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# SOME DIFFERENTIAL-GEOMETRIC PROPERTIES OF $R$-SPACES

By

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## §0. Introduction

Let $G/K$ be an irreducible Riemannian symmetric space, where $G$ is a connected compact semisimple Lie group and $K$ its closed subgroup. The adjoint representation group $\mathrm{Ad}(K)$ acts on the tangent space $T_o(G/K)$ of $G/K$ at the origin $o$ as an isometry group. Let $S$ denote a unit hypersphere in the $T_o(G/K)$ centered at the origin $o$. For each point $a$ of $S$, the orbit $\mathrm{Ad}(K)a$ of $a$ under $\mathrm{Ad}(K)$ is called an *R-space*. The $R$-spaces form an abundant class of homogeneous Riemannian manifolds and have several distinguished properties as submanifolds of $S$, and so they have been investigated by many authors from the point of view of differential geometry. (e.g., [5], [10], [12], [13], [16], [17], [21], [22], [24], [31], [32], [33])

In this paper, for these $R$-spaces we shall study the following:

(I) In the case where $G/K$ is Hermitian, we investigate some relations between the complex structure and the restricted root system with respect to $G/K$.

(II) We express the covariant derivative of the second fundamental form of every $R$-space in $S$ with respect to the Lie brackets in the Lie algebra of $G$.

As an application of (I), we obtain many new examples of homogeneous $CR$-submanifold in a complex projective space, which is stated as Theorem 3.2. As an application of (II), we can give a partial solution to the S. Maeda's Problem, which is stated as Corollary 4.5.

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## §1. Preliminaries

In this paper, let $G/K$ be an irreducible Riemannian symmetric space of compact type once and for all, where $G$ is a connected compact semisimple Lie group

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and $K$ its closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote Lie algebras of $G$ and $K$, respectively. Then $G/K$ gives rise to an involutive automorphism $\theta$ of $\mathfrak{g}$ such that $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$. Put $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \quad \text{(direct sum)}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

We can identify $\mathfrak{p}$ with the tangent space $T_o(G/K)$ of $G/K$ at the origin $o$. Let $B$ denote the Killing form of $\mathfrak{g}$. We may assume that the metric $g$ on $G/K$ is given by $g_o = -B|_{\mathfrak{p}\times\mathfrak{p}}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ denote the dual space of $\mathfrak{a}$. For each $\lambda \in \mathfrak{a}^*$, we define subspaces $\mathfrak{k}_\lambda$ and $\mathfrak{p}_\lambda$ of $\mathfrak{g}$ as follows:

$$\mathfrak{p}_\lambda = \{X \in \mathfrak{p} \mid (\operatorname{ad} H)^2(X) = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\},$$

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} \mid (\operatorname{ad} H)^2(X) = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\}.$$

Then $\mathfrak{p}_\lambda = \mathfrak{p}_{-\lambda}$, $\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}$, $\mathfrak{p}_0 = \mathfrak{a}$ and $\mathfrak{k}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. An element $\lambda$ of $\mathfrak{a}^*$ is called a *restricted root* of $\mathfrak{g}$ with respect to $\mathfrak{a}$ if $\dim \mathfrak{p}_\lambda \neq 0$. We select a suitable ordering in $\mathfrak{a}^*$ and denote by $\Delta$ the set of all positive restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then we have

$$\mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \Delta} \mathfrak{p}_\lambda \quad \text{(orthogonal direct sum)}, \quad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \Delta} \mathfrak{k}_\lambda, \tag{1.1}$$

$$[\mathfrak{a}, \mathfrak{k}_\lambda] = \mathfrak{p}_\lambda \quad \text{and} \quad [\mathfrak{a}, \mathfrak{p}_\lambda] = \mathfrak{k}_\lambda, \quad \lambda \in \Delta. \tag{1.2}$$

The following facts are fundamental (cf. [9]).

If $\lambda, \mu \in \Delta \cup \{0\}$, then

$$\begin{aligned} [\mathfrak{k}_\lambda, \mathfrak{k}_\mu] &\subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}, \\ [\mathfrak{k}_\lambda, \mathfrak{p}_\mu] &\subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}, \\ [\mathfrak{p}_\lambda, \mathfrak{p}_\mu] &\subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}. \end{aligned} \tag{1.3}$$

Moreover, if $\lambda + \mu \in \Delta \cup \{0\}$ or $\lambda - \mu \in \Delta \cup \{0\}$, then

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \neq 0. \tag{1.4}$$

Let $S$ denote a unit hypersphere in $\mathfrak{p}$ centered at the origin $o$. The adjoint representation group $\operatorname{Ad}(K)$ acts on $\mathfrak{p}$ as an isometry group. For any $a \in S$, the orbit $\operatorname{Ad}(K)a$ of $a$ under $\operatorname{Ad}(K)$ is a submanifold in $S$, which is called an *R-space*. For any $a$ $(\neq 0)$ in $\mathfrak{p}$, we put $M_a = \operatorname{Ad}(K)a$ for simplicity. For any real number $\xi \neq 0$, an $R$-space $M_{\xi a}$ is similar to an $R$-space $M_a$. On the other hand, every orbit in $\mathfrak{p}$ under $\mathrm{Ad}(K)$ meets $\mathfrak{a}$ ([32]). Therefore, we can say that all $R$-spaces $M_a$ with $a \in S \cap \mathfrak{a}$ exhaust all $R$-spaces.

For a manifold $L$ and a point $l$ of $L$ we denote by $T_l(L)$ the tangent space of $L$ at $l$. If $Q$ is a submanifold in $L$ and $q$ is a point of $Q$, then we denote by $T_q^N(Q)$ the normal space of $Q$ in $L$ at $q$.

For a point $b$ of $M_a$, let $T_b^N(M_a)$ denote the normal space of $M_a$ in $S$ at $b$. Any vector $X$ in $\mathfrak{p}$ can be uniquely written as $X = A + B + C$, where $A \in \boldsymbol{R}b$, $B \in T_b(M_a)$, $C \in T_b^N(M_a)$. Then we put

$$X_{S_b} = B + C, \quad X_{M_b} = B \quad \text{and} \quad X^{N_b} = C.$$

In particular, we put

$$X_S = X_{S_a}, \quad X_M = X_{M_a} \quad \text{and} \quad X^N = X^{N_a}.$$

Each vector $X$ in $\mathfrak{k}$ induces a vector field $X^*$ on $\mathfrak{p}$ as follows:

$$X_Y^* = \frac{d}{dt}\bigg|_0 \mathrm{Ad}(\exp tX)Y = [X, Y], \quad Y \in \mathfrak{p}. \tag{1.5}$$

Let a symbol $X^*$ stand for a vector field $X^*|_S$ on $S$ or a vector field $X^*|_{M_a}$ on $M_a$ for simplicity. We put

$$a^\perp = \{X \in \mathfrak{a} \mid g_o(X, a) = 0\} \quad \text{and} \quad \Delta_a = \{\lambda \in \Delta \mid \lambda(a) = 0\}.$$

Then from (1.5) we have

$$\begin{cases} T_a(M_a) = [\mathfrak{k}, a] = \sum_{\lambda \in \Delta - \Delta_a} \mathfrak{p}_\lambda, \\ T_a^N(M_a) = a^\perp + \sum_{\lambda \in \Delta_a} \mathfrak{p}_\lambda. \end{cases} \tag{1.6}$$

From the definition of $\Delta_a$ we know easily the following:

(i) If $\lambda, \mu \in \Delta_a$ and $\lambda + \mu \in \Delta$, then

$$\lambda + \mu \in \Delta_a. \tag{1.7}$$

(ii) If $\lambda \in \Delta_a$, $\mu \in \Delta - \Delta_a$ and $\lambda + \mu \in \Delta$, then

$$\lambda + \mu \in \Delta - \Delta_a. \tag{1.8}$$

Let $\nabla$ and $\overline{\nabla}$ denote the Riemannian connections of $M_a$ and $S$, respectively. Let $h$ denote the second fundamental form of $M_a$ in $S$. Then we have the following fundamental formulas:

$$\begin{aligned}\bar{\nabla}_{X_a^*} Y^* &= \left(\frac{d}{dt}\bigg|_0 Y^*_{\mathrm{Ad}(\exp tX)a}\right)_S \\ &= \left(\frac{d}{dt}\bigg|_0 [Y, \mathrm{Ad}(\exp tX)a]\right)_S \\ &= [Y,[X,a]]_S, \\ \nabla_{X_a^*} Y^* &= \left(\frac{d}{dt}\bigg|_0 Y^*_{\mathrm{Ad}(\exp tX)a}\right)_M \\ &= \left(\frac{d}{dt}\bigg|_0 [Y, \mathrm{Ad}(\exp tX)a]\right)_M \\ &= [Y,[X,a]]_M, \quad X, Y \in \mathfrak{k}.\end{aligned}$$

From these we have

$$\begin{aligned}h(X_a^*, Y_a^*) &= (\bar{\nabla}_{X_a^*} Y^*)^N \\ &= [Y,[X,a]]^N, \quad X, Y \in \mathfrak{k}.\end{aligned} \tag{1.9}$$

From now on we assume that a symmetric space $G/K$ is Hermitian, unless otherwise stated. We put $p = \dim \mathfrak{a}$. In the case where $p = 1$, since any $R$-space $M_a$ is very simple, we can easily compute various geometrical quantities on $M_a$ which we want to know in this paper. So we assume that $p \geq 2$.

Now we note the following fact.

Lemma 1.1 ([8, p. 528]). *There are two possibilities $\Delta_1$ and $\Delta_2$ for $\Delta$ as follows. There exists a base $\{\lambda_1, \dots, \lambda_p\}$ of $\mathfrak{a}^*$ such that*

$$\Delta_1 = \{2\lambda_i, \lambda_i \pm \lambda_j \mid 1 \leq i < j \leq p\},$$

$$\Delta_2 = \{\lambda_i, 2\lambda_i, \lambda_i \pm \lambda_j \mid 1 \leq i < j \leq p\}.$$

If $\Delta$ can be expressed as $\Delta_1$ (resp. $\Delta_2$), then $\Delta$ is called of *type C* (resp. *type BC*). We put $I = \{1, \dots, p\}$. Let $I_p$ denote the set of all permutations of $I$. Put

$$\varepsilon_i = \pm 1, \quad 1 \leq i \leq p.$$

For any $\sigma \in I_p$, we put

$$\mu_i = \varepsilon_i \lambda_{\sigma(i)}, \quad 1 \leq i \leq p.$$

Introduce a new lexicographic ordering in $\mathfrak{a}^*$ with respect to the basis $\{\mu_1, \dots, \mu_p\}$. Then the set $\Delta'$ of all new positive restricted roots coincides with the set obtained from $\Delta$ by exchanging every symbol $\lambda$ in $\Delta$ by the symbol $\mu$. In this case, we shall say that *we took a reorder in* $\mathfrak{a}^*$, or *reordered* $\mathfrak{a}^*$.

Let $J$ be the complex structure on $G/K$ at the origin $o$, and put $\dim \mathfrak{p} = 2n+2$. Then we can consider $\mathfrak{p}$ a complex vector space $C^{n+1}$. We denote by $P_n(C)$ the complex projective space, and by $\pi$ the natural projection of $S$ onto $P_n(C)$. The complex structure and the Fubini-Study metric on $P_n(C)$ can be naturally induced from $J$ and $g_o$ through $\pi$. We denote them by $\tilde{J}$ and $\langle\,,\rangle$, respectively. We denote the image $\pi(M_a)$ of an $R$-space $M_a$ under consideration by $\tilde{M}_a$, which we shall call an *$\tilde{R}$-space*. Obviously every $\tilde{R}$-space is a homogeneous submanifold in $P_n(C)$.

Generally, let $\tilde{L}$ be a submanifold of $P_n(C)$ and put $L = \pi^{-1}(\tilde{L})$. Then $L$ is a submanifold in $S$. For $q \in L$ and $\tilde{X} \in T_{\pi(q)}(\tilde{L})$, there exists a unique $\tilde{X}' \in T_q(L)$ such that $\tilde{X}' \in V_q$ and $\pi_{*_q}\tilde{X}' = \tilde{X}$, where $V_q$ denotes the orthogonal complement of $J(q)$ in $T_q(L)$ and $\pi_{*_q}$ the differential map of $\pi$ at $q$. This $\tilde{X}'$ is called the *horizontal lift of* $\tilde{X}$ at $q$. Then we have $(\tilde{J}\tilde{X})' = J\tilde{X}'$. We denote by $T^N_{\pi(q)}(\tilde{L})$ the normal space of $\tilde{L}$ in $P_n(C)$ at $\pi(q)$ and put

$$\tilde{J}\tilde{X} = (\tilde{J}\tilde{X})_{\tilde{L}} + (\tilde{J}\tilde{X})^N,$$

where $(\tilde{J}\tilde{X})_{\tilde{L}} \in T_{\pi(q)}(\tilde{L})$ and $(\tilde{J}\tilde{X})^N \in T^N_{\pi(q)}(\tilde{L})$.

Let $\nabla$ and $\tilde{\nabla}$ denote the Riemannian connections of $L$ and $\tilde{L}$, respectively. We denote by $h$ and $\tilde{h}$ the second fundamental forms of $L$ in $S$ and $\tilde{L}$ in $P_n(C)$, respectively. Then there is a following relation between covariant derivatives of $h$ and $\tilde{h}$ (e.g., cf. [1])

$$(1.10)\qquad (\nabla_{\tilde{X}'}h)(\tilde{Y}', \tilde{Z}') = ((\tilde{\nabla}_{\tilde{X}}\tilde{h})(\tilde{Y}, \tilde{Z}) + \langle(\tilde{J}\tilde{X})_{\tilde{L}}, \tilde{Y}\rangle(\tilde{J}\tilde{Z})^N + \langle(\tilde{J}\tilde{X})_{\tilde{L}}, \tilde{Z}\rangle(\tilde{J}\tilde{Y})^N)', \quad \tilde{X}, \tilde{Y}, \tilde{Z} \in T_{\pi(q)}(\tilde{L}).$$

From this we see

$$(1.11)\qquad \nabla h = 0 \quad \text{on } V_q \Leftrightarrow \mathfrak{S}\tilde{\nabla}\tilde{h} = 0 \quad \text{on } T_{\pi(q)}(\tilde{L})$$

where $\mathfrak{S}$ denotes the cyclic sum.

Now we recall the notation of $CR$-submanifolds owing to A. Bejancu ([1]).

Definition. A submanifold $\tilde{L}$ in $P_n(C)$ is called a *$CR$-submanifold* if there are two subbundles $\mathfrak{D}$ and $\mathfrak{D}^\perp$ of $T(\tilde{L})$ such that

(i) $T_{\tilde{l}}(\tilde{L}) = \mathfrak{D}_{\tilde{l}} + \mathfrak{D}_{\tilde{l}}^{\perp}$ (orthogonal sum) for each $\tilde{l} \in \tilde{L}$,

(ii) $\tilde{J}\mathfrak{D} = \mathfrak{D}$, $\tilde{J}\mathfrak{D}^{\perp} \subset T^N(\tilde{L})$,

where $T^N(\tilde{L})$ denotes the normal bundle of $\tilde{L}$ in $P_n(C)$.

If a $CR$-submanifold $\tilde{L}$ satisfies $\mathfrak{D} = 0$ (resp. $\mathfrak{D}^{\perp} = 0$), then $\tilde{L}$ is called *totally real* (resp. *holomorphic*). If a $CR$-submanifold $\tilde{L}$ satisfies $\tilde{J}\mathfrak{D}^{\perp} = T^N(\tilde{L})$, then $\tilde{L}$ is called *anti-holomorphic*.

## §2. Some Basic Lemmas

Through this paper we preserve notations in §1. First we give some basic Lemmas for later use.

LEMMA 2.1. *Let $G/K$ be a symmetric space and $a$ be any point in $\mathfrak{a} \cap S$. Then the following holds.*

(i) *If $\lambda \in \Delta - \Delta_a$ and $\mu \in \Delta$, then*

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu^N]^N = 0. \tag{2.1}$$

(ii) *If $X, Y \in \sum_{\lambda \in \Delta} \mathfrak{k}_\lambda$ and $Z \in \sum_{\lambda \in \Delta - \Delta_a} \mathfrak{k}_\lambda$, then*

$$[Z, [Y, [X, a]]^N]^N = 0. \tag{2.2}$$

(iii) *If $\lambda + \mu \in \Delta_a$ or $\lambda - \mu \in \Delta_a$, then*

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu]^N \neq 0. \tag{2.3}$$

PROOF. (i) In the case where $\mu \in \Delta_a$ (resp. $\mu \in \Delta - \Delta_a$), from (1.6) we have $\mathfrak{p}_\mu^N = \mathfrak{p}_\mu$ (resp. $\mathfrak{p}_\mu^N = 0$). Hence from (1.3) we have

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu^N] = [\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}.$$

On the other hand, since $\lambda + \mu, \lambda - \mu \in \Delta - \Delta_a$, we have from (1.6)

$$\mathfrak{p}_{\lambda+\mu}^N = 0 = \mathfrak{p}_{\lambda-\mu}^N,$$

which completes the proof of (i).

(ii) It suffices to prove that

$$[Z, [Y, [X, a]]^N]^N = 0 \quad \text{for } X \in \mathfrak{k}_\lambda, \quad Y \in \mathfrak{k}_\mu, \quad Z \in \mathfrak{k}_\nu,$$

where $\lambda, \mu \in \Delta$ and $\nu \in \Delta - \Delta_a$. From (1.3) we have

$$[Y, [X, a]] \in \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}.$$

Moreover, since

$$\mathfrak{p}^N_{\lambda+\mu} = \begin{cases} \mathfrak{p}_{\lambda+\mu} & \text{if } \lambda+\mu \in \Delta_a \\ 0 & \text{if } \lambda+\mu \notin \Delta_a \end{cases}$$

and

$$\mathfrak{p}^N_{\lambda-\mu} = \begin{cases} \mathfrak{p}_{\lambda-\mu} & \text{if } \lambda-\mu \in \Delta_a \\ 0 & \text{if } \lambda-\mu \notin \Delta_a, \end{cases}$$

we have

$$[Y, [X, a]]^N \in \mathfrak{p}^N_{\lambda+\mu} + \mathfrak{p}^N_{\lambda-\mu}.$$

Now (2.2) follows from (i).

(iii) This follows from (1.3), (1.4), and (1.6). (Q.E.D.)

In the case where a symmetric space $G/K$ is Hermitian, we denote by $\mathfrak{z}$ the center of $\mathfrak{k}$. Then, as for a complex structure $J$, the following fact is known ([8], p. 376).

LEMMA 2.2. (i) *There exists a unique (up to sign)* $\tilde{Z} \in \mathfrak{z}$ *such that*

$$J = \operatorname{ad} \tilde{Z}|_{\mathfrak{p}};$$

(ii) *The element* $\tilde{Z}$ *can be written as*

$$\tilde{Z} = Z_0 + \sum_{i=1}^{p} Z_{2\lambda_i},$$

*where* $Z_0 \in \mathfrak{k}_0$ *and* $0 \neq Z_{2\lambda_i} \in \mathfrak{k}_{2\lambda_i}$.

Using this, we shall prove a key lemma.

LEMMA 2.3. *We have the following equations*:

$$J\mathfrak{p}_{\lambda_i \pm \lambda_j} = \mathfrak{p}_{\lambda_i \mp \lambda_j}, \tag{2.4}$$

$$J\mathfrak{p}_{\lambda_i} = \mathfrak{p}_{\lambda_i}, \tag{2.5}$$

$$J\mathfrak{a} = \sum_{i=1}^{p} \mathfrak{p}_{2\lambda_i}, \tag{2.6}$$

$$\sum_{i=1}^{p} J\mathfrak{p}_{2\lambda_i} = \mathfrak{a}. \tag{2.7}$$

PROOF. Since $\tilde{Z}$ satisfies $[\tilde{Z}, \mathfrak{k}] = 0$, we see from Lemma 2.2(ii) that

$$[Z_0, X_\lambda] + \sum_{i=1}^{p} [Z_{2\lambda_i}, X_\lambda] = 0, \quad X_\lambda \in \mathfrak{k}_\lambda.$$

Owing to (1.3), we have

$$[Z_0, X_\lambda] = 0 \quad \text{for } X_\lambda \in \mathfrak{k}_\lambda,$$

where $\lambda \in \Delta - \{\lambda_1, \ldots \lambda_p\}$. From this equation and (1.2), we have

$$0 = [\mathfrak{a}, [Z_0, \mathfrak{k}_\lambda]] = [Z_0, [\mathfrak{a}, \mathfrak{k}_\lambda]] = [Z_0, \mathfrak{p}_\lambda], \quad \lambda \in \Delta - \{\lambda_1, \ldots \lambda_p\}.$$

It follows from Lemma 1.1 and Lemma 2.2 that

$$J\mathfrak{p}_\lambda = \begin{cases} \mathfrak{p}_\lambda & \text{if } \lambda \in \{\lambda_1, \ldots \lambda_p\} \\ \sum_{i=1}^{p} [Z_{2\lambda_i}, \mathfrak{p}_\lambda] & \text{if } \lambda \in \Delta - \{\lambda_1, \ldots \lambda_p\}. \end{cases}$$

Now the Lemma follows from (1.3). (Q.E.D.)

## §3. *CR*-Submanifolds in a Complex Projective Space $P_n(C)$

THEOREM 3.1. *Let $G/K$ be a Hermitian symmetric space and $a$ be any point in $\mathfrak{a} \cap S$. Then an $\tilde{R}$-space $\tilde{M}_a$ is a CR-submanifold in $P_n(C)$.*

REMARK. Y. Shimizu ([31]) showed that an $\tilde{R}$-space $\tilde{M}_a$ is a *CR*-submanifold in $P_n(C)$ if $\Delta_a = \emptyset$.

PROOF OF THEOREM 3.1. From (1.6) we have

$$T_a(M_a) = \sum_{\lambda \in \Delta - \Delta_a} \mathfrak{p}_\lambda.$$

Lemma 2.3 implies that there are elements $\lambda$ and $\mu$ in $\Delta - \Delta_a$ such that

$$J\mathfrak{p}_\lambda = \mathfrak{p}_\lambda \quad \text{and} \quad J\mathfrak{p}_\mu \subset T^N(M_a).$$

Then we put

$$I_\pm = \{(i,j) \mid \lambda_i + \lambda_j, \lambda_i - \lambda_j \in \Delta - \Delta_a, 1 \le i < j \le p\},$$

$$\mathfrak{p}_{(i,j)} = \begin{cases} \mathfrak{p}_{\lambda_i + \lambda_j} + \mathfrak{p}_{\lambda_i - \lambda_j} & \text{if } (i,j) \in I_\pm \\ 0 & \text{if } (i,j) \notin I_\pm. \end{cases}$$

Moreover we put

$$\hat{\mathfrak{D}}_a = \begin{cases} \sum_{(i,j)\in I_\pm} \mathfrak{p}_{(i,j)} & \text{if } \Delta \text{ is of type } C \\ \sum_{(i,j)\in I_\pm} \mathfrak{p}_{(i,j)} + \sum_{\lambda_i\in\Delta-\Delta_a} \mathfrak{p}_{\lambda_i} & \text{if } \Delta \text{ is of type } BC. \end{cases} \tag{3.1}$$

By (1.1) and Lemma 2.2, we see that $\hat{\mathfrak{D}}_a$ and $J(a)$ are mutually orthogonal. Let $\hat{\mathfrak{D}}_a^\perp$ denote the orthogonal complement of $\hat{\mathfrak{D}}_a + J(a)$ in $T_a(M_a)$. Then we have

$$T_a(M_a) = \hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^\perp + \boldsymbol{R}J(a) \quad \text{(direct sum)}. \tag{3.2}$$

Since $\pi$ is a submersion, a space $\hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^\perp$ can be identified with $T_{\pi(a)}(\tilde{M}_a)$. By the action of $\mathrm{Ad}(K)$ we can construct two subbundles $\mathfrak{D}$ and $\mathfrak{D}^\perp$ on $\tilde{M}_a$ such that

$$\begin{gathered} \pi_*(\hat{\mathfrak{D}}_a) = \mathfrak{D}_{\pi(a)}, \quad \pi_*(\hat{\mathfrak{D}}_a^\perp) = \mathfrak{D}_{\pi(a)}^\perp, \\ \tilde{J}\mathfrak{D} = \mathfrak{D}, \quad \tilde{J}\mathfrak{D}^\perp \subset T^N(\tilde{M}_a). \end{gathered} \tag{3.3}$$

Since $J = \mathrm{ad}\,\tilde{Z}|_{\mathfrak{p}}$, the bundles $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are well-defined. These $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are the desired subbundles of $T(\tilde{M}_a)$. (Q.E.D.)

Now we can find a class of $R$-spaces with a distinguished property:

THEOREM 3.2. *Let $G/K$ be a Hermitian symmetric space and $a$ be any point in $\mathfrak{a}\cap S$. Then*

(i) *An $\tilde{R}$-space $\tilde{M}_a$ is anti-holomorphic if and only if for a suitable reordering in $\mathfrak{a}^*$ the set $\Delta_a$ is a subset of $\{\lambda_i - \lambda_j \mid 1 \le i < j \le p\}$.*

(ii) *An $\tilde{R}$-space $\tilde{M}_a$ is totally real if and only if $\Delta$ is of type $C$ and for a suitable reordering in $\mathfrak{a}^*$ the set $\Delta_a$ can be expressed as $\{\lambda_i - \lambda_j \mid 1 \le i < j \le p\}$.*

(iii) *An $\tilde{R}$-space $\tilde{M}_a$ is holomorphic if and only if for a suitable reordering in $\mathfrak{a}^*$ the set $\Delta_a$ is given by*

$$\{2\lambda_i, \lambda_i \pm \lambda_j \mid 2 \le i < j \le p\} \quad \text{if } \Delta \text{ is of type } C$$

$$\{\lambda_i, 2\lambda_i, \lambda_i \pm \lambda_j \mid 2 \le i < j \le p\} \quad \text{if } \Delta \text{ is of type } BC.$$

PROOF. (i) Let $\tilde{M}_a$ be anti-holomorphic. First we assert

$$\lambda_i(a) \neq 0, \quad i = 1, \dots, p. \tag{3.4}$$

In fact, assume that $\lambda_i(a) = 0$ for some index $i$. Then from (1.6) and (3.3) we have

$$\mathfrak{p}_{2\lambda_i} \subset T_a^N(M_a) \quad \text{and} \quad J\mathfrak{p}_{2\lambda_i} \subset JT_a^N(M_a) = \hat{\mathfrak{D}}_a^\perp.$$

On the other hand, since $J\mathfrak{p}_{2\lambda_i} \subset \mathfrak{a}$, we have from (1.1) and (1.6)

$$J\mathfrak{p}_{2\lambda_i} \not\subset \hat{\mathfrak{D}}_a^{\perp},$$

which is a contradiction. Thus (3.4) was proved. Since the case where $\Delta_a = \emptyset$ is trivial, let $\Delta_a \neq \emptyset$. Then by (3.4) there are indices $i$ and $j$ such that

$$\lambda_i + \lambda_j \in \Delta_a \quad \text{or} \quad \lambda_i - \lambda_j \in \Delta_a.$$

For this $i$, we put

$$\Delta' = \{\lambda \in \Delta_a \mid \lambda = \lambda_i + \lambda_j \text{ or } \lambda = \lambda_i - \lambda_j \text{ for some } j\}$$

and denote by $k$ the cardinal number of $\Delta'$. Since for any $i$ and $j$ with $1 \leq i < j \leq p$ the case where both $\lambda_i + \lambda_j$ and $\lambda_i - \lambda_j$ belong to $\Delta_a$ can not occur by (3.4), we can reorder $\mathfrak{a}^*$ so that

$$\Delta' = \{\lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_{k+1}\}.$$

Put

$$\Delta(1) = \{\lambda_i - \lambda_j \mid 1 \leq i < j \leq k+1\}.$$

Then $\Delta(1) \subset \Delta_a$. If $\Delta_a - \Delta(1) \neq \emptyset$, then we can continue this procedure for the set $\Delta_a - \Delta(1)$ and obtain a subset $\Delta(2)$ of $\Delta_a - \Delta(1)$ such that $\Delta(2)$ is given by the form $\{\lambda_i - \lambda_j \mid k+2 \leq i < j \leq l+1\}$, where $l-k-1$ is the cardinal number of $\Delta(2)$. By the induction, $\Delta_a$ is given by the subset of $\{\lambda_i - \lambda_j \mid 1 \leq i < j \leq p\}$. The converse is obvious from Lemma 2.3, (1.6) and (3.3).

(ii) Let $\tilde{M}_a$ be totally real. By (2.5), $\Delta$ is of type $C$. First we assert that $2\lambda_i \in \Delta - \Delta_a$ for any index $i$. In fact, assume that there exists an index $j$ such that $2\lambda_j \in \Delta_a$. Since $a$ is nonzero, there exists an index $k$ such that $2\lambda_k \in \Delta - \Delta_a$. Then for these indices $j$ and $k$, we have from (1.8)

$$\lambda_j + \lambda_k, \ \lambda_j - \lambda_k \in \Delta - \Delta_a,$$

which contradicts (2.5). Thus the assertion was proved. Since for any indices $i$ and $j$

$$\lambda_i + \lambda_j \in \Delta_a \Leftrightarrow \lambda_i - \lambda_j \in \Delta - \Delta_a,$$

we can reorder $\mathfrak{a}^*$ so that $\Delta_a$ is given by

$$\{\lambda_i - \lambda_j \mid 1 \leq i < j \leq p\}.$$

The converse follows from Lemma 2.3, (1.6) and (3.3).

(iii) Let $\tilde{M}_a$ be holomorphic. First we assert that there exists an only index $i$ such that $\lambda_i(a) \neq 0$. In fact, if there exist two indices $i$ and $j$ such that $\lambda_i(a) \neq 0$ and $\lambda_j(a) \neq 0$, a 2-dimensional subspace $J(\mathfrak{p}_{2\lambda_i} + \mathfrak{p}_{2\lambda_j})$ of $T_a(M_a)$ must contain a nonzero element of $\mathfrak{a}$, which contradict (1.6). Hence the assertion was proved. Then we have only to reorder $\mathfrak{a}^*$ so that $\lambda_1(a) \neq 0$. The converse follows from Lemma 2.3, (1.6) and (3.3). (Q.E.D.)

REMARK. Recently Choe, Ki and Takagi ([4]) and Ki, Song and Takagi ([15]) gave some examples of $CR$-submanifolds in $P_n(\boldsymbol{C})$. These examples form a class of $\tilde{R}$-spaces constructed from Theorem 3.2.

REMARK. For every totally real $\tilde{R}$-space $\tilde{M}_a$, we have

$$\dim T(\tilde{M}_a) = \dim T^N(\tilde{M}_a).$$

This is already pointed out by S. Kobayashi ([17]).

## §4. Second Fundamental Forms of $R$-Spaces and Its Covariant Derivatives

For a while, we do not assume that a symmetric space $G/K$ is Hermitian. We define the covariant derivative $\nabla h$ of $h$ on $T_a(S)$ as follows:

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) := (\bar{\nabla}_{X_a^*} h(Y_a^*, Z_a^*))^N - h(\nabla_{X_a^*} Y^*, Z_a^*) - h(Y_a^*, \nabla_{X_a^*} Z^*).$$

THEOREM 4.1. *Let $G/K$ be a symmetric space and $a$ be any point in $\mathfrak{a} \cap S$. Let $\nabla$ and $h$ denote the Riemannian connection of an $R$-space $M_a$ and the second fundamental form of $M_a$ in $S$, respectively. Then we have*

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = -[X, [Z, [Y, a]]_M]^N - [Y, [Z, [X, a]]_M]^N, \tag{4.1}$$

*where $X, Y, Z \in \mathfrak{k}$.*

PROOF. First we calculate $h(\nabla_{X_a^*} Y^*, Z_a^*)$. From (1.9), we have

$$h(\nabla_{X_a^*} Y^*, Z_a^*) = (\bar{\nabla}_L Z^*)^N,$$

where $L = \nabla_{X_a^*} Y^*$. This $L$ can be written as

$$L = \sum_{\lambda \in \Delta - \Delta_a} L_\lambda,$$

where $L_\lambda \in \mathfrak{p}_\lambda$. By Takagi and Takahashi ([32]), we see that

$$L = [Q, a] = [Y, [X, a]]_M,$$

where $Q = \sum_{\lambda\in\Delta-\Delta_a}(1/\lambda(a)^2)[a, L_\lambda]$. From the equation above we have

$$\begin{aligned}\bar{\nabla}_L Z^* &= \left(\frac{d}{dt}\bigg|_0 Z^*_{\mathrm{Ad}(\exp tQ)a}\right)_S \\ &= \left(\frac{d}{dt}\bigg|_0 [Z, \mathrm{Ad}(\exp tQ)a]\right)_S \\ &= [Z, [Q, a]]_S \\ &= [Z, [Y, [X, a]]_M]_S, \quad X, Y, Z \in \mathfrak{k}.\end{aligned}$$

Hence we obtain

$$h(\nabla_{X_a^*} Y^*, Z_a^*) = [Z, [Y, [X, a]]_M]^N.$$

Next, we have

$$\begin{aligned}(\bar{\nabla}_{X_a^*} h(Y_a^*, Z_a^*))^N &= \left(\frac{d}{dt}\bigg|_0 (\bar{\nabla}_{Y^*} Z^* - \nabla_{Y^*} Z^*)_{a(t)}\right)^N \\ &= \left(\frac{d}{dt}\bigg|_0 \bar{\nabla}_{Y^*_{a(t)}} Z^*\right)^N - \left(\frac{d}{dt}\bigg|_0 (\bar{\nabla}_{Y^*_{a(t)}} Z^*)_{M_{a(t)}}\right)^N,\end{aligned}$$

where $a(t) = \mathrm{Ad}(\exp tX)a$. As for the first term, we have

$$\begin{aligned}\left(\frac{d}{dt}\bigg|_0 \bar{\nabla}_{Y^*_{a(t)}} Z^*\right) &= \frac{d}{dt}\bigg|_0 [Z, Y^*_{a(t)}]_{S_{a(t)}} \\ &= \frac{d}{dt}\bigg|_0 [Z, [Y, a(t)]]_{S_{a(t)}} \\ &= [Z, [Y, [X, a]]_S]_S.\end{aligned}$$

As for the second term, we have

$$\begin{aligned}\frac{d}{dt}\bigg|_0 (\bar{\nabla}_{Y^*_{a(t)}} Z^*)_{M_{a(t)}} &= \frac{d}{dt}\bigg|_0 [Z, Y^*_{a(t)}]_{M_{a(t)}} \\ &= \frac{d}{dt}\bigg|_0 [Z, [Y, a(t)]]_{M_{a(t)}} \\ &= \frac{d}{dt}\bigg|_0 \mathrm{Ad}(\exp tX)[\mathrm{Ad}(\exp -tX)Z, [\mathrm{Ad}(\exp -tX)Y, a]]_M \\ &= [X, [Z, [Y, a]]_M]_S - [[X, Z], [Y, a]]_M - [Z, [[X, Y], a]]_M.\end{aligned}$$

Consequently using the equations above, (1.3), (1.6) and (2.2), we have

$$\begin{aligned}(\nabla_{X_a^*}h)(Y_a^*, Z_a^*) &= [Z,[Y,[X,a]]_S]^N - [X,[Z,[Y,a]]_M]^N \\ &\quad - [Z,[Y,[X,a]]_M]^N - [Y,[Z,[X,a]]_M]^N \\ &= -[X,[Z,[Y,a]]_M]^N - [Y,[Z,[X,a]]_M]^N. \qquad \text{(Q.E.D.)}\end{aligned}$$

D. Ferus ([5], [6]) proved the following facts.

(1) Let $a$ be a point on $S$ such that the endomorphism $(\operatorname{ad} a)^2$ of $\mathfrak{p}$ has eigenvalues $0, 1$. Then an $R$-space $M_a$ is a parallel submanifold in $S$.

(2) All $R$-spaces $M_a$ obtained in (1) exhaust all parallel submanifolds in $S$.

Kobayashi and Nagano ([18]) and T. Nagano ([21]) classified completely $R$-spaces satisfying (1). After some time, S. Kobayashi ([17]) realized a various class of symmetric $R$-spaces.

*In the remainder of this paper, we assume that symmetric space $G/K$ is Hermitian.*

From (1), (2) and Theorem 3.2(ii) we have:

LEMMA 4.2. *An $R$-space $M_a$ is parallel in $S$ if and only if the corresponding $\tilde{R}$-space $\tilde{M}_a$ is totally real.*

Here we recall the natural projection $\pi: S \to P_n(\boldsymbol{C})$. For each $a \in \mathfrak{a} \cap S$, we have from $\pi$

$$T_a(M_a) = \boldsymbol{R}J(a) + V_a \quad \text{(orthogonal direct sum)}.$$

From (3.2) we have

$$V = \hat{\mathfrak{D}} + \hat{\mathfrak{D}}^{\perp}. \tag{4.2}$$

If an $R$-space $M_a$ satisfies

$$\nabla h = 0 \quad \text{on } V,$$

then we shall call $M_a$ *almost parallel.*

First we prepare the following Lemma:

LEMMA 4.3. *Let $a \in \mathfrak{a} \cap S$ satisfy*

$$2\lambda_k,\ 2\lambda_l,\ \lambda_k + \lambda_l,\ \lambda_k - \lambda_l \in \Delta - \Delta_a \quad \textit{for some } k, l\ (k \neq l).$$

*Then an $R$-space $M_a$ is not almost parallel.*

PROOF. By (4.2), it suffices to show that there exist elements $X, Y, Z \in \sum_{\lambda \in \Delta - \Delta_a} \mathfrak{k}_\lambda$ such that

$$X_a^*, Y_a^*, Z_a^* \in \hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^\perp \quad \text{and} \quad (\nabla_{X_a^*} h)(Y_a^*, Z_a^*) \neq 0. \tag{4.3}$$

The author could not show the existence of elements $X$, $Y$ and $Z$ of $\sum_{\lambda \in \Delta - \Delta_a} \mathfrak{k}_\lambda$ satisfying (4.3) by a general method. But, according as every Hermitian symmetric space $G/K$ we can find elements $X$, $Y$ and $Z$ of $\sum_{\lambda \in \Delta - \Delta_a} \mathfrak{k}_\lambda$ satisfying (4.3). In the following we show this for a typical Hermitian symmetric space $G/K$ and abbreviate the proofs for every other Hermitian symmetric space since we have only to apply the same method.

Let $0 < p \leq q$ be integers and $M = SU(p+q)/S(U_p \times U_q)$ be a Hermitian symmetric space. Let $E_{ij}$ denote $(p+q) \times (p+q)$ matrix with entry 1 where the $i$th row and $j$th column meet, all other entries being 0. Let $I_p$ denote the unit matrix of order $p$. We put

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

Let $\mathfrak{g} = \mathfrak{su}(p+q)$ denote the Lie algebra of $SU(p+q)$ and $\theta$ the involutive automorphism of $\mathfrak{g}$ defined by $\theta(X) = I_{p,q} X I_{p,q}$ ([8, p. 454 and p. 347–p. 349]). Let $\mathfrak{k}$ (resp. $\mathfrak{p}$) be the eigenspace of $\theta$ for the eigenvalue $+1$ (resp. $-1$). Then

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{matrix} A \in \mathfrak{u}(p), & B \in \mathfrak{u}(q) \\ \operatorname{Tr}(A+B) = 0 & \end{matrix} \right\} \quad \text{and}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix} \middle| Z : p \times q \text{ complex matrix} \right\}.$$

A maximal abelian subspace $\mathfrak{a}$ and the complex structure $J$ on $\mathfrak{p}$ are given by

$$\mathfrak{a} = \sum_{i=1}^{p} \sqrt{-1}\boldsymbol{R}(E_{i,p+i} + E_{p+i,i}) \quad \text{and} \quad J = \operatorname{ad}\left( \sqrt{-1} \begin{pmatrix} \frac{q}{p+q} I_p & 0 \\ 0 & -\frac{p}{p+q} I_q \end{pmatrix} \right).$$

The positive restricted root system $\Delta$ is given by:

$$\{2\lambda_i, \lambda_i \pm \lambda_j \,|\, 1 \leq i < j \leq p\} \qquad \text{if } p = q$$

$$\{\lambda_i, 2\lambda_i, \lambda_i \pm \lambda_j \,|\, 1 \leq i < j \leq p\} \quad \text{if } p < q.$$

Here

$$\lambda_j(\sqrt{-1}(E_{i,p+i} + E_{p+i,i})) = \sqrt{-1}\delta_{ji}, \quad 1 \leq j \leq p.$$

By a direct calculation, we have

$$\mathfrak{k}_{\lambda_i-\lambda_j} = \{x(E_{ij} + E_{p+i,p+j}) - \bar{x}(E_{ji} + E_{p+j,p+i}) \mid x \in \boldsymbol{C}\} \quad (1 \leq i < j \leq p)$$

$$\mathfrak{k}_{\lambda_i+\lambda_j} = \{x(E_{ij} - E_{p+i,p+j}) - \bar{x}(E_{ji} - E_{p+j,p+i}) \mid x \in \boldsymbol{C}\} \quad (1 \leq i < j \leq p)$$

$$\mathfrak{k}_{2\lambda_i} = \sqrt{-1}\boldsymbol{R}(E_{ii} - E_{p+i,p+i}), \quad (1 \leq i \leq p)$$

$$\mathfrak{k}_{\lambda_i} = \sum_{\alpha=1}^{q-p} \boldsymbol{R}(E_{p+i,2p+\alpha} - E_{2p+\alpha,p+i}) + \sum_{\alpha=1}^{q-p} \boldsymbol{R}\sqrt{-1}(E_{p+i,2p+\alpha} + E_{2p+\alpha,p+i}) \quad (1 \leq i \leq p)$$

$$\mathfrak{p}_{\lambda_i-\lambda_j} = \{x(E_{i,p+j} + E_{p+i,j}) - \bar{x}(E_{j,p+i} + E_{p+j,i}) \mid x \in \boldsymbol{C}\} \quad (1 \leq i < j \leq p)$$

$$\mathfrak{p}_{\lambda_i+\lambda_j} = \{x(E_{i,p+j} - E_{p+i,j}) + \bar{x}(E_{j,p+i} - E_{p+j,i}) \mid x \in \boldsymbol{C}\} \quad (1 \leq i < j \leq p)$$

$$\mathfrak{p}_{2\lambda_i} = \boldsymbol{R}(E_{i,p+i} - E_{p+i,i}) \quad (1 \leq i \leq p)$$

$$\mathfrak{p}_{\lambda_i} = \sum_{\alpha=1}^{q-p} \boldsymbol{R}(E_{i,2p+\alpha} - E_{2p+\alpha,i}) + \sum_{\alpha=1}^{q-p} \boldsymbol{R}\sqrt{-1}(E_{i,2p+\alpha} + E_{2p+\alpha,i}) \quad (1 \leq i \leq p).$$

Here we may put $k = 1$ and $l = 2$, that is,

$$a = \sum_{i=1}^{2} \sqrt{-1}a_i(E_{i,p+i} + E_{p+i,i}),$$

where $a_1 \neq 0$, $a_2 \neq 0$, $a_1^2 \neq a_2^2$, $a_i \in \boldsymbol{R}$. Then we see that

$$a^{\perp} = \boldsymbol{R}\sqrt{-1}(a_2(E_{1,p+1} + E_{p+1,1}) - a_1(E_{2,p+2} + E_{p+2,2})) + \sum_{i=3}^{p} \sqrt{-1}\boldsymbol{R}(E_{i,p+i} + E_{p+i,i}),$$

$$Ja^{\perp} = \boldsymbol{R}(a_2(E_{1,p+1} - E_{p+1,1}) - a_1(E_{2,p+2} - E_{p+2,2})) + \sum_{i=3}^{p} \boldsymbol{R}(E_{i,p+i} - E_{p+i,i}),$$

$$\hat{\mathfrak{D}}_a = \sum_{i=2}^{p} \mathfrak{p}_{\lambda_1 \pm \lambda_i} + \sum_{j=3}^{p} \mathfrak{p}_{\lambda_2 \pm \lambda_j}, \quad \hat{\mathfrak{D}}_a^{\perp} = Ja^{\perp}.$$

We put

$$X = a_2^2\sqrt{-1}(E_{11} - E_{p+1,p+1}) - a_1^2\sqrt{-1}(E_{22} - E_{p+2,p+2}) \in \mathfrak{k}_{2\lambda_1} + \mathfrak{k}_{2\lambda_2},$$

$$Y = y(E_{12} + E_{p+1,p+2}) - \bar{y}(E_{21} + E_{p+2,p+1}) \in \mathfrak{k}_{\lambda_1-\lambda_2},$$

$$Z = z(E_{12} - E_{p+1,p+2}) - \bar{z}(E_{21} - E_{p+2,p+1}) \in \mathfrak{k}_{\lambda_1+\lambda_2},$$

where $y\bar{z} \neq \bar{y}z$ and $y, z \in \boldsymbol{C}$. Then we have

$$[Z,[Y,a]]_M = -\sqrt{-1}(a_1 - a_2)(y\bar{z} - \bar{y}z)(E_{1,p+1} - E_{p+1,1} + E_{2,p+2} - E_{p+2,2}),$$

$$[X,[Z,[Y,a]]_M]_S = -2a_2^2(a_1 - a_2)(y\bar{z} - \bar{y}z)(E_{1,p+1} + E_{p+1,1})$$
$$+ 2a_1^2(a_1 - a_2)(y\bar{z} - \bar{y}z)(E_{2,p+2} + E_{p+2,2}),$$

$$[Z,[X,a]]_M = 2a_1a_2z(a_1 - a_2)(E_{1,p+2} + E_{p+1,2})$$
$$- 2a_1a_2\bar{z}(a_1 - a_2)(E_{2,p+1} + E_{p+2,1}),$$

$$[Y,[Z,[X,a]]_M]_S = -2a_1a_2(a_1 - a_2)(y\bar{z} - \bar{y}z)(E_{1,p+1} + E_{p+1,1} - E_{2,p+2} - E_{p+2,2}).$$

Thus we have

$$[X,[Z,[Y,a]]_M]_S + [Y,[Z,[X,a]]_M]_S$$
$$= -2a_2(a_1 - a_2)(a_1 + a_2)(y\bar{z} - \bar{y}z)(E_{1,p+1} + E_{p+1,1})$$
$$+ 2a_1(a_1 - a_2)(a_1 + a_2)(y\bar{z} - \bar{y}z)(E_{2,p+2} + E_{p+2,2})$$
$$\in a^{\perp}.$$

From this and (4.1), we have

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) \neq 0. \qquad \text{(Q.E.D.)}$$

THEOREM 4.4. *Let $G/K$ be a Hermitian symmetric space and $a$ be a point in $\mathfrak{a} \cap S$. Then an R-space $M_a$ is almost parallel but not parallel if and only if the corresponding $\tilde{R}$-space $\tilde{M}_a$ is holomorphic.*

PROOF. Let $M_a$ be almost parallel but not parallel. By Theorem 3.2(ii) and its proof, it suffices to prove that there exists an only index $i$ such that $2\lambda_i \in \Delta - \Delta_a$. For this, we put

$$C_2 = \{i \mid \lambda_i(a) \neq 0\}.$$

It suffices to show that $\sharp C_2 = 1$, where $\sharp C_2$ denotes the cardinal number of $C_2$.

The case where $\Delta$ is of type $C$. Suppose that $\sharp C_2 = p$. Then for any index $i$ we have $2\lambda_i \in \Delta - \Delta_a$. We assert that there exist indices $i$ and $j$ such that $\lambda_i + \lambda_j, \lambda_i - \lambda_j \in \Delta - \Delta_a$. If not so, then for any indices $i$ and $j$ with $1 \leq i < j \leq p$, we have

$$\lambda_i + \lambda_j \in \Delta - \Delta_a, \quad \lambda_i - \lambda_j \in \Delta_a \quad \text{or} \quad \lambda_i - \lambda_j \in \Delta - \Delta_a, \quad \lambda_i + \lambda_j \in \Delta_a.$$

Then for a suitable reordering in $\mathfrak{a}^*$, $\Delta_a$ can be expressed as

$$\{\lambda_i - \lambda_j \mid 1 \leq i < j \leq p\},$$

which contradicts Theorem 3.2(ii) and Lemma 4.2. Thus our assertion was proved. This and Lemma 4.3 imply that $M_a$ is not almost parallel, which is a contradiction. Hence we have $\sharp C_2 < p$. Suppose that $2 \leq \sharp C_2$. Then there exist indices $i$ and $j$ such that $2\lambda_i, 2\lambda_j \in \Delta - \Delta_a$. Since $\sharp C_2 < p$, there exists an index $k$ such that $2\lambda_k \in \Delta_a$. Since $2\lambda_i, 2\lambda_j \in \Delta - \Delta_a$, we choose $0 \neq X \in \mathfrak{k}_{2\lambda_i} + \mathfrak{k}_{2\lambda_j}$ of $X_a^* \in \hat{\mathfrak{D}}_a^{\perp}$. Let $0 \neq Y \in \mathfrak{k}_{\lambda_i + \lambda_k}$. Then from (1.6) and (1.8) we have $0 \neq Y_a^* \in \hat{\mathfrak{D}}_a$. Then we have from (4.1) and (1.3)

$$\begin{aligned}(\nabla_{Y_a^*} h)(Y_a^*, X_a^*) &= -2[Y, [X, [Y, a]]_M]^N \\ &\in [\mathfrak{k}_{\lambda_i + \lambda_k}, \mathfrak{p}_{\lambda_i - \lambda_k}]^N.\end{aligned}$$

Since $\lambda_i + \lambda_k - (\lambda_i - \lambda_k) = 2\lambda_k \in \Delta_a$, from (2.3) we have $(\nabla_{Y_a^*} h)(Y_a^*, X_a^*) \neq 0$, which is a contradiction. Thus we have $\sharp C_2 = 1$.

The case where $\Delta$ is of type $BC$. Suppose that $\sharp C_2 \geq 2$. Then there exist two indices $i$ and $j$ such that $2\lambda_i, 2\lambda_j \in \Delta - \Delta_a$. If both $\lambda_i + \lambda_j$ and $\lambda_i - \lambda_j$ belong to $\Delta - \Delta_a$, we see from Lemma 4.3 that $M_a$ is not almost parallel, which is a contradiction. Hence we may assume that

$$\lambda_i + \lambda_j \in \Delta - \Delta_a \quad \text{and} \quad \lambda_i - \lambda_j \in \Delta_a.$$

Let $X \in \mathfrak{k}_{\lambda_i}$ and $Y \in \mathfrak{k}_{\lambda_i + \lambda_j}$. Then by (2.4) and (2.5), both $X_a^*$ and $Y_a^*$ belong to $\hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^{\perp}$. For these $X$ and $Y$ it follows from (4.1) and (1.3) that

$$\begin{aligned}(\nabla_{X_a^*} h)(X_a^*, Y_a^*) &= -2[X, [Y, [X, a]]_M]^N \\ &\in [\mathfrak{k}_{\lambda_i}, \mathfrak{p}_{\lambda_j}]^N.\end{aligned}$$

On the other hand, since $\lambda_i - \lambda_j \in \Delta_a$, we have from (2.3)

$$(\nabla_{X_a^*} h)(X_a^*, Y_a^*) \neq 0,$$

which is a contradiction. From the facts above, we have $\sharp C_2 = 1$.

Conversely, assume that $\tilde{M}_a$ be holomorphic. First let us prove

$$\nabla h = 0 \quad \text{on } \hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^{\perp}.$$

By Theorem 3.2(iii) and (3.1) we have

$$\hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^{\perp} = \begin{cases} \sum_{i=2}^{p} \mathfrak{p}_{\lambda_1 \pm \lambda_i} & \text{if } \Delta \text{ is of type } C \\ \sum_{i=2}^{p} \mathfrak{p}_{\lambda_1 \pm \lambda_i} + \mathfrak{p}_{\lambda_1} & \text{if } \Delta \text{ is of type } BC. \end{cases}$$

Hence it suffices to prove that:

(a) If $\Delta$ is of type $C$, then for any $X, Y, Z \in \sum_{i=2}^{p} \mathfrak{k}_{\lambda_1 \pm \lambda_i}$

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = 0.$$

(b) If $\Delta$ is of type $BC$, then for any $X, Y, Z \in \mathfrak{k}_{\lambda_1} + \sum_{i=2}^{p} \mathfrak{k}_{\lambda_1 \pm \lambda_i}$,

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = 0.$$

To prove (a), it suffices to prove that

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = 0 \quad \text{for } X \in \mathfrak{k}_{\lambda_1 \pm \lambda_i}, \quad Y \in \mathfrak{k}_{\lambda_1 \pm \lambda_j}, \quad Z \in \mathfrak{k}_{\lambda_1 \pm \lambda_k},$$

where $i, j, k \in \{2, \dots, p\}$. From (4.1) and (1.3) we have

$$\begin{aligned} (\nabla_{X_a^*} h)(Y_a^*, Z_a^*) &= -[X, [Z, [Y, a]]_M]^N - [Y, [Z, [X, a]]_M]^N \\ &\in \mathfrak{p}^N_{\lambda_1 \pm \lambda_i \pm (\lambda_1 \pm \lambda_j) \pm (\lambda_1 \pm \lambda_k)}. \end{aligned}$$

On the other hand, if $\lambda_1 \pm \lambda_i \pm (\lambda_1 \pm \lambda_j) \pm (\lambda_1 \pm \lambda_k)$ is a root, then this root is expressed as $\lambda_1 \pm \lambda_l$, where $l \in \{2, \dots p\}$. Since $\lambda_1 \pm \lambda_l \in \Delta - \Delta_a$, it follows from (1.6) that

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = 0.$$

Using the same method as in the proof of (a), we see that (b) holds. It is immediate from Theorem 3.2(ii) and Lemma 4.2 that $M_a$ is not parallel. (Q.E.D.)

Remark. It is well-known that a parallel submanifold in $P_n(C)$ is either holomorphic or totally real. Holomorphic parallel ones were classified by Nakagawa and Takagi ([26]) and the totally real ones by H. Naitoh ([24]).

On the other hand, S. Maeda proposed the following problem in [20]:

Problem. Is there a submanifold $\tilde{L}$ in $P_n(C)$ such that $\tilde{L}$ is cyclic parallel but not parallel?

We can give a partial answer to the problem above as the following.

Corollary 4.5. *Let $\tilde{M}_a$ be an $\tilde{R}$-space. If $\tilde{M}_a$ is cyclic parallel, then $\tilde{M}_a$ is parallel.*

PROOF. By (1.11), we see that an $R$-space $M_a$ is almost parallel if and only if the corresponding $\tilde{R}$-space $\tilde{M}_a$ is cyclic parallel. Lemma 4.2 and Theorem 4.4 imply that if $M_a$ is almost parallel, then either $\tilde{M}_a$ is totally real or $\tilde{M}_a$ is holomorphic. Applying (1.10) to the both cases above, we see that $\tilde{M}_a$ is parallel. (Q.E.D.)

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