\Omega_1} \psi_1^2(\phi) dS_\phi$, then for $t \in (0, \infty)$

$$\begin{aligned}
 c_3 &\geq M t^{((\sigma+2)/(p-1) - N - \gamma_+)/2} \int_1^\infty \rho^{N + \gamma_+ - \alpha - 1} \exp \left(-\frac{\rho^2}{4t} \right) d\rho \\
 &\geq 2^{N + \gamma_+ - \alpha - 1} M t^{((\sigma+2)/(p-1) - \alpha)/2} \Gamma((N + \gamma_+ - \alpha)/2, 1/(4t)).
 \end{aligned}$$

Letting $t \rightarrow \infty$ we have reached a contradiction. Thus we have proved Theorem 1.1.

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2.

We put

$$(4) \quad \rho(x, t) = \int_D G(x, y, t) \langle y \rangle^{-\alpha} \psi_1 \left(\frac{y}{|y|} \right) dy.$$

From (3), we set for some positive constants c_4 and c_5

$$(5) \quad I_{\nu_1}(z) \leq \begin{cases} c_4 z^{\nu_1} & 0 < z \leq 1, \\ c_5 z^{-1/2} e^z & z > 1. \end{cases}$$

Now we estimate the decay order of $\rho(x, t)$.

LEMMA 3.1. Assume $\alpha > (2 + \sigma)/(p - 1)$. Let $\zeta = \zeta(\alpha) \geq 0$ be

- (i) $\zeta(\alpha) = \alpha - \sigma/(p - 1)$ if $(2 + \sigma)/(p - 1) < \alpha < N + \gamma_+$,
- (ii) $\zeta(\alpha) \in (2/(p - 1), N + \gamma_+ - \sigma/(p - 1))$ if $\alpha = N + \gamma_+$,
- (iii) $\zeta(\alpha) = N + \gamma_+ - \sigma/(p - 1)$ if $\alpha > N + \gamma_+$,

then there exists a positive constant c_6 such that,

$$(6) \quad |x|^{\sigma/(p-1)} \rho(x, t) \leq c_6 (1+t)^{-\zeta/2} \quad \text{for } x \in D, t > 0.$$

PROOF OF LEMMA 3.1. From (4) we have

$$\begin{aligned} & r^{\sigma/(p-1)} \rho(r, \theta, t) \\ &= \int_0^\infty r^{\sigma/(p-1)} \frac{1}{2t} (r\eta)^{-(N-2)/2} \exp\left(-\frac{\eta^2 + r^2}{4t}\right) I_{\nu_1}\left(\frac{r\eta}{2t}\right) \langle \eta \rangle^{-\alpha} \eta^{N-1} d\eta \psi_1(\theta) \\ &= \left\{ \int_0^{2t/r} + \int_{2t/r}^\infty \right\} \psi_1(\theta) \\ &= \{A + B\} \psi_1(\theta). \end{aligned}$$

Then by (5), there exists a constant $c_7 > 0$ such that

$$\begin{aligned} A &\leq c_4 r^{\sigma/(p-1)} \\ &\quad \times \int_0^{2t/r} \frac{1}{2t} (r\eta)^{-(N-2)/2} \exp\left(-\frac{\eta^2 + r^2}{4t}\right) \left(\frac{r\eta}{2t}\right)^{\nu_1} (1 + \eta^2)^{-\alpha/2} \eta^{N-1} d\eta \\ &= c_4 (2t)^{-N/2 - \gamma_+} r^{\sigma/(p-1) + \gamma_+} \exp\left(-\frac{r^2}{4t}\right) \\ &\quad \times \int_0^{2t/r} \exp\left(-\frac{\eta^2}{4t}\right) \eta^{N + \gamma_+ - 1} (1 + \eta^2)^{-\alpha/2} d\eta \\ &\leq c_7 t^{(\sigma/(p-1) - (N + \gamma_+))/2} E_\alpha^{N + \gamma_+}(r, t) \end{aligned}$$

where

$$E_\alpha^\beta(r, t) := \int_0^{2t/r} \exp\left(-\frac{\eta^2}{4t}\right) \eta^{\beta-1} (1 + \eta^2)^{-\alpha/2} d\eta \quad \text{for } r, t > 0.$$

When $(2 + \sigma)/(p - 1) < \alpha \leq N + \gamma_+$,

$$\begin{aligned} E_\alpha^{N+\gamma_+}(r, t) &\leq 2^{\alpha/2} \int_0^{2t/r} \exp\left(-\frac{\eta^2}{4t}\right) (1 + \eta)^{N+\gamma_+-\alpha-1} d\eta \\ &\leq 2^{\alpha/2} \int_1^\infty \eta^{N+\gamma_+-\alpha-1} d\eta \\ &\leq c_8 t^{(N+\gamma_+-\sigma/(p-1)-\zeta)/2} \end{aligned}$$

for some positive constant c_8 .

On the other hand, if $\alpha > N + \gamma_+$,

$$\begin{aligned} E_\alpha^{N+\gamma_+}(r, t) &\leq 2^{2/\alpha} \int_0^{2t/r} \exp\left(-\frac{\eta^2}{4t}\right) (1 + \eta)^{N+\gamma_+-\alpha-1} d\eta \\ &\leq 2^{2/\alpha} \int_1^\infty \eta^{N+\gamma_+-\alpha-1} d\eta \\ &= c_9 < \infty \quad \text{for any } r, t > 0, \end{aligned}$$

where c_9 is a positive constant.

So, we obtain for any $t > 0$

$$A \leq \max\{c_8, c_9\} t^{-\zeta/2}.$$

Now we begin to estimate B . It follows from (5), we have

$$\begin{aligned} B &\leq c_5 \int_{2t/r}^\infty \left(\frac{1}{2t}\right)^{1/2} \exp\left(-\frac{(\eta - r)^2}{4t}\right) r^{-(N-1)/2+\sigma/(p-1)} \eta^{(N-1)/2-\alpha} d\eta \\ &= c_5 \left\{ \int_{[2t/r, \infty) \cap [2r/3, 2r]} + \int_{[2t/r, \infty) \setminus [2r/3, 2r]} \right\} \\ &= c_5 \{J + K\}. \end{aligned}$$

First we estimate J . If $t \geq r^2$ then $J = 0$. When $t < r^2$,

$$\begin{aligned} J &\leq \int_{2r/3}^{2r} \left(\frac{1}{2t}\right)^{1/2} \exp\left(-\frac{(\eta - r)^2}{4t}\right) \left(\frac{\eta}{r}\right)^{(N-1)/2} \left(\frac{r}{\eta}\right)^\alpha r^{\sigma/(p-1)-\alpha} d\eta \\ &\leq 2^{(N-1)/2} \left(\frac{3}{2}\right)^\alpha t^{(\sigma/(p-1)-\alpha)/2} \int_{-\infty}^\infty \left(\frac{1}{2t}\right)^{1/2} \exp\left(-\frac{(\eta - r)^2}{4t}\right) d\eta \\ &\leq c_{10} t^{-(\alpha-\sigma/(p-1))/2}, \end{aligned}$$

where c_{10} is a positive constant.

Next we estimate K . If $\eta \in [2t/r, \infty) \setminus [2r/3, 2r]$, then $|\eta - r| > (r/3)$ and $|\eta - r| > (\eta/2) > (t/r)$. So, there exists a constant $c_{11} > 0$ such that

$$\begin{aligned} K &= \int_{[2t/r, \infty) \setminus [2r/3, 2r]} \left(\frac{1}{2t}\right)^{1/2} \exp\left(-\frac{(\eta - r)^2}{4t}\right) r^{-(N-1)/2 + \sigma/(p-1)} \eta^{(N-1)/2 - \alpha} d\eta \\ &= c_{11} t^{-(\alpha - \sigma/(p-1))/2}. \end{aligned}$$

Then we have

$$B \leq c_5 \{c_{10} + c_{11}\} t^{-(\alpha - \sigma/(p-1))/2}$$

for $t > 0$. On the other hand from the definition of ζ we have $\zeta \leq \alpha - \sigma/(p - 1)$. This completes the proof of Lemma 3.1, since $|x|^{\sigma/(p-1)} \rho(x, t)$ is bounded.

To show the global existence of the solution of (P), we introduce the following norm,

$$\|u\| := \sup_{t \geq 0, x \in D} \frac{u(x, t)}{\rho(x, t)}.$$

For a, u , we define

$$u_0(x, t) = \int_D G(x, y, t) a(y) dy.$$

and

$$\Phi u(x, t) = \int_0^t \int_D G(x, y, t - s) |y|^\sigma u^p(y, s) dy ds$$

Then the following Lemma is immediate from the definition of $\|\cdot\|$.

LEMMA 3.2. *Suppose a satisfies (2) then $\|u_0\| \leq m$.*

LEMMA 3.3. *For some constant $c_{12} > 0$, $\Phi\rho$ satisfies*

$$(7) \quad \|\Phi\rho\| \leq c_{12}.$$

PROOF OF LEMMA 3.3. From (6) we have

$$\begin{aligned} \Phi\rho(x, t) &= \int_0^t \int_D G(x, y, t-s) |y|^\sigma \rho^p(y, s) dy ds \\ &\leq \int_0^t (c_6(1+s)^{-\zeta/2})^{p-1} \int_D G(x, y, t-s) \rho(y, s) dy ds \\ &\leq \int_0^t (c_6(1+s)^{-\zeta/2})^{p-1} \int_D \int_D G(x, y, t-s) G(y, z, s) \langle z \rangle^{-\alpha} \psi_1\left(\frac{z}{|z|}\right) dz dy ds \\ &= \int_0^t (c_6(1+s)^{-\zeta/2})^{p-1} ds \int_D G(x, z, t) \langle z \rangle^{-\alpha} \psi_1\left(\frac{z}{|z|}\right) dz \\ &\leq c_{12} \rho(x, t). \end{aligned}$$

Moreover (7) implies,

$$(8) \quad 0 \leq \Phi u(x, t) \leq \|u\|^p (\Phi\rho)(x, t) \leq c_{12} \|u\|^p \rho(x, t).$$

So the following Lemma holds.

LEMMA 3.4. *If $\|u\| < \infty$ then for some $c_{12} > 0$,*

$$\|\Phi u\| < c_{12} \|u\|^p.$$

LEMMA 3.5. *If there exist $\mathcal{M} < \infty$ such that,*

$$\|u\|, \|v\| \leq \mathcal{M}$$

then,

$$\|\Phi u - \Phi v\| \leq c_{12} p \mathcal{M}^{(p-1)} \|u - v\|.$$

PROOF. By using the mean value theorem

$$\begin{aligned} \left| |y|^\sigma u^p(y, s) - |y|^\sigma v^p(y, s) \right| &\leq p |y|^\sigma (\mathcal{M} \rho(y, s))^{p-1} |u(y, s) - v(y, s)| \\ &\leq p \mathcal{M}^{p-1} \|u - v\| |y|^\sigma \rho^p(y, s). \end{aligned}$$

Thus we obtain,

$$(9) \quad \begin{aligned} |\Phi u(x, t) - \Phi v(x, t)| &\leq p \mathcal{M}^{p-1} \|u - v\| (\Phi\rho)(y, s) \\ &\leq c_{12} p \mathcal{M}^{p-1} \|u - v\| \rho(y, s). \end{aligned}$$

PROOF OF THEOREM 1.2. This proof is motivated by the idea given in [1].
We define

$$u_{n+1} = u_0 + \Phi u_n \quad (n = 0, 1, 2, \dots).$$

By Lemma 3.2 and (8), we can take $m > 0$ such that

$$\|u_n\| \leq 2m \quad (n = 0, 1, 2, \dots),$$

and there exists, $0 < \delta < 1$ such that,

$$(10) \quad \|u_{n+2} - u_{n+1}\| \leq \delta \|u_{n+1} - u_n\| \quad (n = 0, 1, 2, \dots).$$

So, $\sum_n^\infty \|u_{n+1} - u_n\|$ converges. Thus there exists a function u such that

$$\|u_n - u\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The uniqueness is clear from (10).

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