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WAVELETS IN THE GENERALIZED TEMPERED DISTRIBUTIONS

By

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Abstract. We expand the generalized tempered distributions in terms of wavelets of ordinary functions and show the convergence of the wavelet expansions of the generalized tempered distributions.

1. Introduction

It is possible to extend the expansions in orthogonal wavelets from $L^2(R)$ to a certain class of tempered distributions. G. G. Walter has presented a few aspects of the relations between the wavelets and the tempered distributions of polynomial growth. He has found the expansion of the tempered distributions of polynomial growth in terms of regular orthogonal wavelets [6] and the convergence of the wavelet expansions [7].

In the past, the tempered distributions of polynomial growth were extended by various types of the generalized tempered distributions of exponential growth [2] and [5].

In this paper, we will present the expansions of the generalized tempered distributions of exponential growth, that were introduced by G. Sampson and Z. Zielezny [5], in terms of wavelets of ordinary functions, and we will study the convergence of the wavelet expansions of the generalized tempered distributions of exponential growth.

2. The Generalized Tempered Distributions Space $\mathscr{K}_p'(R)$

We denote by $\mathscr{K}_p(R), p \geq 1$, the space of all functions $\phi \in C^\infty(R)$ such that

$$\nu_k(\phi) = \sup_{x \in R, \alpha \leq k} e^{k|x|^p} |D^\alpha \phi(x)| < \infty, \quad k = 1, 2, \ldots \tag{1}$$

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The topology in $\mathcal{K}_p(R)$ is defined by the family of the semi-norms ν_k. Then $\mathcal{K}_p(R)$ becomes a Fréchet space and $\mathcal{D} \hookrightarrow \mathcal{K}_p \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{E}$ are continuous; here $\mathcal{E}$ denotes the spaces of all C^∞-functions, $\mathcal{S}$ the space of rapidly decreasing functions and $\mathcal{D}$ the spaces of C^∞-functions with compact support. By $\mathcal{K}_p'(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_p(R)$. G. Sampson and Z. Zielezny characterized the distributions in $\mathcal{K}_p'(R)$ by the growth at infinity [5]; a distribution $T \in \mathcal{D}'$ is in $\mathcal{K}_p'(R)$ if and only if there exist positive integers α, k_0 and a bounded continuous function $f(x)$ on R such that

$$T = D^\alpha[e^{k_0|x|^p} f(x)]. \tag{2}$$

DEFINITION 1. *We denote by* $\mathcal{K}_p^r(R), p \geq 1$, *the space of all functions* $\phi \in C^r(R)$ *such that*

$$\nu_k^r(\phi) = \sup_{x \in R, \alpha \leq r} e^{k|x|^p} |D^\alpha \phi(x)| < \infty, \quad k = 1, 2, 3, \dots$$

The topology of $\mathcal{K}_p^r(R)$ is defined by the family of semi-norm $\{\nu_k^r\}_{k=1,2,\dots}$. By $\mathcal{K}_p^{r'}(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_p^r(R)$. Each $S \in \mathcal{K}_p^{r'}(R)$ is characterized by

$$S = D^r[e^{K_0|x|^p} f(x)],$$

where $f(x)$ is a bounded continuous function on R and $r, k_0 \in N$, by the same method of the proof of (2) in [5, Theorem 2]. Similarly, we can define

$$\mathcal{S}_r(R) = \{\theta(t) \in C^r(R); |\theta^{(k)}(t)| \leq C_{pk}(1 + |t|)^{-p}, p \in N; k = 0, 1, \dots, r.\}$$

and its dual $\mathcal{S}_r'(R)$. For further details, we refer to [5].

3. Multiresolution Analysis of $L^2(R)$ Associated with $\phi \in \mathcal{K}_p^r(R)$

Let $\phi \in \mathcal{K}_p^r(R)$. In order for it to qualify as a scaling function, there must be associated with ϕ a multiresolution analysis of $L^2(R)$, i.e., a nested sequence of closed subspaces $\{V_m\}_{m \in Z}$ such that

(i) $\{\phi(t - n)\}$ is an orthonormal basis of V_0

(ii) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(R)$

(iii) $f \in V_m \Leftrightarrow f(2\,\cdot) \in V_{m+1}$

(iv) $\bigcap_m V_m = \{0\}, \overline{\bigcup_m V_m} = L^2(R)$.

Then ϕ has an expansion

$$\phi(t) = \sum\nolimits_n c_n \sqrt{2} \phi(2t - n), \quad \{c_n\} \in l^2, \quad t \in R. \tag{3}$$

Once we have the scaling function $\phi \in \mathscr{K}_p^r(R)$, we can obtain mother wavelets $\psi(t)$ such that $\{\psi(t-n)\}$ is an orthogonal basis of the space W_0, given by the orthogonal complement of V_0 in V_1. Also, ψ has an expansion

$$\psi(t) = \sum_n d_n \sqrt{2}\phi(2t-n), \quad \{d_n\} \in l^2, \tag{4}$$

for d_n corresponding to c_n in (3). We will adopt the construction of a mother wavelets defined by $d_n = (-1)^n \overline{c_{1-n}}$. If such a $\psi(t)$ can be found, then $\psi_{nm}(t) = 2^{m/2}\psi(2^m t - n)$ is an orthogonal basis of W_m which is the orthogonal complement of V_m in V_{m+1}.

Example. In [1], Corollary 5.5.3 states that it is impossible that ψ has exponential decay and that $\psi \in C^\infty$, with all derivative bounded, unless $\psi = 0$. Hence there is no mother wavelet $\psi \in \mathscr{K}_p$. So we will restrict our attention to $\mathscr{K}_p^r$. Daubechies' compactly supported wavelets are example of $\mathscr{K}_p^r$ wavelets, but Battle-Lemarié's wavelets (in the page 152 of [1] or Example 4 in [6]) are not $\mathscr{K}_p^r$ wavelets even if they have exponential decay in ordinary sense and have smoothness. However, we did not succeed in constructing the $\mathscr{K}_p^r$ wavelets which are not Daubechies' one.

The reproducing kernel of V_0 is given by

$$q(x,t) = \sum_n \overline{\phi(x-n)}\phi(t-n),$$

where $\phi(x)$ is the scaling fuction. The series and its derivatives with respect to t of order $\le r$ converge uniformly on R because of the regularity of $\phi \in \mathscr{K}_p^r(R)$, i.e.

$$|\phi^{(\alpha)}(x)| \le C_{\alpha k} e^{-k|x|^p}, \quad \alpha = 0, 1, \dots, r; \quad k = 1, 2, \dots \tag{5}$$

We deduce the following properties [4, p33]:

(a) $q(x+k, t+k) = q(x,t)$ for all $k \in Z$.

(b)

$$\begin{aligned}
|q(x,t)| &\le \sum_j |\phi(x-j)|\,|\phi(t-j)| \\
&\le \sum_j c_{k+1}^2 e^{-(k+1)|x-j|^p} e^{-(k+1)|t-j|^p} \\
&\le \sum_j c_{k+1}^2 e^{-|x-j|^p} e^{-|t-j|^p} e^{-k2^{-p}|x-t|^p} \\
&\le c_k' e^{-k2^{-p}|x-t|^p}, \quad k = 1, 2, \dots
\end{aligned}$$

(c) $\int_{-\infty}^{\infty} q(x,t) t^\alpha \, dt = x^\alpha, 0 \le \alpha \le r$.

The reproducing kernel for V_m is given by

$$q_m(x,t) = 2^m q(2^m x, 2^m t).$$

Similarly, we can define the reproducing kernel $r_m(x,t)$ for W_m by

$$r_m(x,t) = 2^m \sum\nolimits_n \overline{\psi(2^m x - n)}\psi(2^m t - n),$$

where $\psi(t)$ is the mother wavelet.

Now, we will extend the expansion in orthogonal wavelets from $L^2(R)$ to $\mathscr{K}_p^r(R)$. Let $\phi \in \mathscr{K}_p^r(R)$. By the orthogonality of $\phi(x)$ and $\phi(2x-n)$ and the regularity (5) of $\phi \in \mathscr{K}_p^r(R)$,

$$|c_n| = \left| \int \phi(x)\overline{\phi(2x-n)}\, dx \right| \le C' e^{-k'|n|^p}, \quad k' = 1,2,\ldots$$

and

$$\begin{aligned} |\phi^{(\alpha)}(2t-n)| &\le C_{\alpha k''} e^{-k''|2t-n|^p} \\ &\le C_{\alpha k''} e^{-k''(|t|^p - |n|^p)}, \quad \alpha = 0,1,\ldots,r; \quad k'' = 1,2,\ldots, \end{aligned}$$

where we used that $|2t|^p = |2t-n+n|^p \le 2^p(|2t-n|^p + |n|^p)$. Since $d_n = (-1)^n \overline{c_{1-n}}$, $|d_n| = |(-1)^n \overline{c_{1-n}}| = |c_{1-n}| \le C' e^{-k'|1-n|^p}$. Hence, if we take k' which is sufficiently larger than k'',

$$\begin{aligned} \sum_n |d_n \sqrt{2}\phi^{(\alpha)}(2t-n)| &\le \sum_n \sqrt{2} C' C_{\alpha k''} e^{-k'|1-n|^p} e^{k''|n|^p} e^{-k''|t|^p} \\ &\le \sum_n C'_{\alpha k''} e^{-|n|^p} e^{-k''|t|^p} \\ &= C''_{\alpha k''} e^{-k''|t|^p}, \quad \alpha = 0,1,2,\ldots,r; \quad k'' = 1,2,\ldots \end{aligned}$$

Hence by (4), $\psi \in \mathscr{K}_p^r(R)$. Then expansion coefficients with respect to both $\{\phi(t-n)\}$ and $\{\psi(t-n)\}$ are well defined. Indeed, since $f \in \mathscr{K}_p^{r'}(R)$ is also characterized by

$$f = D^r\{e^{k_0|x|^p}\mu\}, \quad k_0 \in N,$$

for a bounded measure μ on R, coefficients a_n may be found which satisfy

$$\begin{aligned} a_n &= (f, \phi(\cdot - n)) \\ &= (D^r(e^{k_0|t|^p}\mu), \phi(\cdot - n)) \\ &= \int_{-\infty}^{\infty} e^{k_0|t|^p}(-1)^r \phi^{(r)}(t-n)\, d\mu. \end{aligned}$$

Hence

$$|a_n| \leq \int_{-\infty}^{\infty} e^{2^p k_0 |t-n|^p} e^{2^p k_0 |n|^p} |\phi^{(r)}(t-n)| d|\mu|$$
$$= \mathcal{O}(e^{k_1 |n|^p}). \tag{6}$$

Similarly, $b_n = (f, \psi(t-n))$ satisfies the same kind of growth condition. Then, since

$$\sum_n |a_n \phi^{(j)}(t-n)| \leq C \sum_n e^{k_1 |n|^p} e^{-2^p (k_1+1)|t-n|^p}$$
$$\leq C \sum_n e^{-2^p |t-n|^p} e^{2^p k_1 |t|^p}$$
$$\leq C' e^{k_2 |t|^p},$$

$\sum_n a_n \phi(t-n)$ converges uniformly on bounded sets as do its first r derivatives. In fact, we have shown that the limiting function and its r derivatives are continuous functions of $e^{k_0|x|^p}$ growth. These results enable us to imitate the multiresolution analysis of $L^2(R)$ in $\mathcal{K}_p^{r'}(R)$.

Definition 2. *Let $\{a_n\}$ be a sequence of complex numbers with $a_n = \mathcal{O}(e^{k_1|n|^p})$ for some $k_1 \in N$; then $T_0 = \{f : f(t) = \sum_n a_n \phi(t-n)\}$ and $U_0 = \{g : g(t) = \sum_n a_n \psi(t-n)\}$. We denote by T_m and U_m their corresponding dilation spaces, i.e., $f \in T_0 \Leftrightarrow f(2^m t) \in T_m$ and $g \in U_0 \Leftrightarrow g(2^m t) \in U_m$.*

Then, we may expect that a multiresolution analysis of $\mathcal{K}_p^{r'}(R)$ exists, namely,

$$\cdots \subset T_{-m} \cdots \subset T_{-1} \subset T_0 \subset T_1 \cdots \subset T_m \subset \cdots \subset \mathcal{K}_p^{r'}(R) \tag{7}$$

and

$$\overline{\bigcup_m T_m} = \mathcal{K}_p^{r'}(R),$$

where the closure is in the topology of $\mathcal{K}_p^{r'}(R)$.

Theorem 3. *Let the scaling function $\phi \in \mathcal{K}_p^r(R)$ satisfy a dilation equation (3) with $c_k = \mathcal{O}(e^{-l|k|^p})$ for all $l \in N$, and let ϕ have an associated multiresolution analysis in $L^2(R)$; let $\psi \in \mathcal{K}_p^r(R)$ be the mother wavelet given in (4). Then there exists a multiresolution analysis (7) of closed dilation subspaces $\{T_m\}$ whose union is dense in $\mathcal{K}_p^{r'}(R)$; the closed subspaces U_m of Definition 2 are complementary* *subspaces of* T_m *in* T_{m+1} *and*

$$T_m = U_0 \oplus U_1 \oplus \cdots \oplus U_m \oplus T_0,$$

where $\oplus$ *denotes the nonorthogonal direct sum.*

PROOF. We will only prove that T_0 is closed in the sense of $\mathscr{K}_p^{r'}(R)$ and $\overline{\bigcup_m T_m} = \mathscr{K}_p^{r'}(R)$. The other statements are the same as in the case of $\mathscr{S}_r'(R)$ [6]. It is clear that $f_m \to 0$ in $\mathscr{K}_p^{r'}(R)$ is equivalent to

$$f_m = D^r[e^{k_0|x|^p} v_m], r, k_0 \in N; \quad \int d|v_m| \to 0,$$

where $\{v_m\}$ is a sequence of bounded measures on R. If $f_m \in T_0$, then we have that $a_{nm} = (f_m, \phi(\cdot - n)) \to 0$ as $m \to \infty$ and $|a_{nm}| \leq ce^{k_1|n|^p}$ by (6). Hence if $f_m \to f$ in $\mathscr{K}_p^{r'}(R)$, the coefficient of f, a_n, is of $\mathcal{O}(e^{k_1|n|^p})$ and hence its series $\sum_n a_n\phi(t-n) = f \in T_0$. Thus T_0 is closed in the sense of $\mathscr{K}_p^{r'}(R)$. Now by the facts that $\mathscr{S}_r'(R)$ is dense in $\mathscr{K}_p^{r'}(R)$, $L^2 = \overline{\bigcup_m V_m}$ is dense in $\mathscr{S}_r'(R)$ and $\bigcup_m V_m \subset \bigcup_m T_m$, $\bigcup_m T_m$ is dense in $\mathscr{K}_p^{r'}(R)$.

REMARK. As in the case of $\mathscr{S}_r'(R)$ [6], the property $\bigcap_m V_m = \{0\}$ of the usual multiresolution analysis is lacking. By the moment property of the reproducing kernel $q_m(x, t)$, any polynomial of degree $\leq r$ belongs to $\bigcap_m T_m$.

4. Convergence of the Expansions of $\mathscr{K}_p^{r'}(R)$

A *quasi-positive delta sequence* is a sequence $\{\delta_m(\cdot, y)\}$ of functions in $L^1(R)$ with parameter $y \in R$ which satisfies the following:

(a) there is a $C > 0$ such that

$$\int_{-\infty}^{\infty} |\delta_m(x, y)|\, dx \leq C, \quad y \in R, m \in N;$$

(b) there is a $c > 0$ such that

$$\int_{y-c}^{y+c} \delta_m(x, y)\, dx \to 1$$

uniformly on compact subsets of R, as $m \to \infty$;

(c) for each $\gamma > 0$,

$$\sup_{|x-y| \leq \gamma} |\delta_m(x, y)| \to 0 \quad \text{as } m \to \infty.$$

Then since $\mathcal{K}_p^r(R) \subset \mathcal{S}_r(R)$, we have following important two Lemmas as in [7]:

LEMMA 4. *Let $\{\delta_m(x, y)\}$ be a quasi-positive delta sequence and let $f \in L^1(R)$ be continuous on (a, b); then*

$$f_m(y) = \int_{-\infty}^{\infty} \delta_m(x, y) f(x)\, dx \to f(y) \quad \text{as } m \to \infty$$

uniformly on compact subsets of (a, b).

LEMMA 5. *If $q_m(x, y)$ is the reproducing kernel of V_m, $\phi \in \mathcal{K}_p^r(R)$, then $q_m(x, y)$ and $K_m(x, t) = ((x - t)/\alpha!)(\partial^\alpha/\partial t^\alpha) q_m(x, t)$ for $\alpha \in N, 0 \le \alpha \le r$, are quasi-positive delta sequences on R.*

By the remark in section 3, the pure wavelet series of $f \in \mathcal{K}_p^{r'}(R)$, $\sum_{n,m} b_{mn}\psi_{mn}(t)$ does not necessarily converge to f. However, the mixed expansion

$$f = \sum\nolimits_n a_n \phi(\cdot - n) + \sum\nolimits_{m=0}^{\infty} \sum\nolimits_{n=-\infty}^{\infty} b_{mn}\psi_{mn},$$

converges to f in the sense of $\mathcal{K}_p^{r'}(R)$. This global convergence of the expansion of $\mathcal{K}_p^{r'}(R)$ is important for theoretical purposes. For computational purposes, we will study some sort of local convergence in the sense of S. Lojasiewicz [3].

DEFINITION 6. *Let $f \in \mathcal{K}_p'(R)$. f is said to have a value γ of order α at x_0 if there exists a continuous function $F(x)$ of exponential growth of $e^{k_0|x|^p}$ for some $k_0 \in N$ such that $D^\alpha F = f$ in some neighborhood of x_0 and*

$$\lim_{x \to x_0} \frac{F(x)}{(x - x_0)^\alpha} = \frac{\gamma}{\alpha!}.$$

THEOREM 7. *Let $f \in \mathcal{K}_p^{r'}(R)$ and have a value γ of order $\alpha \le r$ at $x = x_0$. Then the function f_m given by $f_m(x) = (f(\cdot), q_m(x, \cdot))$ satisfies*

$$f_m(x_0) \to \gamma \quad \text{as } m \to \infty$$

PROOF. Although each $f \in \mathcal{K}_p^r(R)$ is a derivative of order $\beta \le r$ of a continuous function G of exponential growth, we may obtain that $G = F$ and $\alpha = \beta$, where F is in the definition 6. In fact, if we take $G(x) = (x - x_0)^{\beta-\alpha}F(x)$ when $\beta \ge \alpha$ and $G(x) = (x - x_0)^{\alpha-\beta}F(x)$ when $\alpha > \beta$, then $\lim_{x\to x_0} F(x)/(x - x_0)^\alpha = \lim_{x\to x_0} G(x)/(x - x_0)^\beta$ and $\lim_{x\to x_0} F(x)/(x - x_0)^\beta = \lim_{x\to x_0} G(x)/(x - x_0)^\alpha$, respectively. We may assume $\alpha > \beta$.

Using integration by parts, for some $A > 0$,

$$\begin{aligned} f_m(x) &= \int_{-\infty}^{\infty} (-1)^{\alpha} \partial_y^{\alpha} q_m(x, y) F(y)\, dy \\ &= \int_{-\infty}^{\infty} \frac{(x-y)^{\alpha}}{\alpha!} \partial_y^{\alpha} q_m(x, y) \frac{F(y)\alpha!}{(y-x)^{\alpha}}\, dy \\ &= \int_{x-A}^{x+A} + \int_{x+A}^{\infty} + \int_{-\infty}^{x-A} . \end{aligned}$$

Now, we claim that

$$\int_{x+A}^{\infty} + \int_{-\infty}^{x-A} \to 0 \quad \text{as } m \to \infty$$

First,

$$\begin{aligned} &\int_{x+A}^{\infty} \frac{(x-y)^{\alpha}}{\alpha!} \partial_y^{\alpha} q_m(x, y) \frac{F(y)\alpha!}{(y-x)^{\alpha}}\, dy \\ &= \sum_n \sum_{k=0}^{\alpha} (2^m x - n)^k \binom{\alpha}{k} \overline{\phi(2^m x - n)} \\ &\quad \times \int_{2^m(x+A)}^{\infty} \frac{\phi^{(\alpha)}(y-n)}{\alpha!} (n-y)^{\alpha-k} \frac{2^{m\alpha} F(y/2^m)\alpha!}{(y-2^m x)^{\alpha}}\, dy. \end{aligned}$$

If we denote by I the last integral above, then

$$\begin{aligned} I &= \left| \int_{2^m(x+A)}^{\infty} \phi^{(\alpha)}(y-n)(y-n)^{(\alpha-k)} \frac{2^{m\alpha} F(y/2^m)}{(y-2^m x)^{\alpha}}\, dy \right| \\ &\le \int_{2^m(x+A)}^{\infty} C_{\alpha,j} e^{-j|y-n|^p} |y-n|^{(\alpha-k)} \cdot e^{k_0 |y/2^m|^p}\, dy \\ &\le \int_{2^m(x+A)}^{\infty} C_{\alpha,j} e^{-j|y-n|^p} |y-n|^{(\alpha-k)} e^{2^p k_0 |y-n|^p} e^{2^p k_0 |n|^p}\, dy \\ &\le \int_{2^m(x+A)}^{\infty} C_{\alpha,j} e^{-j|y-n|^p} e^{c_{\alpha,k}|y-n|^p} e^{2^p k_0 |y-n|^p + 2^p k_0 |n|^p} e^{2|y|^p - 2|y|^p}\, dy \\ &\le C_{\alpha,j} e^{(2^p k_0 + 2^{(p+1)})|n|^p} e^{(-j + c_{\alpha,k} + 2^p k_0 + 2^{(p+1)})|2^m(x+A)-n|^p} \int_{2^m(x+A)}^{\infty} e^{-2|y|^p}\, dy \\ &= C'_{\alpha,j} e^{(2^p k_0 + 2^{(p+1)})|n|^p} e^{(-j + c_{\alpha,k} + 2^p k_0 + 2^{(p+1)})|2^m(x+A)-n|^p}, \end{aligned}$$

for $j > c_{\alpha,k} + 2^p k_0 + 2^{(p+1)}$.

Hence

$$\left|\int_{x+A}^{\infty} \frac{(x-y)^\alpha}{\alpha!}\partial_y^\alpha q_m(x,y)\frac{F(y)\alpha!}{(y-x)^\alpha}dy\right|$$
$$\leq \frac{C'_{\alpha,l,l'}}{2^m}\sum\nolimits_n \frac{e^{k_3|n|^p}}{e^{l|2^m x-n|^p}e^{l'|2^m(x+A)-n|^p}}, \tag{8}$$

for $l=1,2,3,\ldots$ and sufficiently large l'. In the inequality $2^{-p}|a+b|^p-|b|^p\leq |a|^p$, substitute a by $a-b$, we have $-|a-b|^p\leq |b|^p-2^{-p}|a|^p$. Let us estimate the exponent of the term of the summation in (8). Assume that $l-l'>0$. Then we have

$$\begin{aligned}
&k_3|n|^p - l|2^m x-n|^p - l'|2^m A+2^m x-n|^p\\
&\quad\leq k_3|n|^p-(l-l')|2^m x-n|^p-2^{-p}l'|2^m A|^p\\
&\quad\leq (k_3-2^{-p}(l-l'))|n|^p+(l-l')|2^m x|^p-2^{-p}l'|2^m A|^p.
\end{aligned}$$

Take l satisfying $k_3-2^{-p}(l-l')=-1$. Then $l-l'=2^p(k_3+1)$, whose right-hand side is a constant. Take l' such that $2^{-p}l'|A|^p>2^p(k_3+1)|x|^p$. Then the right-hand side of (8) is estimated by

$$\leq C_{\alpha,l,l'}e^{-(2^{-p}l'|A|^p-2^p(k_3+1)|x|^p)2^{mp}}\sum_n e^{-|n|^p}$$
$$= C'_{\alpha,l,l'}e^{-(2^{-p}l'|A|^p-2^p(k_3+1)|x|^p)2^{mp}}$$

The same method for the estimation of $|\int_{-\infty}^{x-A}|$ induces

$$\left|\int_{x+A}^{\infty}\right|+\left|\int_{-\infty}^{x-A}\right|\leq C''_{\alpha,l,l'}e^{-(2^{-p}l'|A|^p-2^p(k_3+1)|x|^p)2^{mp}}$$

Hence

$$\int_{x+A}^{\infty}+\int_{-\infty}^{x-A}\to 0 \quad \text{as } m\to\infty.$$

Thus we can express f_m as

$$f_m(x)=\int_{-\infty}^{\infty}K_m(x,y)F_A(x,y)\,dy+o(1),$$

where $K_m(x,y)=((x-y)^\alpha/\alpha!)(\partial^\alpha/\partial y)q_m(x,y)$ and $F_A(x,y)$ is continuous for all y except for $y=x\pm A$ and has a compact support. Hence F_A is bounded. Since

$K_m(x, y)$ is a quasi-positive delta sequence by Lemma 5, Lemma 4 implies that

$$f_m(x_0) \to F_A(x_0, x_0) = \gamma \quad \text{as } m \to \infty.$$

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