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ON TRIPLE COVERINGS OF IRRATIONAL CURVES

Dedicated to Professor Robert D. M. Accola on the occasion of his retirement

By

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Abstract. Given a triple covering X of genus g of a general (in the sense of Brill-Noether) curve C of genus h, we show the existence of base-point-free pencils of degree d which are not composed with the triple covering for any $d \geq g - [(3h+1)/2] - 1$ by utilizing some enumerative methods and computations. We also discuss about the sharpness of our main result and the so-called Castelnuovo-Severi bound by exhibiting some examples.

0. Introduction

In this paper, we investigate the problem of the existence of base-point-free pencils of relatively low degree on a triple covering X of genus g of a general curve C of genus $h > 0$. Such a problem is classical and the picture is rather well known for the degree range close to the genus g. On the other hand, by a simple application of the Castelnuovo-Severi inequality one can easily see that there does not exist a base-point-free pencil of degree less than or equal to $(g-3h)/2$ other than the pull-backs from the base curve C; while for the degree beyond this range not many things have been known about the existence of such a pencil which is not composed with the given triple covering. The main result of this paper is the following.

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THEOREM A. *Let X be a smooth algebraic curve of genus g, over an algebraically closed field k with char$(k)=0$, which admits a three sheeted covering onto a general curve C of genus $h \geq 1$, $g \geq (2[(3h+1)/2]+1)([(3h+1)/2]+1)$. Then there exists a base-point-free pencil of any degree $d \geq g-[(3h+1)/2]-1$ which is not composed with the given triple covering.*

This paper is organized as follows. In § 1, we prove Theorem A by using the enumerative methods and computations in $H^*(X_\alpha, Q)$ of various sub-loci of the symmetric product X_α of the given triple covering X, while we defer proving the key lemma which are necessary to prove Theorem A. Specifically we compare the fundamental class of $X_\alpha^1 := \{D \in X_\alpha : \dim |D| \geq 1\}$ with the class of all irreducible components of X_α^1 whose general elements correspond to pencils on X with base points. This argument works because the latter components are all induced from the base curve C and X_α^1 has the expected dimension. In § 2, we proceed to prove Lemma 2.1. In proving Lemma 2.1, we need to utilize and carry out some computations based on the fundamental result of Rick Miranda about the triple covering of algebraic varieties in general [M]. In the last section we discuss about the sharpness of our main result and the so-called Castelnuovo-Severi bound by considering some examples.

1. Existence of base-point-free pencils on triple coverings

The purpose of this section is to prove Theorem A. We begin with the following dimension theoretic statements about the variety of special linear systems on an algebraic curve; cf. [CKM] Corollary 3.1.2.

LEMMA 1.1. *Let X be an algebraic curve of genus g. Let $n \in N$, $g \geq (2n-3)(n-1)$ (and if $n \leq 2$ let $g \geq 2n-1$). Assume that* $\dim W_{n+1}^1(X) < 1$. *Then $W_d^1(X)$ is equi-dimensional of the minimal possible dimension $\rho(d,g,1) := 2d-2-g$ for all d such that $g-n < d \leq g$.*

The following lemma is a special case of the Castelnuovo-Severi bound; for the proof, see [ACGH] p. 366 Exercise C-1 or [A], p. 21.

LEMMA 1.2 (Castelnuovo-Severi bound). *Let X be a smooth algebraic curve of genus g which admits a triple covering onto a curve C of genus h. Let $\pi : X \to C$ be the triple covering. For any integer $n \leq (g-3h)/2$, a base-point-free pencil g_n^1 (possibly incomplete) is composed with π, or equivalently the morphism $X \to P^1$ induced by the g_n^1 always factors through π.*

Proof of Theorem A. Let $n = [(3h+1)/2] + 2$. Then $n + 1 = [(3h+1)/2] + 3 \leq (g - 3h)/2$ by the numerical hypothesis on the genus g of X. Assume that there exists a g^1_{n+1} on X. By Lemma 1.2, it follows that every g^1_{n+1} is composed with π and

$$W^1_{n+1}(X) = \pi^* W^1_m(C) + W_{n+1-3m}(X), \tag{A.1}$$

for some m with $n + 1 - 3m \geq 0$. On the other hand, since C is general

$$\dim W^1_m(C) = \dim \pi^* W^1_m(C) = \rho(m, h, 1) := 2m - 2 - h \geq 0. \tag{A.2}$$

By setting $u := n + 1 - 3m (\geq 0)$ and $v := 2m - 2 - h (\geq 0)$, we have

$$2u + 3v = \begin{cases} 1 & \text{if } h \text{ is odd} \\ 0 & \text{if } h \text{ is even.} \end{cases}$$

Since $u \geq 0$, $v \geq 0$ and $\dim W^1_{n+1}(X) = u + v$ by (A.1) and (A.2), we have $\dim W^1_{n+1}(X) = 0$. Accordingly by Lemma 1.1, $W^1_d(X)$ is equi-dimensional of the minimal possible dimension $\rho(d, g, 1) := 2d - 2 - g$ for all d such that $g - n < d \leq g$. And it follows that for $d \geq g - [(3h+1)/2]$, $W^1_d(X) \supsetneq W^1_{d-1}(X) + W_1(X)$ by dimension count, and hence there exists a base-point-free pencil of degree d on X, which may possibly be composed with π. Furthermore, for any $d \geq g - [(3h+1)/2]$ and m with $\rho(m, h, 1) \geq 0$ (i.e. $m \geq [(h+3)/2]$), one has

$$\begin{aligned}
\dim W^1_d(X) &- \dim(\pi^* W^1_m(C) + W_{d-3m}(X)) \\
&= \rho(d, g, 1) - \rho(m, h, 1) - d + 3m = d - g + m + h \\
&\geq \left(g - \left[\frac{3h+1}{2}\right]\right) - g + \left[\frac{h+3}{2}\right] + h \\
&= \left[\frac{h+3}{2}\right] + h - \left[\frac{3h+1}{2}\right] = 1.
\end{aligned}$$

Hence it follows that

$$W^1_d(X) \supsetneq \pi^* W^1_m(C) + W_{d-3m}(X) \tag{A.3}$$

and we conclude that there exists a base-point-free complete pencil of any degree $d \geq g - [(3h+1)/2]$ which is not composed with π.

Thus it remains to prove the theorem only for the case $d = g - [(3h+1)/2] - 1$. In order to avoid the unnecessary symbol [ ], we consider the two cases according to the parity of h, the genus of the base curve C.

(a) *h is even*: Set $h = 2e$, then $d = g - [(3h+1)/2] - 1 = g - 3e - 1$. The proof for this case is divided into three steps.

(a-1) Let Σ be a component of $W^1_{g-3e-1}(X)$ whose general element has a base point. Then $\Sigma = \Sigma^1_\beta + W_{g-3e-1-\beta}(X)$ for some β with $4 \le \beta \le g - 3e - 2$, where Σ^1_β is a subvariety of $W^1_\beta(X)$ whose general element is base-point-free. In this step we will show that β is relatively small compared to g, i.e. $\beta \le 9e + 4$.

We first note that $\dim \Sigma = \rho(g - 3e - 1, g, 1) = g - 6e - 4$, hence $\dim \Sigma^1_\beta = \beta - 3e - 3 \ge 0$. Let $L \in \Sigma^1_\beta$ be a general element. By the standard description of the Zariski tangent space to the variety $W^r_d(X)$ of any curve X in general, we have

$$\dim(\operatorname{Im}\mu_0)^\perp = \dim T_L(W^1_\beta(X)) \ge \dim \Sigma^1_\beta = \beta - 3e - 3$$

where $\mu_0 : H^0(X, L) \otimes H^0(X, K_X \otimes L^{-1}) \to H^0(X, K_X)$ is the usual cup product map. By the base-point-free pencil trick ([ACGH]), p. 126), we have

$$\begin{aligned}\dim(\operatorname{Im}\mu_0)^\perp &= g - \dim(\operatorname{Im}\mu_0) = g - h^0(X, L)h^1(X, L) + \dim(\operatorname{Ker}\mu_0)\\ &= g - 2(g - \beta + 1) + h^0(X, K_X \otimes L^{-2})\\ &= h^0(X, L^2) - 3 \ge \beta - 3e - 3.\end{aligned}$$

Hence $h^0(X, L^2) \ge \beta - 3e \ge 3$ which implies $\dim W^{\beta-3e-1}_{2\beta}(X) \ge \beta - 3e - 3$. By reducing to pencils if necessary we have

$$\begin{aligned}\dim W^1_{\beta+3e+2}(X) &= \dim W^1_{2\beta-(\beta-3e-2)}(X) \ge \beta - 3e - 3 + (\beta - 3e - 2)\\ &= 2(\beta - 3e) - 5.\end{aligned}$$

Note that $\beta \le g - 3e - 2$, hence $\beta + 3e + 2 \le g$. We now consider the following two cases:

(1) If $\beta + 3e + 2 = g$, then by passing to residual series

$$\dim W^1_{\beta+3e+2}(X) = \dim W_{g-2}(X) = g - 2 \ge 2(\beta - 3e) - 5,$$

hence, $12e \ge g - 7$ which contradicts the genus assumption.

(2) If $\beta + 3e + 2 \le g - 1$, we have

$$2(\beta - 3e) - 5 \le \dim W^1_{\beta+3e+2}(X) \le (\beta + 3e + 2) - 2 - 1,$$

hence, $\beta \le 9e + 4$ by H. Martens-Mumford's theorem; cf. [ACGH] Ch. 4, §5.

(a-2) In this step, we will show that Σ is of very special type; specifically,

$$\Sigma = \pi^*\Sigma' + W_{g-6e-4}(X), \tag{A.4}$$

where Σ' is a component of $W^1_{e+1}(C)$.

First note that by the assumption on h and g by step (a-1), we have $\beta \leq 9e + 4 \leq (g - 3h)/2$. Hence by Lemma 1.2, every element of Σ^1_β is a pull-back of a $g^1_{\beta/3}$ on C, i.e. $\Sigma^1_\beta = \pi^*(\Sigma^1_{\beta/3}(C))$, where $\Sigma^1_{\beta/3}(C)$ is a component of $W^1_{\beta/3}(C)$, i.e.

$$\Sigma^1_\beta \subset W^1_\beta(X) = \pi^* W^1_{\beta/3}(C)$$

and $W^1_\beta(X) = \pi^* W^1_{\beta/3}(C)$ has dimension $\rho(\beta/3, h, 1) = 2 \cdot (\beta/3) - 2 - 2e$, since C is general. We also have

$$\begin{aligned} g - 6e - 4 = \dim \Sigma &= \dim(\Sigma^1_\beta + W_{g-3e-1-\beta}(X)) \\ &\leq \dim(W^1_\beta(X) + W_{g-3e-1-\beta}(X)) \\ &= \left(2 \cdot \frac{\beta}{3} - 2 - 2e\right) + (g - 3e - 1 - \beta) \\ &\leq \dim W^1_{g-3e-1}(X) = g - 6e - 4. \end{aligned}$$

From this we get $(\beta/3) = e + 1$ and every component of $W^1_{g-3e-1}(X)$ whose general element has a base point is of the form $\pi^*\Sigma' + W_{g-6e-4}(X)$, where Σ' is a component of $W^1_{e+1}(C)$.

(a-3) In this step, we will show that there exists a component of $W^1_{g-3e-1}(X)$ which is not of the form (A.4), which will in turn imply that there exists a base-point-free pencil of degree $g - 3e - 1$ on X which is not composed with π.

By recalling the fact that C is general and the Brill-Noether number $\rho(e + 1, h, 1)$ is zero, Σ' is a single point $g^1_{e+1}(C)$ of $W^1_{e+1}(C)$ and there are exactly $s := h!(1/(h - e)! \cdot 1/(h - e + 1)!)$ reduced points in $W^1_{e+1}(C)$, which is Castelnuovo's count for the number of g^r_d's when $\rho = 0$; cf. [ACGH] Theorem 1.3, p. 212.

Let $\tilde{\Sigma}'$ be the locus in C^1_{e+1} corresponding to $g^1_{e+1}(C) = \Sigma'$. We will show in the second section (Lemma 2.1) that $W^1_{g-3e-1}(X)$ is reduced at a general point of $\pi^*(W^1_{e+1}(C)) + W_{g-6e-4}(X)$, which in turn says that X^1_{g-3e-1} is reduced at a general point of $\pi^*(\tilde{\Sigma}') + X_{g-6e-4}$.

We now recall some of the notations and conventions used in [ACGH], especially in Chapter VIII. Let $u : X_d \to J(X)$ be the abelian sum map and let θ be the class of the theta divisor in $J(X)$. Let $u^* : H^*(J(X), Q) \to H^*(X_d, Q)$ be the homomorphism induced by u. By abusing notation, we use the same letter θ for the class $u^*\theta$. By fixing a point P on X, one has the map $\imath : X_{d-1} \to X_d$ defined by $\imath(D) = D + P$. We denote the class of $\imath(X_{d-1})$ in X_d by x.

By the steps (a-1) and (a-2), the only components of $W^1_{g-3e-1}(X)$ whose general element has a base point are of the form $\pi^*\Sigma' + W_{g-6e-4}(X)$, where Σ' is a component of $W^1_{e+1}(C)$.

We denote by σ and $\tilde{\sigma}$ the class of $\pi^*C^1_{e+1} + X_{g-6e-4}$ in X_{g-3e-1} and the class of $\pi^*C^1_{e+1}$ in X_{3e+3} respectively. Because X^1_{g-3e-1} is of pure and expected dimension $\rho(g-3e-1, g, 1) + 1 = g - 6e - 3$, the class x^1_{g-3e-1} of X^1_{g-3e-1} in X_{g-3e-1} is well known (cf. [ACGH] p. 326), namely

$$\text{(A.5)} \qquad x^1_{g-3e-1} = \frac{1}{(3e+1)!(3e+2)!}((3e+1)!\theta^{3e+2} - (3e+2)!x\theta^{3e+1}).$$

Let's also recall that for a cycle Z in X_d, the assignments

$$Z \mapsto A_k(Z) := \{E \in X_{d+k} : E - D \geq 0 \text{ for some } D \in Z\}$$

$$Z \mapsto B_k(Z) := \{E \in X_{d-k} : D - E \geq 0 \text{ for some } D \in Z\}$$

induce maps

$$A_k : H^{2m}(X_d, \boldsymbol{Q}) \to H^{2m}(X_{d+k}, \boldsymbol{Q})$$

$$B_k : H^{2m}(X_d, \boldsymbol{Q}) \to H^{2m-2k}(X_{d-k}, \boldsymbol{Q})$$

and the so called push-pull formulas for symmetric products hold (cf. [ACGH], p. 367–369).

Note the fact that X^1_{g-3e-1} and $\pi^*C^1_{e+1} + X_{g-6e-4}$ have dimension $g - 6e - 3$ in X_{g-3e-1}. On the other hand, by the push-pull formulas

$$B_{g-6e-4}(x^{g-6e-3}) = (g - 6e - 3)x$$

and

$$\begin{aligned}(\sigma \cdot x^{g-6e-3})_{X_{g-3e-1}} &= (A_{g-6e-4}(\tilde{\sigma}) \cdot x^{g-6e-3})_{X_{g-3e-1}} \\ &= (\tilde{\sigma} \cdot B_{g-6e-4}(x^{g-6e-3}))_{X_{3e+3}} = (g - 6e - 3)(\tilde{\sigma} \cdot x)_{X_{3e+3}} \\ &= (g - 6e - 3)s\end{aligned}$$

by noting the fact that $(\tilde{\sigma} \cdot x)_{X_{3e+3}} = s = h!(1/(h-e)! \cdot 1/(h-e+1)!)$.

By (A.5)

$$
\begin{aligned}
&(x^1_{g-3e-1}\cdot x^{g-6e-3})_{X_{g-3e-1}}\\
&=\left(\frac{1}{(3e+1)!(3e+2)!}(3e+1)!\theta^{3e+2}-(3e+2)!x\theta^{3e+1}\right)\cdot x^{g-6e-3}\\
&=\frac{1}{(3e+2)!}x^{g-6e-3}\cdot\theta^{3e+2}-\frac{1}{(3e+1)!}x^{g-6e-2}\cdot\theta^{3e+1}\\
&=\frac{1}{(3e+2)!}\cdot\frac{g!}{(g-3e-2)!}-\frac{1}{(3e+1)!}\cdot\frac{g!}{(g-3e-1)!}
\end{aligned}
$$

where the last equality comes from the fact that $(x^{d-\alpha}\cdot\theta^{\alpha})_{X_d}=g!/(g-\alpha)!$, which is a consequence of Poincaré's formula; cf. [ACGH], p. 328.

Finally a tedious but straightforward calculation yields

$$(x^1_{g-3e-1}\cdot x^{g-6e-3})_{X_{g-3e-1}}>(\sigma\cdot x^{g-6e-3})_{X_{g-3e-1}}$$

and this shows that there exists a component in X^1_{g-3e-1} other than those of the form $\pi^*(C^1_{e+1})+X_{g-6e-4}$, which is equivalent to the fact that there exists a component of $W^1_{g-3e-1}(X)$ which is not of the form (A.4). And this finishes the proof of our assertion that there exists a base-point-free pencil of degree $g-3e-1$ on X which is not composed with π.

We now proceed to handle the case when $h=2e+1$. Even though the method is almost the same, we will present somewhat detailed argument and the computations for the convenience of the reader.

(b) *h is odd*: Set $h=2e+1$, then $d=g-[(3h+1)/2]-1=g-3e-3$. The proof for this case is also divided into three steps.

(b-1) Let Σ be a component of $W^1_{g-3e-3}(X)$ whose general element has a base point. Then $\Sigma=\Sigma^1_\beta+W_{g-3e-3-\beta}(X)$ for some β with $4\le\beta\le g-3e-4$, where Σ^1_β is a subvariety of $W^1_\beta(X)$ whose general element is base-point-free. In this step we will show that β is relatively small compared to g, i.e. $\beta\le 9e+10$.

We first note that $\dim\Sigma=\rho(g-3e-3,g,1)=g-6e-8$, hence $\dim\Sigma^1_\beta=\beta-3e-5\ge 0$. Let $L\in\Sigma^1_\beta$ be a general element. Again by the description of the Zariski tangent space to the variety of special linear systems, we have

$$\dim(\operatorname{Im}\mu_0)^\perp=\dim T_L(W^1_\beta(X))\ge\dim\Sigma^1_\beta\ge\beta-3e-5.$$

By the base-point-free pencil trick, we have

$$\begin{aligned}\dim(\operatorname{Im}\mu_0)^{\perp} &= g - \dim(\operatorname{Im}\mu_0) = g - h^0(X,L)h^1(X,L) + \dim(\operatorname{Ker}\mu_0)\\ &= g - 2(g-\beta+1) + h^0(X, K_X \otimes L^{-2})\\ &= h^0(X,L^2) - 3 \geq \beta - 3e - 5.\end{aligned}$$

Hence $h^0(X,L^2) \geq \beta - 3e - 2 \geq 3$ which implies $W_{2\beta}^{\beta-3e-3}(X) \geq \beta - 3e - 5$. By reducing to pencils if necessary we have

$$\begin{aligned}\dim W^1_{\beta+3e+4}(X) = \dim W^1_{2\beta-(\beta-3e-4)}(X) &\geq \beta - 3e - 5 + (\beta - 3e - 4)\\ &= 2(\beta - 3e) - 9.\end{aligned}$$

Note that $\beta \leq g - 3e - 4$ hence $\beta + 3e + 4 \leq g$. We consider the following two cases:

(1) If $\beta + 3e + 4 = g$, then by passing to residual series

$$\dim W^1_{\beta+3e+4}(X) = \dim W_{g-2}(X) = g - 2 \geq 2(\beta - 3e) - 9,$$

hence $12e \geq g - 15$, contradictory to the genus assumption.

(2) If $\beta + 3e + 4 \leq g - 1$, we have

$$2(\beta - 3e) - 9 \leq \dim W^1_{\beta+3e+4}(X) \leq (\beta + 3e + 4) - 2 - 1,$$

hence, $\beta \leq 9e + 10$ by H. Martens-Mumford's theorem.

(b-2) In this step, we will show that Σ is of very special type; specifically,

$$\Sigma = \pi^*\Sigma' + W_{g-6e-9}(X), \tag{A.6}$$

where Σ' is a component of $W^1_{e+2}(C)$.

First note that by the assumption on h and g and by step (b-1), we have $\beta \leq 9e + 10 \leq (g - 3h)/2$. Hence by Lemma 1.2, every element of Σ^1_β is a pull-back of a $g^1_{\beta/3}$ on C, i.e. $\Sigma^1_\beta = \pi^*(\Sigma^1_{\beta/3}(C))$, where $\Sigma^1_{\beta/3}(C)$ is a component of $W^1_{\beta/3}(C)$,

$$\Sigma^1_\beta \subset W^1_\beta(X) = \pi^* W^1_{\beta/3}(C)$$

and $W^1_\beta(X) = \pi^* W^1_{\beta/3}(C)$ has dimension $\rho(\beta/3, h, 1) = 2 \cdot (\beta/3) - 2 - (2e + 1)$

since C is general. We also have

$$\begin{aligned} g-6e-8 = \dim \Sigma &= \dim(\Sigma^1_\beta + W_{g-3e-3-\beta}(X)) \\ &\le \dim(W^1_\beta(X) + W_{g-3e-3-\beta}(X)) \\ &= \left(2\cdot\frac{\beta}{3} - 2 - 2e - 1\right) + (g-3e-3-\beta) \\ &\le \dim W^1_{g-3e-3}(X) = g-6e-8 \end{aligned}$$

From this we get $\beta/3 = e+2$ and every component of $W^1_{g-3e-3}(X)$ whose general element has a base point is of the form $\pi^*\Sigma' + W_{g-6e-9}(X)$, where Σ' is a component of $W^1_{e+2}(C)$.

(b-3) In this step, we will show that there exists a component of $W^1_{g-3e-3}(X)$ which is not of the form (A.6), which will in turn imply that there exists a base-point-free pencil of degree $g-3e-3$ on X which is not composed with π.

Since C is general and the Brill-Noether number $\rho(e+2,h,1)=1>0$, $W^1_{e+2}(C)$ is irreducible of dimension 1 which follows from Gieseker's theorem by Fulton and Lazarsfeld's connectedness theorem together with Brill-Noether theorem by Griffiths-Harris; cf. [ACGH] p. 214. Hence we have

$$\Sigma' = W^1_{e+2}(C) \quad \text{and} \quad \Sigma = \pi^* W^1_{e+1}(C) + W_{g-6e-9}(X).$$

By Lemma 2.1, $W^1_{g-3e-3}(X)$ is reduced at a general point of Σ, which in turn says that X^1_{g-3e-3} is reduced at a general point of $\pi^* C^1_{e+2} + X_{g-6e-9}$.

We denote σ and $\tilde{\sigma}$ by the class of $\pi^* C^1_{e+2} + X_{g-6e-9}$ in X_{g-3e-3} and the class of $\pi^* C^1_{e+2}$ in X_{3e+6} respectively. Because X^1_{g-3e-3} is of pure and expected dimension $\rho(g-3e-3,g,1)+1 = g-6e-7$, the fundamental class x^1_{g-3e-3} of X^1_{g-3e-3} in X_{g-3e-3} is

$$x^1_{g-3e-3} = \frac{1}{(3e+3)!(3e+4)!}((3e+3)!\theta^{3e+4} - (3e+4)!x\theta^{3e+3}). \tag{A.7}$$

Note the fact that X^1_{g-3e-3} and $\pi^* C^1_{e+2} + X_{g-6e-9}$ have dimension $g-6e-7$ in X_{g-3e-3}. Hence, as in the even genus case, we have

$$B_{g-6e-9}(x^{g-6e-7}) = \binom{g-6e-7}{g-6e-9} x^2$$

and

$$\begin{aligned}(\sigma\cdot x^{g-6e-7})_{X_{g-3e-3}} &= (A_{g-6e-9}(\tilde{\sigma})\cdot x^{g-6e-7})_{X_{g-3e-3}}\\ &= (\tilde{\sigma}\cdot B_{g-6e-9}(x^{g-6e-7}))_{X_{3e+6}} = \binom{g-6e-7}{g-6e-9}(\tilde{\sigma}\cdot x^2)_{X_{3e+6}}\\ &= \binom{g-6e-7}{g-6e-9}(c^1_{e+2}\cdot x^2)_{C_{e+2}}\\ &= \binom{g-6e-7}{g-6e-9}\left(\frac{(2e+1)!}{e!(e+1)!}-\frac{(2e+1)!}{(e-1)!(e+2)!}\right),\end{aligned}$$

where the last equality comes from Poincaré's formula, the fact that C is general and the class c^1_{e+2} of C^1_{e+2} in C_{e+2} is

$$c^1_{e+2} = \frac{\theta^e}{e!} - \frac{x\theta^{e-1}}{(e-1)!}.$$

On the other hand, by (A.7) and Poincaré's formula

$$\begin{aligned}&(x^1_{g-3e-3}\cdot x^{g-6e-7})_{X_{g-3e-3}}\\ &= \frac{1}{(3e+3)!(3e+4)!}((3e+3)!\theta^{3e+4} - (3e+4)!x\theta^{3e+3})\cdot x^{g-6e-7}\\ &= \frac{g!(g-6e-7)}{(3e+4)!(g-3e-3)!}.\end{aligned}$$

Finally a tedious but strightforward calculation yields

$$(x^1_{g-3e-3}\cdot x^{g-6e-7})_{X_{g-3e-3}} > (\sigma\cdot x^{g-6e-7})_{X_{g-3e-3}}$$

and this shows that there exists a component in X^1_{g-3e-3} other than those of the form $\pi^*(C^1_{e+2}) + X_{g-6e-9}$, which is equivalent to the fact that there exists a component of $W^1_{g-3e-3}(X)$ which is not of the form (A.6). Whence this finishes the proof of our assertion that there exists a base-point-free pencil of degree $g-3e-3$ on X which is not composed with π. □

2. The proof of Lemma 2.1

In this section, our aim is to prove the following key lemma.

LEMMA 2.1. *Assume that C is a general curve of genus h.*

(i) *Assume* $h = 2e$ *and* $g > 6h + 4$. *Then a general element* $\mathscr{L}$ *of* $\pi^*(W^1_{e+1}(C)) + W_{g-6e-4}(X)$ *is a reduced point of* $W^1_{g-3e-1}(X)$.

(ii) *Assume* $h = 2e + 1$ *and* $g > 6h + 7$. *Then a general element* $\mathscr{L}$ *of* $\pi^*(W^1_{e+2}(C)) + W_{g-6e-9}(X)$ *is a reduced point of* $W^1_{g-3e-3}(X)$.

Assume that $\pi : X \to C$ is a finite morphism of degree n. Because $\mathcal{O}_C \to \pi_*\mathcal{O}_X$ is an algebra homomorphism, the trace map gives a splitting. Hence we have $\pi_*\mathcal{O}_X = \mathcal{O}_C \oplus \mathscr{E}$ where $\mathscr{E}$ is defined by

$$\mathscr{E} = \{a \in \pi_*(\mathcal{O}_X) | a^n + c_1 a^{n-2} + \cdots + c_{n-2}a + c_{n-1} = 0 \text{ and } c_1, \ldots, c_{n-1} \in \mathcal{O}_C\}.$$

This $\mathscr{E}$ is clearly a locally free sheaf of rank $n - 1$.

REMARK 2.2. For the rest of this section, we assume, again, that $n = 3$, i.e. a triple covering. Therefore π is a finite separable morphism of degree 3 and $\mathscr{E}$ is a locally free $\mathcal{O}_C$-module of rank 2. As $\pi_*(\mathcal{O}_X)$ is a commutative $\mathcal{O}_C$-algebra, we have a map defined by the multiplication,

$$\pi_*(\mathcal{O}_X) \otimes_{\mathcal{O}_C} \pi_*(\mathcal{O}_X) \to \pi_*(\mathcal{O}_X).$$

Hence the multiplication induces $\mathcal{O}_C$-module homomorphisms

$$\phi_1 : S^2\mathscr{E} \to \mathcal{O}_C \quad \text{and} \quad \phi_2 : S^2\mathscr{E} \to \mathscr{E}.$$

Furthermore, the algebra structure on $\pi_*(\mathcal{O}_X) = \mathcal{O}_C \oplus \mathscr{E}$ is written as follows:

$$(a, b) \cdot (a', b') = (aa' + \phi_1(bb'), ab' + a'b + \phi_2(bb')), \tag{2.A}$$

where $a, a' \in \mathcal{O}_C$ and $b, b' \in \mathscr{E}$. Locally, ϕ_2 is written as the following form [M]:

$$\phi_2(z^2) = az + bw, \quad \phi_2(zw) = -dz - aw, \quad \phi_2(w^2) = cz + dw$$

where $z, w \in \mathscr{E}$ is a local frame of $\mathscr{E}$ and $a, b, c, d \in \mathcal{O}_C$. Such a map ϕ_2 is called a triple covering homomorphism due to R. Miranda. We denote by $\mathrm{TCHom}_{\mathcal{O}_C}(S^2\mathscr{E}, \mathscr{E})$ the set of all triple covering homomorphisms on $\mathscr{E}$ and there is a functorial homomorphism

$$F : \mathrm{TCHom}_{\mathcal{O}_C}(S^2\mathscr{E}, \mathscr{E}) \to \mathrm{Hom}_{\mathcal{O}_C}(S^2\mathscr{E}, \mathcal{O}_C)$$

such that $\phi_2 \in \mathrm{TCHom}_{\mathcal{O}_C}(S^2\mathscr{E}, \mathscr{E})$ and $\phi_1 = F(\phi_2)$ gives an $\mathcal{O}_C$-algebra structure on $\mathcal{O}_C \oplus \mathscr{E}$ by the relation (2.A) [M], p. 1131 Proposition 3.5.

Also for the triple covering $\pi : X \to C$ there is a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\pi^*\pi_*(\mathcal{O}_X), \mathcal{O}_X) \cong \mathrm{Hom}_{\mathcal{O}_C}(\pi_*(\mathcal{O}_X), \pi_*(\mathcal{O}_X)).$$

We take an element $\sigma \in \mathrm{Hom}_{\mathcal{O}_X}(\pi^*\pi_*(\mathcal{O}_X), \mathcal{O}_X)$ which corresponds to the identity map: $\pi_*(\mathcal{O}_X) \to \pi_*(\mathcal{O}_X)$. Since $\pi_*(\mathcal{O}_X) = \mathcal{O}_C \oplus \mathscr{E}$, we can write $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1 : \mathcal{O}_X \to \mathcal{O}_X$ and $\sigma_2 : \pi^*\mathscr{E} \to \mathcal{O}_X$. Because σ is an $\mathcal{O}_X$-algebra homomorphism, σ_1 is the identity map. As for σ_2, we have the following result.

LEMMA 2.3. *The image of σ_2 is an invertible sheaf.*

PROOF. Let $U =$ Spec(A) be an affine open subset and let $\pi^{-1}(U) =$ Spec(B); note that π is a finite morphism therefore is an affine morphism. Let $\Gamma(U, \mathscr{E}) = M$. The morphism σ_2 on $\pi^{-1}(U)$ is written as follows:

$$\begin{tikzcd} M \otimes_A B \arrow[r, "\sigma_2"] \arrow[d, phantom, "\wr\|"] & B \\ M \otimes_A (A \oplus M) \arrow[ur, "\tau"'] & \end{tikzcd}$$

where $\tau = \mu \oplus (\phi_1, \phi_2)$, $\mu : M \otimes_A A \to B$ is a multiplication map, ϕ_2 is the triple covering homomorphism corresponding to the A-algebra structure of B and ϕ_1 is the homomorphism defined in Remark 2.2. Clearly μ is a non-zero map, therefore σ_2 is not a zero map either and it follows that Im(σ_2) is an invertible sheaf. $\square$

In general, one knows that to give a morphism of X to the projective bundle $\boldsymbol{P}(\mathscr{E})$ over C is equivalent to give an invertible sheaf $\mathscr{L}$ on X and a surjective homomorphism $\pi^*\mathscr{E} \to \mathscr{L}$; cf. [H], p. 162 Proposition 7.12. Hence by Lemma 2.3, we have a morphism $f : X \to \boldsymbol{P}(\mathscr{E})$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowleft"] & \boldsymbol{P}(\mathscr{E}) \arrow[dl, "\varphi"] \\ & C & \end{tikzcd}$$

With these preparations we now prove the following lemma.

LEMMA 2.4. *f is an embedding.*

PROOF. Let $U =$ Spec(A) be an affine open subset of C. Let $Z = \pi^{-1}(U) =$ Spec(B), $\Gamma(U, \pi_*(\mathcal{O}_X)) = A \oplus As_1 \oplus As_2$ and $Z_i = \{P \in Z | s_i(P) \neq 0\}$, $i = 1, 2$. Because π is a finite morphism, Z is an affine open set and hence Z_i is also an affine open set for $i = 1, 2$. Moreover $Z_i =$ Spec(B_{s_i}). Therefore B_{s_i} is generated by $1, s_j/s_i$ where $j \neq i$. By a criterion for a morphism to a projective space to be an embedding ([H] p. 151 Proposition 7.2), $f|_{\pi^{-1}(U)}$ is an embedding for every affine open set $U \subset C$ and hence f is an embedding. $\square$

We now consider the ruled surface $\varphi : \boldsymbol{P}(\mathscr{E}) \to C$. Let C_0 be a divisor on $\boldsymbol{P}(\mathscr{E})$ such that $\varphi_*(\mathscr{O}_{\boldsymbol{P}(\mathscr{E})}(C_0)) = \mathscr{E}$. Let $\bar{C}$ be a minimal section of $\varphi : \boldsymbol{P}(\mathscr{E}) \to C$, $\bar{\mathscr{E}} = \varphi_*(\mathscr{O}_{\boldsymbol{P}(\mathscr{E})}(\bar{C}))$ and $\delta = -(\bar{C}^2)$. Then there exist an exact sequence

$$0 \to \mathscr{O}_C \to \bar{\mathscr{E}} \to \bigwedge^2 \bar{\mathscr{E}} \to 0 \tag{2.B}$$

and an isomorphism $\mathscr{E} \cong \bar{\mathscr{E}} \otimes_{\mathscr{O}_C} \mathscr{M}$ for some line bundle $\mathscr{M}$ on C.

We now take an invertible sheaf $\mathscr{M}$ such that $\mathscr{E} \cong \bar{\mathscr{E}} \otimes \mathscr{M}$ and let $n = \deg(\mathscr{M})$. We then have $C_0 \sim \bar{C} + \pi^*(\mathscr{M})$ (linearly equivalent). Therefore $C_0 \equiv \bar{C} + nF$ (where $\equiv$ means numerical equivalence and F is a fibre of φ) and $(C_0^2) = 2n - \delta = \deg(\bigwedge^2 \mathscr{E})$. One also knows the following numerical equivalence;

$$K_{\boldsymbol{P}(\mathscr{E})} \equiv -2C_0 + ((2h-2) + \deg(\bigwedge^2 \mathscr{E}))F, \tag{2.C}$$

see [H], p. 372 Proposition 2.8 and p. 374 Corollary 2.11.

By (2.C), we have the following:

LEMMA 2.5. *We have* $n = ((C_0^2) + \delta)/2 = (\delta - g + 3h - 2)/2$ *and we also have*

$$f(X) \equiv 3C_0 - 2\deg(\bigwedge^2 \mathscr{E})F \equiv 3\bar{C} + \frac{3\delta + g - 3h + 2}{2} F$$

for numerical equivalence.

PROOF. By the Riemann-Hurwitz relation ([H], p. 127 Exercise 5.16), we have

$$2g - 2 = 3(2h-2) + \deg(\bigwedge^2 \mathscr{E}^{\otimes -2})$$

and hence

$$\deg(\bigwedge^2 \mathscr{E}) = \frac{3(2h-2) - (2g-2)}{2}.$$

As $C_0 \equiv \bar{C} + nF$, we have $n = (\delta - g + 3h - 2)/2$ because $(C_0^2) = 2n - \delta = \deg(\bigwedge^2 \mathscr{E})$. By Lemma 2.4, $f : X \to \boldsymbol{P}(\mathscr{E})$ is an embedding and $\pi : X \to C$ is a finite morphism of degree 3. Therefore we have $f(X) \sim 3C_0 + \varphi^*(T)$ for some divisor T on C and $2g - 2 = (K_{\boldsymbol{P}(\mathscr{E})} + f(X)).f(X)$ by the adjunction formula. Hence

$\deg(T) = -2\deg(\bigwedge^2 \mathscr{E})$ by (2.C) and we have

$$\begin{aligned} f(X) &\equiv 3C_0 - 2\deg(\textstyle\bigwedge^2 \mathscr{E})F \\ &\equiv 3\left(\bar{C} + \frac{\delta - g + 3h - 2}{2} F\right) - 2\left(\frac{3(2h-2) - (2g-2)}{2}\right) F \\ &\equiv 3\bar{C} + \frac{3\delta + g - 3h + 2}{2} F. \end{aligned}$$

□

PROPOSITION 2.6. *There is an exact sequence*

$$0 \to \mathscr{M} \to \mathscr{E} \to \mathscr{L} \to 0,$$

where $\deg(\mathscr{M}) = -(g-3h)/2 - (2-\delta)/2$ *and* $\deg(\mathscr{L}) = -(g-3h)/2 - (2+\delta)/2$.

PROOF. Since $\mathscr{E} \cong \bar{\mathscr{E}} \otimes \mathscr{M}$, one has $\bigwedge^2 \mathscr{E} \cong \bigwedge^2 \bar{\mathscr{E}} \otimes \mathscr{M}^{\otimes 2}$ and $\deg(\bigwedge^2 \bar{\mathscr{E}}) = -\delta$. Again by the Riemann-Hurwitz relation,

$$\deg(\textstyle\bigwedge^2 \mathscr{E}) = \frac{3(2h-2) - (2g-2)}{2}.$$

Therefore we have $\deg(\mathscr{M}) = -(g-3h)/2 - (2-\delta)/2$. By tensoring $\mathscr{M}$ to the exact sequence (2.B) and taking $\mathscr{L} = \bigwedge^2 \bar{\mathscr{E}} \otimes \mathscr{M}$, we have the exact sequence

$$0 \to \mathscr{M} \to \mathscr{E} \to \mathscr{L} \to 0$$

and $\deg(\mathscr{L}) = -(g-3h)/2 - (2+\delta)/2$. □

LEMMA 2.7. $-h \le \delta \le (g-3h+2)/3$.

PROOF. If $\delta \le 0$, then $-h \le \delta$ by a theorem of Nagata; [N] p. 191 Theorem 1. Therefore we may assume that $\delta \ge 0$. As $f(X) \equiv 3\bar{C} + (3\delta + g - 3h + 2)/2F$ by Lemma 2.5 and $f(X)$ is irreducible, we have $3\delta \le (3\delta + g - 3h + 2)/2$ by [H] p.382 Proposition 2.20. Therefore we have $\delta \le (g-3h+2)/3$ □

PROPOSITION 2.8. $\deg(\mathscr{M}) \le (-g+3h-2)/3$ *and* $\deg(\mathscr{L}) \le (-g+4h-2)/2$.

PROOF. By Proposition 2.6, we have the exact sequence

$$0 \to \mathscr{M} \to \mathscr{E} \to \mathscr{L} \to 0$$

such that $\deg(\mathscr{M}) = -((g-3h)/2 + (2-\delta)/2)$ and $\deg(\mathscr{L}) = -((g-3h)/2 +$

$(2+\delta)/2)$. Therefore we have

$$\begin{aligned}\deg(\mathscr{M}) &= -\left(\frac{g-3h}{2}+\frac{2-\delta}{2}\right) = -\frac{g-3h+2}{2}+\frac{\delta}{2}\\ &\le -\frac{g-3h+2}{2}+\frac{g-3h+2}{6} \quad \text{by Lemma 2.7}\\ &= \frac{-2g+6h-4}{6} = \frac{-g+3h-2}{3}\end{aligned}$$

and

$$\begin{aligned}\deg(\mathscr{L}) &= -\left(\frac{g-3h}{2}+\frac{2+\delta}{2}\right) = -\frac{g-3h+2}{2}-\frac{\delta}{2}\\ &\le -\frac{g-3h+2}{2}+\frac{h}{2} \quad \text{by Lemma 2.7}\\ &= \frac{-g+4h-2}{2}.\end{aligned}$$

□

With all these preparations we now start to prove Lemma 2.1.

Proof of Lemma 2.1. We will only provide the proof of the case $h=2e$ because the odd genus case can be done in the same way.

Let $\mathscr{N} = \mathscr{O}_X(\pi^*(D)+\Delta)$ be a general element of a component of $\pi^*(W^1_{e+1}(C)) + W_{g-6e-4}(X)$, where $\mathscr{O}_X(\Delta) \in W_{g-6e-4}(X)$ is general and $\mathscr{O}_C(D) \in W^1_{e+1}(C)$. Since C is general, $\mathscr{O}_C(D)$ is a base-point-free pencil by the Brill-Noether theorem for general curves. Hence we deduce that $\mathscr{O}_X(\pi^*(D))$ is also base-point-free and the base locus of $\mathscr{N}$ is just Δ by the general choice of Δ. Since every component of $\pi^*(W^1_{e+1}(C)) + W_{g-6e-4}(X)$ has the expected dimension, to show the reducedness at $\mathscr{N}$ it is sufficient to show the injectivity of the Brill-Noether map $\Gamma(X,\mathscr{N}) \otimes \Gamma(X,\omega_X \otimes \mathscr{N}^{\otimes -1}) \to \Gamma(X,\omega_X)$ by the description of the Zariski tangent space to the variety $W^r_d(X)$. Hence by the base-point-pencil trick, it will be enough to show that $\Gamma(X,\omega_X \otimes \mathscr{N}^{\otimes -2} \otimes \mathscr{O}_X(\Delta)) = 0$.

Note that

$$\Gamma(X,\omega_X \otimes \mathscr{N}^{\otimes -2} \otimes \mathscr{O}_X(\Delta)) \cong \Gamma(X,\omega_X \otimes \pi^*(\mathscr{O}_C(-D)^{\otimes 2}) \otimes \mathscr{O}_X(-\Delta)).$$

We will show that

$$\dim_k(\Gamma(X,\omega_X \otimes \pi^*(\mathscr{O}_C(-D)^{\otimes -2}))) \le g-6e-4,$$

which will imply $\Gamma(X,\omega_X \otimes \mathscr{N}^{\otimes -2} \otimes \mathscr{O}_X(\Delta)) = 0$ by the general choice of Δ.

On the other hand, since C is a general curve $\mathcal{O}_C(D)$ is a reduced point of $W^1_{e+1}(C)$ (cf. [ACGH] p. 214) and hence by the base-point-free pencil trick again, we have

$$\dim_k(\Gamma(C, \mathcal{O}_C(D)^{\otimes 2})) = 3.$$

Since $\pi^*(\mathcal{O}_X) \cong \mathcal{O}_C \oplus \mathcal{E}$, we have

$$\pi_*(\pi^*\mathcal{O}_C(D)^{\otimes 2}) \cong \mathcal{O}_C(2D) \oplus (\mathcal{E} \otimes \mathcal{O}_C(2D)).$$

Moreover we have the exact sequence

$$0 \to \mathcal{M} \otimes \mathcal{O}_C(2D) \to \mathcal{E} \otimes \mathcal{O}_C(2D) \to \mathcal{L} \otimes \mathcal{O}_C(2D) \to 0$$

and

$$\deg(\mathcal{M}) \leq \frac{-g+3h-2}{3} \quad \text{and} \quad \deg(\mathcal{L}) \leq \frac{-g+4h-2}{2}$$

by Proposition 2.6 and Proposition 2.8. Therefore we have

$$\deg(\mathcal{M} \otimes \mathcal{O}_C(2D)) \leq \frac{-g+6h+4}{3} < 0$$

and

$$\deg(\mathcal{L} \otimes \mathcal{O}_C(2D)) \leq \frac{-g+6h+2}{2} < 0$$

by the assumption on g and h. Therefore we have $\Gamma(C, \mathcal{E} \otimes \mathcal{O}_C(2D)) = 0$. This implies $\dim_k(\Gamma(X, \pi^*(\mathcal{O}_C(2D)))) = 3$ and this finishes the proof of Lemma 2.1 by the Riemann-Roch theorem. □

REMARK 2.9. (i) One may show the result of Lemma 2.1 (with a slightly weaker bound on g) in the following way. Here we sketch an alternative argument for $h = 2e+1$ only; the $h = 2e$ case is similar. $W^1_{e+2}(C)$ is reduced at a general point $\mathcal{L}$, hence $h^0(C, 2\mathcal{L}) = 4$. Suppose $\pi^*\mathcal{L}$ is a non-reduced point on $\pi^*(W^1_{e+2}(C)) = W^1_{3e+6}(X)$. Then $h^0(X, 2\pi^*\mathcal{L}) \geq 5$ and since $\pi^*|2\mathcal{L}| \subsetneq |2\pi^*\mathcal{L}|$, it follows that $|2\pi^*\mathcal{L}|$ is not composed with π. Thus X has a base-point-free g^1_x with $x \leq 6e+9$ not composed with π by subtracting 3 generically chosen points on X. And this is contradictory to Lemma 1.2 if $g \geq 18e+21$. Hence $W^1_{3e+6}(X)$ is reduced at a general point $\pi^*\mathcal{L}$. Finally, one can assert that $\pi^*\mathcal{L} \otimes \mathcal{O}_X(\Delta)$ is a reduced point of $W^1_{g-3e-3}(X)$ where $\mathcal{O}_X(\Delta) \in W_{g-6e-9}$ is general, by invoking a result of Coppens [C], Corollary 8.

(ii) On the other hand, one may recognize the importance of the method (i.e. utilizing the Rick Miranda's triple covering result) adopted in the proof of Lemma 2.1 as follows:

(1) For $d \geq g - [(3h+1)/2] - 1$, $W_d^1(X)$ has expected dimension and hence the enumerative method works to prove the existence of a base-point-free pencil which is not composed with the triple covering. On the other hand, for the degree outside this range $W_d^1(X)$ does not have the expected dimension and one should find a totally different way to investigate the problem. This constitutes a part of our forthcoming paper where the methods we used in Lemma 2.1 play some role in the paper.

(2) The reducedness result we proved in Lemma 2.1 has somewhat better bound (of the genus g of the curve X with respect to the genus h of the base curve C) than the one we outlined in (i) above; $g \geq 12e + 14$ vs. $g \geq 18e + 21$ for $h = 2e + 1$ and $g \geq 12e + 5$ vs. $g \geq 18e + 6$ for $h = 2e$.

3. Examples

In this section, we shall give examples of cyclic triple coverings of curves. One example suggests that there is a possibility of an improvement of our bounds given in Section 1. Another assures that the Castelnuovo-Severi bound is not sharp in some sense, i.e. the bound of degree of base-point-free pencils, which is not composed with the triple covering projection, is greater than the Castelnuovo-Severi bound. Both examples are given by cyclic triple coverings. We begin with the following remark about cyclic triple coverings.

Remark 3.1. Let C be a curve of genus h and X be a cyclic triple covering of C. Let X be of genus g and let σ be the automorphism of X of order 3 so that $X/\langle\sigma\rangle$ is equivalent to C. Let π be the projection map of X to C. Since the order of σ is prime, every fixed point of σ is a total ramification point of π and they are all. By the Riemann-Hurwitz formula, there are $g - 3h + 2$ fixed points, $P_1, \dots, P_{g-3h+2} \in X$, of σ.

Since the order of σ is 3, the space $H = H^1(X, \mathcal{O}_X)^\vee$ can be decomposed into the direct sum $H = H_0 \oplus H_1 \oplus H_2$, where

$$H_j = \{\omega \in H : \omega \circ \sigma = \zeta^j \omega\}, \quad (j = 0, 1, 2), \quad \zeta = e^{2\pi\sqrt{-1}/3}.$$

It is known that every $\omega \in H_0$ is the pullback $\pi^*\omega'$ of some $\omega' \in H^1(C, \mathcal{O}_C)^\vee$. So $\dim H_0 = h$ and for $\omega \in H_0$, the divisor $(\omega) = 2P_1 + \cdots + 2P_{g-3h+2} + \pi^*(Q_1 + \cdots + Q_{2h-2})$, where $Q_1 + \cdots + Q_{2h-2}$ is a canonical divisor on C (cf. [L]).

For a fixed point $P_i \in X$ of σ, one can take a local parameter z_i at P_i such that $\sigma : z_i \mapsto \zeta z_i$ or $\sigma : z_i \mapsto \zeta^2 z_i$. The exponent of ζ only depends on P_i.

For $\eta \in H_1$, let the Taylor expansion of η near P_i be given by

$$\eta = (a_0 + a_1 z_i + a_2 z_i^2 + \cdots)dz_i.$$

In case $\sigma : z_i \mapsto \zeta z_i$,

$$\eta \circ \sigma = (a_0 + a_1 \zeta z_i + a_2 \zeta^2 z_i^2 + \cdots)\zeta dz_i.$$

By the definition of H_1, $\eta \circ \sigma = \zeta\eta$, hence we have $a_j = 0$ if $j \equiv 1, 2 \pmod 3$. Hence, η has a zero of order 0 modulo 3 at P_i.

In case $\sigma : z_i \mapsto \zeta^2 z_i$, using a similar argument as above, one obtains that η has a zero of order 1 modulo 3 at P_i.

Hence, there is an integer t such that for every $\eta \in H_1$, renumbering the suffixes if necessary,

$$(\eta) = P_1 + \cdots + P_t + \pi^*(D_1), \tag{3.1.1}$$

where D_1 is an effective divisor on C which depends only on η. Similarly, for every $\tau \in H_2$,

$$(\tau) = P_{t+1} + \cdots + P_{g-3h+2} + \pi^*(D_2), \tag{3.1.2}$$

where D_2 is an effective divisor on C which depends only on τ.

Let $k_j = \deg D_j$. Since η, $\tau \in H$, $\deg \eta = \deg \tau = 2g - 2$. Hence, $t + 3k_1 = g - 3h + 2 - t + 3k_2 = 2g - 2$. Since $0 \le t \le g - 3h + 2$,

$$3k_j \ge 2g - 2 - (g - 3h + 2) = g + 3h - 4.$$

If $g + 3h - 4 \ge 6h - 5$, i.e. $k_j > 2h - 2$, then $h^0(C, D_j) = \dim H_j = k_j - h + 1$. We may also assume $t \ge g - 3h + 2 - t$, i.e. $2t \ge g - 3h + 2$. If it is not the case, we take $\zeta = e^{4\pi i/3}$ instead of $e^{2\pi i/3}$

EXAMPLE 3.2. *Let X, C, π, σ, t be as above. If $g \ge 3h - 1$, then there exists a complete base-point-free pencil g_n^1 which is not composed with π, for some integer $n \le \max\{t, g + 2 - t\}$.*

PROOF. It is enough to show the existence of a linear series $g^1_{\max\{t, g+2-t\}}$ which is not composed with π.

Take an $\eta \in H_1$. Let the divisor of η be

$$(\eta) = P_1 + \cdots + P_t + \pi^*(Q_1 + \cdots + Q_{k_1}),$$

$Q_1, \dots, Q_{k_1} \in C$. By (3.1.2), for every $\tau \in H_2$, the divisor of τ is of the form $P_{t+1} + \cdots + P_{g-3h+2} + \pi^*(D_2)$, $\deg D_2 = k_2$ and $h^0(D_2) = k_2 - h + 1$.

In case $t \le (g+2)/2$, we have $2t = g + 2 - 3h + 3k_2 - 3k_1 \le g + 2$, hence $k_2 - h \le k_1$. So we may assume that

$$D_2 = Q_1 + \cdots + Q_{k_2-h} + Q'_{k_2-h+1} + \cdots + Q'_{k_2},$$

for $Q'_{k_2-h+1}, \dots, Q'_{k_2} \in C$. Then, η/τ is a meromorphic function on X whose divisor is

$$\begin{aligned} &P_1 + \cdots + P_t + \pi^*(Q_{k_2-h+1} + \cdots + Q_{k_1}) \\ &\quad - (P_{t+1} + \cdots + P_{g-3h+2}) - \pi^*(Q'_{k_2-h+1} + \cdots + Q'_{k_2}). \end{aligned}$$

Hence, the linear series $|P_1 + \cdots + P_t + \pi^*(Q_{k_2-h+1} + \cdots + Q_{k_1})|$ is of degree $t + 3(k_1 - k_2 + h) = g + 2 - t$ and is not composed with π.

In case $t > (g+2)/2$, we have $k_2 - h > k_1$. In this case, we take

$$D_2 = Q_1 + \cdots + Q_{k_1} + Q'_{k_1+1} + \cdots + Q'_{k_2},$$

Then, the divisor of η/τ is

$$P_1 + \cdots + P_t - (P_{t+1} + \cdots + P_{g-3h+2}) - \pi^*(Q'_{k_1+1} + \cdots + Q'_{k_2}).$$

Then, $|P_1 + \cdots + P_t|$ is of degree t and is not composed with π. This completes the proof. □

EXAMPLE 3.3. *Let X, C, π, σ, t be as above. Assume that $g \ge 3h - 1$. If $n < (g - 3h + 2 + t)/3$, then every g^1_n is composed with π. Furthermore, if $g \ge 7h - 4$, then for every C and every t such that $(g - 3h + 2)/2 \le t \le g - 3h + 2$ and $t \equiv 2g - 2 \bmod 3$, there exists a cyclic triple covering X of C on which there exists a complete base point free linear series g^1_t not composed with the triple covering.*

PROOF. Let f be a meromorphic function on X such that $f \circ \sigma = \zeta f$, $\deg(f)_\infty = N_1$ and the support of $(f)_\infty$ does not contain any P_j. Then $f(P_j) = 0$. For $\omega \in H_0$, $\omega' = f\omega$ is a 1-form which satisfies $\omega' \circ \sigma = \zeta\omega'$. Hence, the order of zero of ω' at P_j, denoted by $\mathrm{ord}_{P_j}\omega'$, is

$$\mathrm{ord}_{P_j}\omega' \equiv \begin{cases} 1 & (j = 1, \dots, t) \\ 0 & (j = t+1, \dots, g - 3h + 2) \end{cases} \mod 3.$$

Hence,

$$\operatorname{ord}_{P_j} f \equiv \begin{cases} 2 & (j = 1, \dots, t) \\ 1 & (j = t+1, \dots, g-3h+2) \end{cases} \mod 3.$$

So, $N_1 = \deg(f)_\infty = \deg(f)_0 \geq g - 3h + 2 + t$.

Applying a similar argument for a meromorphic function f on X such that $f \circ \sigma = \zeta^2 f$, $\deg(f)_\infty = N_2$ and the support of $(f)_\infty$ does not contain any P_j, we have $N_2 \geq 2(g - 3h + 2) - t$.

Let f be a meromorphic function on X such that $\deg(f)_\infty = m$ and $f \circ \sigma \neq f$. Considering a linear fraction, if necessary, we may assume neither the support of $(f)_\infty$ nor $(f)_0$ contains any P_j.

Let $f_i = f + \zeta^{2i} f \circ \sigma + \zeta^i f \circ \sigma^2$, $(i = 0, 1, 2)$. Then $f_i \circ \sigma = \zeta^i f_i$, $(i = 0, 1, 2)$ and $3f = f_0 + f_1 + f_2$. Since $f(P_j) \neq 0$ and $f_i(P_j) = 0$ $(i = 1, 2,\ j = 1, \dots, g - 3h + 2)$, we have $f_0(P_j) \neq 0$. Further we may assume neither f_1 nor f_2 vanishes identically. If it is not the case, say f_2 vanishes identically (denoted by $f_2 \equiv 0$), one can consider $F = (1/f)$ instead of f. Again, F can be decomposed as $F = F_0 + F_1 + F_2$ such that $F_i \circ \sigma = \zeta^i F_i$ $(i = 0, 1, 2)$. Assume $F_1 \equiv 0$, then

$$3 = 3fF = (f_0 + f_1)(F_0 + F_2) = (f_0 F_0 + f_1 F_2) + f_1 F_0 + f_0 F_2.$$

Here, $(f_0 F_0 + f_1 F_2) \circ \sigma = f_0 F_0 + f_1 F_2$, $f_1 F_0 \circ \sigma = \zeta f_1 F_0$ and $f_0 F_2 \circ \sigma = \zeta^2 f_0 F_2$. Hence, $f_1 F_0 \equiv f_0 F_2 \equiv 0$ and $f_1 \equiv F_2 \equiv 0$. A contradiction. In case $F_2 \equiv 0$, again we have a contradiction.

By the definition of f_i, we have $(f_i)_\infty \leq (f)_\infty + (f \circ \sigma)_\infty + (f \circ \sigma^2)_\infty$ $(i = 0, 1, 2)$. Hence, $\deg(f_i)_\infty \leq 3m$. This implies that

$$3m \geq \max\{g - 3h + 2 + t, 2(g - 3h + 2) - t\}.$$

As in Example 3.2, we may assume $2t \geq g - 3h + 2$. Hence, $m \geq (g - 3h + 2 + t)/3$. This implies the first assertion.

Next, we shall show the second assertion. Let $K(C)$ be the meromorphic function field on C. Since $t \equiv 2g - 2 \mod 3$, $t - (g - 3h + 2 - t) = 2t - g + 3h - 2 \equiv 0 \mod 3$. Let $\ell = (2t - g + 3h - 2)/3 \geq 0$. Take $g - 3h + 2 - t + \ell$ general points $P_{t+1}, \dots, P_{g-3h+2}, Q_1, \dots, Q_\ell$ on C. Since $g - 3h + 2 - t + 3\ell = t \geq (g - 3h + 2)/2 \geq 2h - 1$, there is a $y \in K(C)$ such that

$$(y)_\infty = P_{t+1} + \cdots + P_{g-3h+2} + 3(Q_1 + \cdots + Q_\ell).$$

We may assume the support of the zero divisor of y consists of distinct t points $P_1, \dots, P_t \in C$. Let x be a triple valued function on C satisfying $y = x^3$ and let X

be a curve so that $K(X) = K(C)(x)$. For $P \in C$, taking a local parameter z_P at P, we have an automorphism σ of X such that $\sigma : (z_P, x(z_P)) \mapsto (z_P, \zeta x(z_P))$. Then $X/\langle\sigma\rangle$ is equivalent to C. Thus, X is a cyclic triple covering of C which is branched over $P_1, \dots, P_{g-3h+2}$.

Since $x \in K(X)$ satisfies $x \circ \sigma = \zeta x$, we can divide $P_1, \dots, P_{g-3h+2}$ into $P_1, \dots, P_t$ and $P_{t+1}, \dots, P_{g-3h+2}$ as desired. It is noted that in this case we have g_t^1 which is not composed with π. It is a better estimate in case $t < (g+2)/2$. $\square$

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