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PARTIALLY LOCALLY ATOMIC MODELS

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1. Introduction

Various atomic models are useful tools for stability theory. We know the following about atomic models.

1) If T is ω-stable then for every set there is an atomic model over it.

2) If T is superstable then for every set there is an a-atomic model over it.

3) If T is stable then for every set there is a locally atomic model over it.

In the present paper we introduce a condition on a theory called partial stability (see definition 1), which is a generalization of stability. We also define the notion of a partially locally atomic model (see definition 5). Then we have

THEOREM 6. *If T is partially stable (stable in P), then there is a partially locally atomic model (locally atomic with respect to P) over any countable set in P.*

2. Preliminaries

Let T be a countable theory in a language L and $P(x)$ an L-formula with only one free variable. As usual, we work in the big model $\mathscr{M}$ of T. M denotes an elementary submodel of $\mathscr{M}$. We write $A, B, \ldots$ for subsets of $\mathscr{M}$. λ denotes an infinite cardinal.

We write P^A for $\{a \in A : \mathscr{M} \models P(a)\}$. We abbreviate $P(x_1) \wedge \cdots \wedge P(x_n)$ as $P(\bar{x})$ where $\bar{x} = x_1 \ldots x_n$. We abbreviate $(\forall \bar{x})[P(\bar{x}) \rightarrow \varphi(\bar{x}\bar{y})]$ as $(\forall \bar{x} \in P)\varphi(\bar{x}\bar{y})$. Let $p(\bar{x})$ be a type (could be partial). We write $p_{\varphi(\bar{x}\bar{y})}(\bar{x})$ for the restriction of $p(\bar{x})$ to $\{\varphi(\bar{x}\bar{y}), \neg\varphi(\bar{x}\bar{y})\}$ and $S_{\varphi(\bar{x}\bar{y})}(A)$ for $\{p_{\varphi(\bar{x}\bar{y})}(\bar{x}) : p(\bar{x}) \in S(A)\}$. We write $p_{\varphi(\bar{x}\bar{y})}$ as p_φ and $S_{\varphi(\bar{x}\bar{y})}$ as S_φ, if there is no confusion.

We define two partial stablity conditions and show related lemmas.

DEFINITION 1. (1) *T is λ-stable in P if for all A, $|A| \leq \lambda$ implies $|\{p(\bar{x}) \in S(A) : P(\bar{x}) \in p(\bar{x})\}| \leq \lambda$. T is stable in P if T is λ-stable in P for some λ.*

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2) *T is λ-stable over P if for all A, $|A| \leq \lambda$ and $A \subset P^{\mathscr{M}}$ imply $|S(A)| \leq \lambda$. T is stable over P if T is λ-stable over P for some λ.*

FACT 2 (The unstable formula Theorem [2] II.2.2). *The following properties of a formula $\varphi(\bar{x}\bar{y})$ are equivalent:*

(1) *$\varphi(\bar{x}\bar{y})$ is unstable;*

i.e., for every $\lambda \geq \aleph_0$, there is a set A such that $|S_\varphi(A)| > \lambda \geq |A|$.

(2) *$\varphi(\bar{x}\bar{y})$ has the order property;*

i.e., $\{\varphi(\bar{x}_i\bar{y}_j) : i < j < \omega\} \cup \{\neg\varphi(\bar{x}_i\bar{y}_j) : j \leq i < \omega\}$ is consistent.

(3) *There are a set A and a type $p \in S_\varphi(A)$ such that p is not A-definable.*

LEMMA 3. *T is stable in P if and only if T is stable over P.*

PROOF. T is unstable in P

$\Leftrightarrow$ There is a formula $\varphi(\bar{x}\bar{y})$ such that $\varphi(\bar{x}\bar{y}) \wedge P(\bar{x})$ is unstable

Let $\psi(\bar{x}\bar{y}) = \varphi(\bar{x}\bar{y}) \wedge P(\bar{x})$

$\Leftrightarrow$ $\psi(\bar{x}\bar{y})$ (parameters are in $\bar{y}$) has the order property (By Fact 2)

$\Leftrightarrow$ $\psi(\bar{y}\bar{x})$ (parameters are in $\bar{y}$) has the order property

$\Leftrightarrow$ $\psi(\bar{y}\bar{x})$ is unstable (By Fact 2)

$\Leftrightarrow$ T is unstable over P. □

LEMMA 4. *Let T be stable in P. Then for each L-formula $\varphi(\bar{x}\bar{y})$, for each tuple $\bar{a} \in \mathscr{M}$ and for each set A, there is an $L(P^A)$-formula $\psi(\bar{x})$ such that $\mathscr{M} \models [\varphi(\bar{b}\bar{a}) \leftrightarrow \psi(\bar{b})]$ for all $\bar{b} \in P^A$. If $A = M$ and $\bar{a} \in M$ then $M \models \forall \bar{x} \in P[\varphi(\bar{x}\bar{a}) \leftrightarrow \psi(\bar{x})]$.*

PROOF. Since T is stable in P, $\varphi(\bar{x}\bar{y}) \wedge P(\bar{x})$ is stable. By the similar argument in Lemma 3, $\varphi(\bar{y}\bar{x}) \wedge P(\bar{y})$ also does not have the order property. By (3) $\Leftrightarrow$ (2) of Fact 2, $p(\bar{x}) = tp_{\varphi(\bar{y}\bar{x}) \wedge P(\bar{y})}(\bar{a}/P^A)$ is P^A-definable. Hence there is an $L(P^A)$-formula $\psi(\bar{y})$ such that for all $\bar{b} \in P^A$, $\varphi(\bar{b}\bar{x}) \wedge P(\bar{b}) \in p(\bar{x})$ if and only if $\mathscr{M} \models \psi(\bar{b})$. Hence $\mathscr{M} \models \varphi(\bar{b}\bar{a}) \leftrightarrow \psi(\bar{b})$ for all $\bar{b} \in P^A$. □

3. Main Theorem

First recall the definitions concerning local atomicity. Let $p(\bar{x})$ be a type over A. The type $p(\bar{x})$ is locally isolated if for each L-formula $\theta(\bar{x}\bar{y})$, there is an $L(A)$-formula $\varphi(\bar{x}) \in p(\bar{x})$ such that $\varphi(\bar{x}) \vdash p_{\theta(\bar{x}\bar{y})}(\bar{x})$. Let $B \subset A$. The set A is said to be locally atomic over B if $tp(\bar{a}/B)$ is locally isolated for any $\bar{a} \in A$. We now define the notion of local atomicity with repsect to P.

DEFINITION 5. *Let $B \subset A$. A is locally atomic over B with respect to P if P^A is locally atomic over P^B.*

The next theorem is our main theorem. From now on, by a subset of $P^{\mathscr{M}}$, we mean a subset of $(P^{\mathscr{M}})^n$ for some natural number n if the value of n is irrelevant for the augument.

THEOREM 6. *Let T be stable in P and A a countable set. If every $L(A)$-definable subset of $P^{\mathscr{M}}$ is $L(P^A)$-definable then there is a countable model $M \supset A$ such that M is locally atomic over A with respect to P.*

To prove Theorem 6 we need a fact and a lemma.

FACT 7 (Transitivity of local atomicity [1] IX. 5.7 Lemma). *If $A \subset B \subset C$, C is locally atomic over B and B is locally atomic over A then C is locally atomic over A.*

LEMMA 8. *Let T be stable in P, B a countable set and $\varphi(\bar{x}) \in L(B)$. If every $L(B)$-definable subset of $P^{\mathscr{M}}$ is $L(P^B)$-definable then there is a countable set $C \supset B$ such that:*

(1) *φ has a realization in C;*

(2) *C is locally atomic over B with respect to P;*

(3) *Every $L(C)$-definable subset of $P^{\mathscr{M}}$ is $L(P^C)$-definable.*

PROOF OF LEMMA. Since T is stable in P, $R(P(\bar{x}), \theta(\bar{x}\bar{y}), 2) < \omega$ for any L-formula $\theta(\bar{x}\bar{y})$. So we can choose a type containing $\psi(\bar{x}) \wedge P(\bar{x})$ and is locally isolated for every formula $\psi(\bar{x})$ which is consistent with $P(\bar{x})$.

By the consistency of $\varphi(\bar{x}) \wedge P(\bar{x})$, we consider two cases.

(Case 1) $\varphi(\bar{x}) \wedge P(\bar{x})$ is consistent.

This is the easier case. We can choose a tuple $\bar{a} \in P^{\mathscr{M}}$ such that $\varphi(\bar{x}) \wedge P(\bar{x}) \in tp(\bar{a}/B)$ and $tp(\bar{a}/B)$ is locally isolated. So for every L-formula $\theta(\bar{x}\bar{y})$, $tp_\theta(\bar{a}/P^B)$ is isolated by an $L(B)$-formula which is in $tp(\bar{a}/B)$. Since every $L(B)$-definable subset of $P^{\mathscr{M}}$ is $L(P^B)$-definable, $tp_\theta(\bar{a}/P^B)$ is isolated by an $L(P^B)$-formula which is in $tp(\bar{a}/B)$. So $tp(\bar{a}/P^B)$ is also locally isolated. Let $C = B\bar{a}$.

By the construction of the set C, C satisfies (1) and (2). We show that C satisfies (3). Let $\psi(\bar{x}\bar{y})$ be an L-formula and $\bar{c} \in C$. Suppose that $\psi(\bar{x}\bar{c})$ defines a subset of $P^{\mathscr{M}}$. And suppose that $\bar{c} = \bar{a}'\hat{\ }\bar{b}$ for some $\bar{a}' \subset \bar{a}$ and $\bar{b} \in B$ and $\psi(\bar{x}\bar{a}'\bar{b})$ defines a subset of $P^{\mathscr{M}}$. Since every $L(B)$-definable subset of $P^{\mathscr{M}}$ is $L(P^B)$-definable, there is an $L(P^B)$-formula $\theta(\bar{x}\bar{y}')$ such that $\mathscr{M} \models \forall \bar{x}\bar{y}' \in P[\psi(\bar{x}\bar{y}'\bar{b}) \leftrightarrow \theta(\bar{x}\bar{y}')]$. Since $\bar{a}' \in P^C$, $\mathscr{M} \models \forall \bar{x} \in P[\psi(\bar{x}\bar{a}'\bar{b}) \leftrightarrow \theta(\bar{x}\bar{a}')]$. Thus the set C satisfies (3).

(Case 2) $\varphi(\bar{x}) \wedge P(\bar{x})$ is inconsistent.
This is the harder one. We construct a set C and a realization $\bar{a}$ of $\varphi(\bar{x})$ so that every $L(B\bar{a})$-definable subset of $P^{\mathscr{M}}$ is $L(P^C)$-definable. Since B is countable, let $\{\psi_i(\bar{x}\bar{y}_i\bar{b}_i) : i < \omega\}$ be an enumeration of all consistent $L(B)$-formulas. By induction on i, we construct a sequence $\bar{c}_0, \bar{c}_1, \ldots$, and L-formulas $\theta_i(\bar{y}_i\bar{z}_i)$ $(i < \omega)$ satisfying the following properties:

(i) $C_i = B \cup \{\bar{c}_j : j < i\}$;

(ii) C_{i+1} is locally atomic over C_i with respect to P;

(iii) $\bar{c}_i$ is a realization $Q_i(\bar{z}_i)$ where

$$Q_i(\bar{z}_i) = P(\bar{z}_i) \wedge \exists \bar{x}\Bigg[\varphi(\bar{x}) \wedge \bigwedge_{j<i} \forall \bar{y}_j \in P[\psi_j(\bar{x}\bar{y}_j\bar{b}_j) \leftrightarrow \theta_j(\bar{y}_j\bar{c}_j)]$$

$$\wedge \forall \bar{y}_i \in P[\psi_i(\bar{x}\bar{y}_i\bar{b}_i) \leftrightarrow \theta_i(\bar{y}_i\bar{z}_i)]\Bigg].$$

Suppose that $\bar{c}_j$ and θ_j are defined for $j < i$. Let $\bar{a}_i$ be a realization of $R_i(\bar{x})$ where

$$R_i(\bar{x}) = \varphi(\bar{x}) \wedge \bigwedge_{j<i} \forall \bar{y}_j \in P[\psi_j(\bar{x}\bar{y}_j\bar{b}_j) \leftrightarrow \theta_j(\bar{y}_j\bar{c}_j)].$$

By applying Lemma 4 to $\psi_i(\bar{a}_i\bar{y}_i\bar{b}_i)$, we can choose an L-formula $\theta_i(\bar{y}_i\bar{z}_i)$ and $\bar{c} \in P^{\mathscr{M}}$ such that $\mathscr{M} \models \forall \bar{y}_i \in P[\psi_i(\bar{a}_i\bar{y}_i\bar{b}_i) \leftrightarrow \theta_i(\bar{y}_i\bar{c})]$. Hence we get an L-formula $\theta_i(\bar{y}_i\bar{z}_i)$ such that $Q_i(\bar{z}_i)$ is consistent. Then we can choose a type $p(\bar{z}_i) \in S(C_i)$ which contains $Q_i(\bar{z}_i)$ and is locally isolated. Let $\bar{c}_i$ be a realization of $p(\bar{z}_i)$. Thus $\bar{c}_i$ and θ_i are defined. Then

$$\Psi(\bar{x}) = \{\varphi(\bar{x})\} \cup \{\forall \bar{y}_i \in P[\psi_i(\bar{x}\bar{y}_i\bar{b}_i) \leftrightarrow \theta_i(\bar{y}_i\bar{c}_i)] : i < \omega\}$$

is consistent. Let $\bar{a}$ be a realization of $\Psi(\bar{x})$. Put $C = \bigcup_{i<\omega} C_i \cup \bar{a}$.

The set C clearly satisfies (1). Indeed $tp(\bar{c}_i/P^{C_i})$ is also locally isolated for all $i < \omega$. So C satisfies (2) by Fact 7. To show that C satisfies (3), we may assume the following without loss of generality. Let $\bar{c} \in C$ and $\psi(\bar{x}\bar{y})$ an L-formula.

Suppose that $\bar{c} = \bar{a}'\hat{}\bar{b}\hat{}\bar{c}'$ for some $\bar{a}' \subset \bar{a}$, $\bar{b} \in B$ and $\bar{c}' \in P^C \backslash P^B$ and suppose that $\psi(\bar{x}\bar{a}'\bar{b}\bar{c}')$ defines a subset of $P^{\mathscr{M}}$. It suffices to show that $\psi(\bar{x}\bar{a}'\bar{b}\bar{c}')$ is equivalent to an $L(P^C)$-formula. By the choice of $\{\psi_i(\bar{x}_i\bar{y}_i\bar{b}_i) : i < \omega\}$ and $\bar{a}$, every $L(B\bar{a})$-definable subset of $P^{\mathscr{M}}$ is $L(P^C)$-definable. Then for the $L(B\bar{a})$-formula $\psi(\bar{x}\bar{a}'\bar{b}\bar{y}')$, there is an $L(P^C)$-formula $\theta(\bar{x}\bar{y}')$ such that $\mathscr{M} \models \forall\bar{x}\bar{y}' \in P[\psi(\bar{x}\bar{a}'\bar{b}\bar{y}') \leftrightarrow \theta(\bar{x}\bar{y}')]$. Since $\bar{c}' \in P^C$, $\mathscr{M} \models \forall\bar{x} \in P[\psi(\bar{x}\bar{a}'\bar{b}\bar{c}') \leftrightarrow \theta(\bar{x}\bar{c}')]$. So C satisfies (3). □

With Fact 7 and Lemma 8, we now prove Theorem 6.

Proof of Theorem 6. We construct countable sets A_i $(i < \omega)$ by induction on i.

(Step 0) $A_0 = A$

(Step i) Suppose that A_i is defined. We construct countable sets B_i^j $(i, j < \omega)$ by induction on j.

(Substep 0) Let $B_i^0 = A_i$ and $\{\varphi_i^j(x) \in L(A_i) : j < \omega\}$ be an enumeration of consistent $L(A_i)$-formulas.

(Substep j) Suppose that B_i^j is defined.

(Case 1 of Substep j) B_i^j has a realization of $\varphi_i^j(x)$.
Put $B_i^{j+1} = B_i^j$

(Case 2 of Substep j) B_i^j has no realization of $\varphi_i^j(x)$.
By Lemma 8, there is a set $C \supset B_i^j$ with following properties:
(i) $\varphi_i^j(x)$ has a realization in C;
(ii) C is locally atomic over B_i^j with respect to P;
(iii) Every $L(C)$-definable subset of $P^{\mathscr{M}}$ is $L(P^C)$-definable.
Put $B_i^{j+1} = C$.
Thus $B_i^j (j < \omega)$ are defined. Let $A_{i+1} = \bigcup_{j<\omega} B_i^j$. This completes Step i.

Put $M = \bigcup_{i<\omega} A_i$. By the construction of M, it is easy to show that M is a countable model of T by Tarski-Vaught test. By Fact 7 and the construction of A_{i+1}, $P^{A_{i+1}}$ is locally atomic over P^{A_i}. Then again by Fact 7, P^M is locally atomic over P^A. □

Corollary 9. *If T is stable in P and not ω-stable in P, then there is a model M of cardinality $\aleph_1$ which has no indiscernible set of cardinality $\aleph_1$ in P^M.*

SKETCH OF PROOF. This augument is a simple generalization of Theorem A in [3]. We use partially locally atomic models instead of locally atomic models. Start with a certain countable set. By Theorem 6, we can construct an increasing chain of length ω_1 consisting of M_i's such that each M_{i+1} is countable and locally atomic over $M_i\bar{a}_i$ with respect to P where $\bar{a}_i$ is a suitable tuple. Then the union of this chain is a model we have been looking for. □

Question. Is the countability condition on A in Theorem 6 necessary?

References

[1] J. T. Baldwin. Fundamentals of Stability Theory. Springer-Verlag 1988.

[2] S. Shelah. Classification Theory and the Number of Non-isomorphic models. North-Holland 1978.

[3] A. Tsuboi. Large indiscernible sets of a Structure. Kobe J. Math., **10** (1993) 173–178.

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