TSUKUBA J. MATH.
Vol. 21 No. 3 (1997), 629–670

ON A LOCAL ENERGY DECAY OF SOLUTIONS FOR THE EQUATIONS OF MOTION OF COMPRESSIBLE VISCOUS AND HEAT-CONDUCTIVE GASES IN AN EXTERIOR DOMAIN IN R^3

By

Takayuki KOBAYASHI

Abstract. We consider the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in R^3. We prove the local energy decay of solutions to the linearized evolution problem in L_p framework.

§0. Introduction

Let Ω be an exterior domain in R^3 with compact smooth boundary $\partial\Omega$. The motion of a compressible viscous and heat-conductive fluid is described by the following system

$$
(0.1)\quad
\begin{aligned}
&\rho_t + (v\cdot\nabla)\rho + \rho\cdot \operatorname{div} v = 0 && \text{in } [0,\infty)\times\Omega,\\
&v_t + (v\cdot\nabla)v = \frac{\mu}{\rho}\cdot\Delta v + \frac{\mu+\mu'}{\rho}\cdot\nabla(\operatorname{div} v) - \frac{\nabla P(\rho,\theta)}{\rho} && \text{in } [0,\infty)\times\Omega,\\
&\theta_t + (v\cdot\nabla)\theta + \frac{\theta\cdot\partial_\theta P}{\rho\cdot c}\cdot\operatorname{div} v = \frac{k}{\rho\cdot c}\Delta\theta + \frac{\Psi}{\rho\cdot c} && \text{in } [0,\infty)\times\Omega,\\
&v|_{\partial\Omega} = v|_\infty = 0,\ \theta|_{\partial\Omega} = \theta|_\infty = \bar\theta && \text{on } [0,\infty)\times\partial\Omega,\\
&(\rho,v,\theta)(0,x) = (\rho_0,v_0,\theta_0)(x) && \text{in } \Omega,
\end{aligned}
$$

where ρ is the density, $v=(v_1,v_2,v_3)$ the velocity, θ the absolute temperature, $P=P(\rho,\theta)$ the pressure, μ and μ' the viscosity coefficients, k the coefficient of the heat conduction, c the heat capacity at constant volume and Ψ is the dissipation

MOS Subject Classification: 35Q30, 76N10
Keywords: compressible viscous and heat conductive fluid, analytic semigroup, the local energy decay.
Received November 12, 1995.
Revised September 13, 1996.

function:

$$\Psi = \frac{\mu}{2}(\partial_k v_j + \partial_j v_k)^2 + \mu'(\partial_j v_j)^2.$$

The existence theorems of unique solution local in time for the system (0.1) were obtained by Nash [15], Itaya [7,8] for the initial value problem, and by Tani [22] for the initial boundary value problem. On the other hand the existence theorem of global solution in time for the system (0.1) were obtained by Matsumura and Nishida [12,13], Ponce [17] for the initial value problem, and by Matsumura and Nishida [14] for the initial boundary value problem in L_2-framework for sufficiently small initial data. Also Ströhmer [21] proved the global in time existence theorem for small initial data in a bounded domain in L_q-framework. In particular, Matsumura and Nishida [14] showed that this solution approaches the stationary state as $t \to \infty$, and also Deckelnick [3,4] gave some estimates for the dacay rate in an exterior domain. But this decay rate is weaker than that of Matsumura and Nishida [12] and Ponce [17] in Cauchy problem.

In this paper, we shall give the local energy decay of solutions for the linearlized equations of nonlinear problem (0.1). Although this system has a hyperbolic part that is the density ρ, these solutions have the same decay rate as well-known results of the local energy decay of some parabolic equations, for example Stokes operator and Oseen operator. (cf. Iwashita [9], Kobayashi and Shibata [11], Iwashita and Shibata [10] and Shibata [18].) In particular, this decay rate corresponds to that of Matsumura and Nishida [12] and Ponce [17].

Now, we introduce the linearized equations for the system (0.1) below.

$$(0.2)\qquad \begin{array}{ll} \rho_t + \gamma \operatorname{div} v = f_1 & \text{in } [0,\infty) \times \Omega, \\ v_t - \alpha\Delta v - \beta\nabla(\operatorname{div} v) + \gamma\nabla\rho + \omega\nabla\theta = f_2 & \text{in } [0,\infty) \times \Omega, \\ \theta_t - \kappa\Delta\theta + \omega \operatorname{div} v = f_3 & \text{in } [0,\infty) \times \Omega, \\ v|_{\partial\Omega} = v|_{\infty} = 0, \ \theta|_{\partial\Omega} = \theta|_{\infty} = 0 & \text{on } [0,\infty) \times \partial\Omega, \\ (\rho, v, \theta)(0,x) = (\rho_0, v_0, \theta_0)(x) & \text{in } \Omega, \end{array}$$

where $\alpha, \gamma, \kappa, \omega$ are positive numbers and β is a nonnegative number.

System (0.2) was given by Matsumura and Nishida [12] and Ponce [17]. They seek solutions for the system (0.1) in a neighborhood of a constant state $(\rho, v, \theta) = (\bar{\rho}_0, 0, \bar{\theta}_0)$ where $\bar{\rho}_0, \bar{\theta}_0$ are positive constants under the following assumptions:

(1) μ, μ' are constants $\mu > 0$ and $\dfrac{2}{3}\mu + \mu' \geq 0$.

(2) c, k are positive constants.

(3) P is a known function of ρ, θ, smooth in a neighborhood of $(\bar{\rho}_0, \bar{\theta}_0)$ where $\dfrac{\partial P}{\partial \rho}, \dfrac{\partial P}{\partial \theta} > 0$.

Note that the assumption (1) is stronger than ours because they also study the Neumann boundary condition.

In equations (0.1), put $\alpha = (\mu/\bar{\rho}_0)$, $\beta = (\mu + \mu')/\bar{\rho}_0$, $\gamma = \{(\partial P/\partial \rho)(\bar{\rho}_0, \bar{\theta}_0)\}^{1/2}$, $\kappa = (k/c\bar{\rho}_0)$ and put $\omega = (1/\bar{\rho}_0) \cdot (\partial P/\partial \theta)(\bar{\rho}_0, \bar{\theta}_0)\{\bar{\theta}_0/c\}^{1/2}$. Then using the notation (ρ, v, θ) for the vector $(1/\bar{\rho}_0)\{(\partial P/\partial \rho)(\bar{\rho}_0, \bar{\theta}_0)\}^{1/2}\rho, v, \{c/\bar{\theta}_0\}^{1/2}\theta)$, we can obtain the equations (0.2).

Concerning the linearized equations (0.2), Matsumura and Nishida [12] gave the spectral analysis and energy estimates of solutions in L_2-sense and Ponce [17] the $L_p - L_q$ estimates for solutions in R^3, respectively. Ströhmer [20] showed that the operator $-A$ generates an analytic semigroup in a bounded domain. But the results for the case of an exterior domin were not known. Therefore we shall start with a result for the case of an exterior domain.

Our main results are the following. Let $1 < q < \infty$, m be an integer and set

$$X_q^m(\Omega) = \{{}^T\boldsymbol{u}; \boldsymbol{u} \in W_q^{m+1}(\Omega) \times W_q^m(\Omega) \times W_q^m(\Omega)\}, \quad X_q(\Omega) = X_q^0(\Omega),$$

where ${}^T\boldsymbol{u}$ means the transposed $\boldsymbol{u}$. Define the 5×5 matrix operator A by the relation:

$$A = \begin{pmatrix} 0 & \gamma\,\mathrm{div} & 0 \\ \gamma\nabla & -\alpha\Delta - \beta\nabla\,\mathrm{div} & \omega\nabla \\ 0 & \omega\,\mathrm{div} & -\kappa\Delta \end{pmatrix}, \tag{0.3}$$

with the domain:

$$\mathscr{D}(A) = \{{}^T\boldsymbol{u}; \boldsymbol{u} = \{\rho, v, \theta\} \in W_q^1(\Omega) \times W_q^2(\Omega) \times W_q^2(\Omega),$$
$$v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}.$$

Let $\boldsymbol{P}$ be projection from $\mathscr{D}(A)$ into $\{{}^T\{v, \theta\}; \{v, \theta\} \in W_q^2(\Omega) \times W_q^2(\Omega), v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}$ and $\rho(-A)$ be the resolvent set of the operator $-A$. Then

THEOREM A. *Let* $1 < q < \infty$. *Then* $-A$ is a closed linear operator in $X_q(\Omega)$ *and*

$$\rho(-A) \supset \Sigma = \{\lambda \in C; C\,\mathrm{Re}\,\lambda + (\mathrm{Im}\,\lambda)^2 > 0\}, \tag{0.4}$$

where C is a constant depending only on $\alpha, \beta, \gamma, \kappa$ and ω. Moreover, the following properties are valid: There exist positive constants λ_0 and $\delta < (\pi/2)$ such that

$$|\lambda| \, \|(\lambda + A)^{-1} f\|_{X_q(\Omega)} + \|P(\lambda + A)^{-1} f\|_{2,q,\Omega} \leq C(\lambda_0, \delta, m)\|f\|_{X_q(\Omega)}$$

for any $\lambda - \lambda_0 \in \sum_\delta = \{\lambda \in C; |\arg \lambda| \leq \pi - \delta\}$ and any $f \in X_q(\Omega)$.

Theorem A means that $-A$ generates an analytic semigroup e^{-tA} on $X_q(\Omega)$. Then let $B_b = \{x \in R; |x| < b\}$, $\Omega_b = \Omega \cap B_b$ and setting

(0.5)
$$Y_{q,b}(\Omega) = \left\{ u = {}^T\{\rho, v, \theta\} \in X_q(\Omega); \quad u(x) = 0 \text{ for } x \in R^3 \backslash B_b, \int_{\Omega_b} \rho(x)\, dx = 0 \right\},$$

we have

THEOREM B (local energy decay). *Let $1 < q < \infty$ and let b_0 be a fixed number such that $B_{b_0} \supset R^3 \backslash \Omega$. Suppose that $b > b_0, u = {}^T\{\rho, v, \theta\} \in Y_{q,b}(\Omega)$. Then the following estimates are valid: for $M \geq 0$ integer, $u \in Y_{q,b}(\Omega)$ and $t > 0$*

$$\|\partial_t^M e^{-tA} u\|_{X_q(\Omega_b)} + \|\partial_t^M e^{-tA} u\|_{2,q,\Omega_b} \leq C(q, b, M) t^{-3/2-M} \|u\|_{X_q(\Omega)}. \tag{0.6}$$

REMARK. In dealing with the system (0.2), it is natural to introduce the bace space $X_q(\Omega)$ without the condition $\int_\Omega \rho(x)\, dx = 0$ because the Stokes formula does not hold in an exterior domain. Hence we shall treat the case $\int_\Omega \rho(x)\, dx \neq 0$ also. In this case, roughly speaking, since $\lambda = 0$ seems to be a pole in the sence of § 1 (1.22), it is difficult to expect the same results in Theorem B. Therefore, we decompose the semigroup e^{-tA} as the following and by using Theorem B we have

COROLLARY C. *Let*

$$X_{q,b}(\Omega) = \{u \in X_q(\Omega); u(x) = 0 \textit{ for } x \in R^3 \backslash B_b\}. \tag{0.7}$$

Taking $\varphi \in C_0^\infty(\Omega_b)$ so that $\int_{\Omega_b} \varphi(x)\, dx = 1$, for $u = {}^T\{\rho, v, \theta\} \in X_{q,b}(\Omega)$, we have the following representation

$$e^{-tA} u = T_1(b, \varphi, t) u + T_2(b, \varphi, t) u \tag{0.8}$$

where e_j $(j = 1, 2, \dots, 5)$ *are unit row vectors in* R^5, $N_D\boldsymbol{u} = \int_D \rho(x)\,dx$ *and*

$$T_1(b, \varphi, t)\boldsymbol{u} = e^{-tA}\{\boldsymbol{u} - (N_{\Omega_b}\boldsymbol{u}) \cdot \varphi e_1\},$$

$$T_2(b, \varphi, t)\boldsymbol{u} = (N_{\Omega_b}\boldsymbol{u})\left\{\varphi \cdot e_1 - \gamma \int_0^t e^{-sA}\begin{pmatrix} 0 \\ \nabla\varphi \\ 0 \end{pmatrix} ds\right\}.$$

Moreover, the following estimates are valid: for $M \geq 0$ *integer,* $\boldsymbol{u} \in X_{q,b}(\Omega)$ *and* $t > 0$

$$\begin{aligned} &\|\partial_t^M T_1(b, \varphi, t)\boldsymbol{u}\|_{X_q(\Omega_b)} + \|\partial_t^M \boldsymbol{P}T_1(b, \varphi, t)\boldsymbol{u}\|_{2,q,\Omega_b} \\ &\quad \leq C(q, b, \varphi, M)t^{-3/2-M}\|\boldsymbol{u}\|_{X_q(\Omega)}, \end{aligned} \tag{0.9}$$

$$\begin{aligned} &\|\partial_t^{M+1} T_2(b, \varphi, t)\boldsymbol{u}\|_{X_q(\Omega_b)} + \|\partial_t^{M+1} \boldsymbol{P}T_2(b, \varphi, t)\boldsymbol{u}\|_{2,q,\Omega_b} \\ &\quad \leq C(q, b, \varphi, M)t^{-3/2-M}\|\boldsymbol{u}\|_{X_q(\Omega)}. \end{aligned} \tag{0.10}$$

The most important part of the proof of our main results is the cutoff technique in Shibata [18]. In § 1, the same resolvent estimates of the operator $-A$ in a bounded domain as in Ströhmer [20] are proved. The difference between ours and Ströhmer [20] are the following:

(i) We shall show that the resolvent set of the operator $-A$ contains a parabolic region,

(ii) We do not assume that $\int_\Omega \rho(x)\,dx = 0$. (see Remark.)

The regularity of resolvent $(\lambda + A)^{-1}$ in R^3 near $\lambda = 0$ is investigated in § 2, which is the essential point of our proof of Theorem B. The proof of Theorem A in § 3 and a costruction of a parametrix of the exterior stationary problem in § 4 are done by the method of cutoff technique. And then, with the help of a theorem concerning the relationship between the regularity of functions and the decay rate of their Fourier image, which was given by Shibata [18], we prove Theorem B in § 5. Since the resolvent set contains a parabolic region, we can not take the same path of integration for the Laplace tranceform between the resolvent and semigroup as in Iwashita [9] etc. Hence we shall use the same way as in Kobayashi and Shibata [11].

Notations. Three dimensional row vector valued functions are denoted with bold-face letter, for example $\boldsymbol{u} = (u_1, u_2, u_3)$. As usual, we put

$$\partial_t = \partial/\partial t; \quad \partial_j = \partial/\partial_j; \quad \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2;$$

$$\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3;$$

$$\partial_x^m p = (\partial_x^\alpha p; |\alpha| = m); \quad \bar{\partial}_x^m p = (\partial_x^\alpha p; |\alpha| \leq m);$$

$$\operatorname{div} \boldsymbol{u} = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3;$$

Sobolev spaces of vector valued functions are used, as well as of scalar valued functions. Thus, if D is any domain in R^3, we put

$$\|u\|_{q,D} = \left(\int_D |u(x)|^q \, dx\right)^{1/q}; \quad \|\boldsymbol{u}\|_{q,D} = \left(\sum_{j=1}^{3} \|u_j\|_{q,D}^q\right)^{1/q};$$

$$\|u\|_{m,q,D} = \|\bar{\partial}_x^m u\|_{q,D}; \quad \|\boldsymbol{u}\|_{m,q,D} = \|\bar{\partial}_x^m \boldsymbol{u}\|_{q,D}; \quad (\boldsymbol{u}, \boldsymbol{v}) = \int_D \boldsymbol{u}(x) \cdot \overline{\boldsymbol{v}(x)} \, dx.$$

$L_q(D)$ denotes the usual L_q space on D, $W_q^m(D) = \{u \in L_q(D); \|u\|_{m,q,D} < \infty\}$, $\mathscr{S}'$ the set of all tempered distributions on R^3 and $C_0^\infty(D)$ the set of all functions of $C^\infty(R^3)$ whose support is contained in D. For function spaces of three dimensional vector valued functions, we use the bold letters, that is for example, $\boldsymbol{L}_q(D) = \{L_q(D)\}^3$ likewise for $\boldsymbol{W}_q^m(D)$. To denote various constants, we use the same letter C and $C(A, B, \ldots)$ means that the constant depends on the qualities $A, B, \ldots$. For two Banach spaces X and Y, $\mathscr{B}(X, Y)$ denotes the set of all bounded linear operators from X into Y and $\|\cdot\|_{\mathscr{B}(X,Y)}$ means its operator norm.

§1. Stationary problem in a bounded domain

In this section we consider the stationary problem in a bounded domain D in R^3 with smooth boundary ∂D;

$$\lambda\rho + \gamma \cdot \operatorname{div} v = f_1 \quad \text{in } D, \tag{1.1a}$$

$$\lambda v - \alpha\Delta v - \beta\nabla(\operatorname{div} v) + \gamma \cdot \nabla\rho + \omega \cdot \nabla\theta = f_2 \quad \text{in } D, \tag{1.1b}$$

$$\lambda\theta - \kappa\Delta\theta + \omega \cdot \operatorname{div} v = f_3 \quad \text{in } D, \tag{1.1c}$$

$$v|_{\partial D} = 0 \quad \text{on } \partial D, \tag{1.1d}$$

$$\theta|_{\partial D} = 0 \quad \text{on } \partial D. \tag{1.1e}$$

here λ is a complex parameter.

We shall prepare some results to show a unique existence of solutions to (1.1). The following proposition is concerned the existence theorem of solutions to the Stokes equations.

PROPOSITION 1.1 ([2]). *Let $1 < q < \infty$, m be an integer ≥ 0 and let $D \subset R^3$ be a bounded domain with smooth boundary ∂D. Then for every $\boldsymbol{f} \in \boldsymbol{W}_q^m(D)$ and every $g \in W_q^{m+1}(D)$ with $\int_D g(x)\, dx = 0$ there exists a unique $\boldsymbol{u} \in \boldsymbol{W}_q^{m+2}(D)$ which* *together with some* $p \in W_q^{m+1}(D)$ *satisfies*

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \operatorname{div} \boldsymbol{u} = g \text{ in } D,$$

$$\boldsymbol{u} = 0 \text{ on } \partial D.$$

Here p *is unique up to an additive constant. Furthermore, the following estimate is valid:*

$$\|\boldsymbol{u}\|_{m+2,q,D} + \|\nabla p\|_{m,q,D} \leq C\{\|\boldsymbol{f}\|_{m,q,D} + \|g\|_{m+1,q,D}\},$$

where $C = C(D, q, \varepsilon)$ *is a constant.*

The following proposition is well-known as a general Poincaré's inequality.

PROPOSITION 1.2 (cf., eg. [5]). *Let* $1 \leq q < \infty$. *There exists a constant* $C > 0$ *such that the inequality*

$$\|u\|_{q,D} \leq C\left\{\|\nabla u\|_{q,D} + \left|\int_D u(x)\,dx\right|\right\},$$

holds for any $u \in W_q^1(D)$. *Furthermore, if* $q \neq 1$, D *is bounded and if* $u \in W_q^1(D)$ *with* $u = 0$ *on* ∂D, *then we have*

$$\|u\|_{q,D} \leq C\|\nabla u\|_{q,D}.$$

The next result is well-known as the system of Laplacian with Dirichlet boundary conditions.

PROPOSITION 1.3. *Let* $1 < q < \infty$ *and let* $D \subset R^3$ *be a bounded domain (or exterior domain) with smooth boundary* ∂D. *Let* $0 < \delta < (\pi/2)$ *and* $\kappa > 0$. *Then for every* $\lambda \in \sum_\delta$, *every* $\boldsymbol{f} \in L_q(D)$ *there exists a unique solution* $\boldsymbol{u} \in W_q^2(D)$ *such that*

$$\lambda \boldsymbol{u} - \kappa \Delta \boldsymbol{u} = \boldsymbol{f} \text{ in } D, \quad \boldsymbol{u} = 0 \text{ on } \partial D.$$

Furthermore, the following estimate is valid:

$$|\lambda|\,\|\boldsymbol{u}\|_{q,D} + \|\boldsymbol{u}\|_{2,q,D} \leq C\|\boldsymbol{f}\|_{q,D}, \quad \|\boldsymbol{u}\|_{3,q,D} \leq C(\lambda)\{\|\boldsymbol{f}\|_{1,q,D} + \|\boldsymbol{u}\|_{q,D}\},$$

where $C = C(D, q, \delta)$ *is a constant.*

The following proposition is concerned the existence theorem of solutions to the elastic equations.

PROPOSITION 1.4. *Let $1 < q < \infty$ and let $D \subset R^3$ be a bounded domain (on exterior domain) with smooth boundary ∂D. Let α be a positive number, η be a complex number such that* $\mathrm{Re}\{\alpha+\eta\} > 0$. *Then there exist positive numbers λ_0 and $\delta < (\pi/2)$ satisfying the following conditions: For every $\lambda - \lambda_0 \in \sum_\delta$, every $f \in L_q(D)$ there exists a unique $u \in W_q^2(D)$ such that*

$$\lambda u - \alpha\Delta u - \eta\nabla\mathrm{div}\, u = f \text{ in } D, \quad u|_{\partial D} = 0 \text{ on } \partial D. \tag{1.2}$$

Furthermore the following estimates is valid:

$$|\lambda|\, \|u\|_{q,D} + \|u\|_{2,q,D} \leq C\|f\|_{q,D}, \quad \|u\|_{3,q,D} \leq C(\lambda)\{\|f\|_{1,q,D} + \|u\|_{q,D}\}, \tag{1.3}$$

where $C = C(D, q, \delta, \lambda_0, \alpha, \eta)$ is a constant.

PROOF. Since

$$\det\begin{pmatrix} -\alpha|\xi|^2 - \eta\xi_1^2 & -\eta\xi_1\xi_2 & -\eta\xi_1\xi_3 \\ -\eta\xi_1\xi_2 & -\alpha|\xi|^2 - \eta\xi_2^2 & -\eta\xi_2\xi_3 \\ -\eta\xi_1\xi_3 & -\eta\xi_2\xi_3 & -\alpha|\xi| - \eta\xi_3^2 \end{pmatrix} = -(\alpha+\eta)\alpha^2|\xi|^6, \tag{1.4}$$

(1.2) is the elliptic when $\mathrm{Re}(\alpha+\eta) > 0$, which means that a priori estimate:

$$|\lambda|\, \|u\|_{q,D} + \|u\|_{2,q,D} \leq C\{\|f\|_{q,D} + \|u\|_{q,D}\}, \quad \|u\|_{3,q,D} \leq C(\lambda)\{\|f\|_{1,q,D} + \|u\|_{q,D}\},$$

is valid for $\lambda - \lambda_0 \in \sum_\delta$. Taking sufficiently large number λ_0, we have (1.3). Define the operator $T(\lambda;\eta)$ by the relation:

$$T(\lambda;\eta)u = \lambda u - \alpha\Delta u - \eta\nabla\,\mathrm{div}\, u, \tag{1.5}$$

with the domain: $\mathscr{D}(T(\lambda;\eta)) = \{u \in W_q^2(D); u|_{\partial D} = 0\}$. Then, by (1.3) $T(\lambda;\eta)$ is densely defined closed operator in $L_q(D)$ and the range of $T(\lambda;\eta)$ is closed in $L_q(D)$. Since the dual operator of $T(\lambda;\eta)$ in $L_q(D)$ is $T(\bar{\lambda};\bar{\eta})$ in $L_p(D)$ where $(1/p)+(1/q) = 1$, the closed range theorem means that a unique solution for (1.2) exists in $L_q(D)$. Combining this with a priori estimate (1.3), the proof is completed.

Now we will lead to the main theorem in this section. Let $1 < q < \infty, m$ be an integer and let

$$Y_q^m(D) = \left\{{}^T\{f_1, f_2, f_3\} \in X_q^m(D); \int_D f_1(x)\, dx = 0\right\}, \quad Y_q(D) = Y_q^0(D). \tag{1.6}$$

Define the 5×5 matrix operator A_D by the relation:

$$A_D = \begin{pmatrix} 0 & \gamma \operatorname{div} & 0 \\ \gamma \nabla & -\alpha \Delta - \beta \nabla \operatorname{div} & \omega \nabla \\ 0 & \omega \operatorname{div} & -\kappa \Delta \end{pmatrix},$$

with the domain: $\mathscr{D}(A_D) = Y_q(D) \cap \mathscr{D}(A)$ i.e, A_D is the maximal restriction to closed subspace $Y_q(D)$. Applying this notation to (1.1), we have

$$(\lambda + A_D)\boldsymbol{u} = \boldsymbol{f}$$

where $\boldsymbol{u} = {}^T\{\rho, v, \theta\}$ and $\boldsymbol{f} = {}^T\{f_1, f_2, f_3\}$. Then

THEOREM 1.5. *Let* $1 < q < \infty$ and let $D \subset R^3$ *be a bounded domain with smooth boundary* ∂D. *Then,* A_D *is a closed linear operator in* $Y_q(D)$ *and*

$$\rho(-A_D) \supset \{0\} \cup \Sigma'$$

where $\Sigma' = \{\lambda \in C; 6(\gamma^2 + \omega^2)\operatorname{Re}\lambda + \alpha(\operatorname{Im}\lambda)^2 > 0\}$. *Moreover, the following properties are valid: There exists a number* $0 < \delta < (\pi/2)$ *such that*

$$|\lambda| \, \|(\lambda + A_D)^{-1}\boldsymbol{f}\|_{Y_q(D)} + \|P(\lambda + A_D)^{-1}\boldsymbol{f}\|_{2,q,D} \leq C(q, \delta, D)\|\boldsymbol{f}\|_{Y_q(D)} \tag{1.7}$$

for any $\lambda \in \Sigma_\delta \cup \{0\}$ *and any* $\boldsymbol{f} \in Y_q(\Omega)$.

PROOF. We shall prepare the following three lemmas to prove this theorem.

LEMMA 1.6. *Let* $1 < q < \infty$, *and* $D \subset R^3$ *be a bounded domain or an exterior domain with smooth boundary* ∂D. *Let* A *be the operators defined in* (0.3) *with* $\Omega = D$. *Then there exist positive numbers* λ_0 *and* $0 < \delta < (\pi/2)$ *such that if* $\boldsymbol{u} \in \mathscr{D}(A)$ *satisfies* $(\lambda + A)\boldsymbol{u} = \boldsymbol{f}$ *with* $\boldsymbol{f} \in X_q(D)$, *then the following estimate is valid:*

$$|\lambda| \, \|\boldsymbol{u}\|_{X_q(D)} + \|P\boldsymbol{u}\|_{2,q,D} \leq C(q, \lambda_0, \delta, D)\|\boldsymbol{f}\|_{X_q(D)}$$

for $\lambda - \lambda_0 \in \Sigma_\delta$.

PROOF OF LEMMA 1.6. Let $\boldsymbol{u} = {}^T\{\rho, v, \theta\}$ and let $\boldsymbol{f} = {}^T\{f_1, f_2, f_3\}$. Recall that the equation $(\lambda + A)\boldsymbol{u} = \boldsymbol{f}$ means that the equations (1.1) hold. Applying Propositions 1.3 and 1.4 to the system $\lambda - \kappa\Delta$ and $\lambda - \alpha\Delta - \beta\nabla \operatorname{div}$ in (1.1), we see that there exist positive number λ_1 and $0 < \delta_1 < (\pi/2)$ such that

$$|\lambda| \, \|\theta\|_{q,D} + |\lambda|^{1/2} \|\theta\|_{1,q,D} + \|\theta\|_{2,q,D} \leq C\{\|f_3 - \omega \operatorname{div} v\|_{q,D} + \|\theta\|_{q,D}\}, \tag{1.8a}$$

and

$$|\lambda| \, \|v\|_{q,D} + |\lambda|^{1/2} \|v\|_{1,q,D} + \|v\|_{2,q,D} \leq C\{\|f_2 - \gamma \nabla p - \omega \nabla \theta\|_{q,D} + \|v\|_{q,D}\}, \tag{1.8b}$$

hold for $\lambda - \lambda_1 \in \Sigma_{\delta_1}$ with C depending only on q, λ_1 and δ_1. Furthermore it follows from the equations (1.1a) that

$$|\lambda| \, \|p\|_{q,D} \leq \gamma \|v\|_{1,q,D} + \|f_1\|_{q,D}, \tag{1.9}$$

and

$$|\lambda| \, \|\nabla p\|_{q,D} \leq \gamma \|v\|_{2,q,D} + \|f_1\|_{1,q,D}. \tag{1.10}$$

Combining (1.8a), (1.8b), (1.9) and (1.10), and taking sufficiently large number λ_0, we have lemma 1.6.

LEMMA 1.7. *Let $1 < q < \infty$, m be an integer ≥ 0 and D be a bounded domain in R^3 with smooth boundary ∂D. Then, $(-A_D)^{-1}$ exists. Furthermore, the following estimate is valid:*

$$\|(-A_D)^{-1} f\|_{Y_q^m(D)} + \|P(-A_D)^{-1} f\|_{m+2,q,D} \leq C(q, m, D) \|f\|_{Y_q^m(D)}$$

for $f \in Y_q^m(D)$.

PROOF OF LEMMA 1.7. Putting $u = {}^T\{p, v, \theta\}$ and $f = {}^T\{f_1, f_2, f_3\}$, we consider the system (1.1) with $\lambda = 0$ in stead of the equation $A_D u = f$ in Lemma 1.7. Since it follows from (1.1a), (1.1c) and (1.1e) that

$$-\kappa \Delta \theta = f_3 - \frac{\omega}{\gamma} f_1 \text{ in } D, \quad \theta|_{\partial D} = 0 \text{ on } \partial D, \tag{1.11}$$

and since D is a bounded domain, there exists a unique solution $\theta \in W_q^{m+2}(D)$ to (1.11) such that

$$\|\theta\|_{m+2,q,D} \leq C \left\| f_3 - \frac{\omega}{\gamma} f_1 \right\|_{m,q,D}. \tag{1.12}$$

We have by (1.1a), (1.1b) and by (1.1d)

$$-\alpha\Delta v+\nabla(\gamma\rho)=f_2+\frac{1}{\gamma}\beta\nabla f_1-\omega\cdot\nabla\theta \text{ in } D, \tag{1.13}$$
$$\operatorname{div} v=\frac{f_1}{\gamma} \text{ in } D, \quad v|_{\partial D}=0 \text{ on } \partial D.$$

Applying Proposition 1.1 to the system (1.13), there exists a unique pair $(v,\rho)\in W_q^{m+2}(D)\times W_q^{m+1}(D)$ with $\int_D \rho(x)\,dx=0$ satisfying (1.13) such that

$$\begin{aligned}&\|v\|_{m+2,q,D}+\|\rho\|_{m+1,q,D}\\&\quad\le C\left\{\left\|f_2+\frac{1}{\gamma}\beta\nabla f_1-\omega\cdot\nabla\theta\right\|_{m,q,D}+\left\|\nabla\frac{f_1}{\gamma}\right\|_{m,q,D}\right\}.\end{aligned} \tag{1.14}$$

Combining (1.12) with (1.14) implies that this lemma holds.

Lemma 1.8. *Let* $1<q<\infty$, $\lambda\in\Sigma'\cup\{0\}$ *and* $D\subset R^3$ *be a bounded domain with smooth boundary* ∂D. *Let* A *be the operators defined in* (0.3) *with* $\Omega=D$. *Then*

$$\operatorname{Ker}(\lambda+A)=\{0\},$$

where $\operatorname{Ker} T$ *is the kernel of the operator* T.

Proof of Lemma 1.8. Let $(\lambda+A)\boldsymbol{u}=0$, $\boldsymbol{u}={}^T\{\rho,v,\theta\}\in\mathscr{D}(A)$. Then we have

$$\lambda\rho+\gamma\cdot\operatorname{div} v=0 \quad \text{in } D, \tag{1.15a}$$
$$\lambda v-\alpha\Delta v-\beta\nabla(\operatorname{div} v)+\gamma\cdot\nabla\rho+\omega\cdot\nabla\theta=0 \quad \text{in } D, \tag{1.15b}$$
$$\lambda\theta-\kappa\Delta\theta+\omega\cdot\operatorname{div} v=0 \quad \text{in } D, \tag{1.15c}$$
$$v|_{\partial D}=0 \quad \text{on } \partial D, \tag{1.15d}$$
$$\theta|_{\partial D}=0 \quad \text{on } \partial D. \tag{1.15d}$$

We can assume that $\lambda\neq 0$ by Lemma 1.7. Noting that $\operatorname{Re}\{\alpha+\beta+(\gamma^2/\lambda)\}>0$ when $\alpha>0$, $\beta\ge 0$ and $\lambda\in\Sigma'$, in view of (1.4), since the systems $-\kappa\Delta$ and $-\alpha\Delta-(\beta+(\gamma^2/\lambda))\nabla\operatorname{div}$ with Dirichlet boundary conditions are elliptic, by boot-strap argument, we see that $\{\rho,v,\theta\}\in W_q^{\ell+1}(D)\times W_q^{\ell+2}(D)\times W_q^{\ell+2}(D)$ for all integers $\ell\ge 0$. When $2\le q<\infty$, since D is a bounded domain, we see that $\{\rho,v,\theta\}\in W_2^1(D)\times W_2^2(D)\times W_2^2(D)$. When $1<q<2$, by Sobolev's imbedding theorem,

$\{\rho, v, \theta\} \in W_2^1(D) \times W_2^2(D) \times W_2^2(D)$. Thus, multiplying (1.15b) by $\bar{v}$, integrating the resulting relation over D and using integration by parts, we have by (1.15a)

$$(1.16) \qquad \lambda\|v\|_{2,D}^2 + \alpha\|\nabla v\|_{2,D}^2 + \left(\beta + \frac{\gamma^2}{\lambda}\right)\|\operatorname{div} v\|_{2,D}^2 + \omega(\nabla\theta, v) = 0.$$

Similarly, multiplying (1.15c) by $\bar{\theta}$, we have

$$(1.17) \qquad \lambda\|\theta\|_{2,D}^2 + \kappa\|\nabla\theta\|_{2,D}^2 + \omega(\operatorname{div} v, \theta) = 0.$$

Since $\operatorname{Re}\{\omega(\operatorname{div} v, \theta)\} = -\operatorname{Re}\{\omega(\nabla\theta, v)\}$ and since $\operatorname{Im}\{\omega(\operatorname{div} v, \theta)\} = \operatorname{Im}\{\omega(\nabla\theta, v)\}$, it follows from (1.16), (1.17) and Schwartz's inequality that

$$(1.18) \qquad \operatorname{Re}\lambda \cdot (\|v\|_{2,D}^2 + \|\theta\|_{2,D}^2) + \alpha\|\nabla v\|_{2,D}^2 + \kappa\|\theta\|_{2,D}^2 + \left(\beta + \frac{\operatorname{Re}\lambda \cdot \gamma^2}{|\lambda|^2}\right)\|\operatorname{div} v\|_{2,D}^2 = 0,$$

$$(1.19) \qquad \|v\|_{2,D}^2 = \frac{\gamma^2}{|\lambda|^2}\|\operatorname{div} v\|_{2,D}^2 + \|\theta\|_{2,D}^2 \text{ if } \operatorname{Im}\lambda \neq 0,$$

and

$$(1.20) \qquad |\operatorname{Im}\lambda|\, \|\theta\|_{2,D} \leq \omega\|\operatorname{div} v\|_{2,D}.$$

When $\operatorname{Re}\lambda \geq 0$, by (1.18) and (1.19) we have $\theta = 0$, $v = 0$ in D because $\theta = 0$, $v = 0$ on ∂D, which implies $\rho = 0$ in D by (1.15a). When $\operatorname{Re}\lambda < 0$, since $\operatorname{Im}\lambda \neq 0$, it follows from (1.18), (1.19) and (1.20) that

$$\alpha\|\nabla v\|_{2,D}^2 + \kappa\|\nabla\theta\|_{2,D}^2 + \beta\|\operatorname{div} v\|_{2,D}^2 \leq -2\operatorname{Re}\lambda\left\{\frac{\gamma^2}{|\lambda|^2} + \frac{\omega^2}{|\operatorname{Im}\lambda|^2}\right\}\|\operatorname{div} v\|_{2,D}^2.$$

Noting that $\|\operatorname{div} v\|_{2,D}^2 \leq 3\|\nabla v\|_{2,D}^2$ and $6(\gamma^2 + \omega^2)\operatorname{Re}\lambda + \alpha(\operatorname{Im}\lambda)^2 \geq 0$ when $\lambda \in \Sigma'$, we have $\nabla v = 0$ in D. Combining this with (1.19) and (1.20) implies that $\theta = 0$, $v = 0$ in D and that $\rho = 0$ in D by (1.15a). This completes the proof of Lemma 1.8.

WE ARE NOW IN THE POSITION TO PROVE THEOREM 1.5. Note that Lemma 1.7 allows us to show the case $\lambda \neq 0$. Putting $\boldsymbol{u} = {}^T\{\rho, v, \theta\}$ and $\boldsymbol{f} = {}^T\{f_1, f_2, f_3\}$, we consider the system (1.1) in stead of the equation $(\lambda + A_D)\boldsymbol{u} = \boldsymbol{f}$. In view of Proposition 1.3 and 1.4, fixing a complex number $\lambda_1 \in \sum_\delta + \lambda_0$, it follows from (1.1) that for $\lambda \in \Sigma'$

$$(I + P(\lambda))v = T\left(\lambda_1; \beta + \frac{\gamma^2}{\lambda}\right)^{-1}\left[-\frac{\gamma}{\lambda}\nabla f_1 + f_2 - \omega\nabla(\lambda - \kappa\Delta)^{-1}f_3\right],$$

where I is the identity operator,

$$P(\lambda) = T\left(\lambda_1; \beta + \frac{\gamma^2}{\lambda}\right)^{-1} [(\lambda - \lambda_1) - \omega^2 \nabla(\lambda - \kappa\Delta)^{-1} \mathrm{div}],$$

$$T\left(\lambda_1; \beta + \frac{\gamma^2}{\lambda}\right) = \text{the operator defined in (1.5)},$$

and

$$(\lambda - \kappa\Delta)^{-1} = \text{the resolvent for the system in Proposition 1.3.}$$

By Proposition 1.3 and 1.4 $P(\lambda)$ is a bounded linear operator from $\{u \in W_q^2(D); u|_{\partial D} = 0\}$ into $W_q^3(D) \cap \{u \in W_q^2(D); u|_{\partial D} = 0\}$ which is compactly imbedded into $\{u \in W_q^2(D); u|_{\partial D} = 0\}$ as follows from Rellich's compactness theorem, and hence $P(\lambda)$ is a compact operator from $\{u \in W_q^2(D); u|_{\partial D} = 0\}$ into itself. Noting that by Lemma 1.8 we know that $I + P(\lambda)$ is injective, by Fredholm's alternative theorem we see that $I + P(\lambda)$ has the bounded inverse. Hence, setting

$$v = (I + P(\lambda))^{-1} T\left(\lambda_1; \beta + \frac{\gamma^2}{\lambda}\right)^{-1} \left[-\frac{\gamma}{\lambda} \nabla f_1 + f_2 - \omega \nabla(\lambda - \kappa\Delta)^{-1} f_3\right],$$

$$\theta = (\lambda - \kappa\Delta)^{-1}[f_3 - \omega \,\mathrm{div}\, v], \quad \rho = \frac{1}{\lambda}[f_1 - \gamma \,\mathrm{div}\, v],$$

implies that

$$\rho(-A_D) \supset \Sigma' \cup \{0\}.$$

Furthermore, since the resolvent $(\lambda + A_D)^{-1}$ is analytic in $\lambda \in \rho(-A_D)$, Lemma 1.6 and Lemma 1.7 mean that the estimates (1.7) is valid, which reach the desired conclusion.

REMARK 1.9. In Theorem 1.5 we assume that $\int_D f_1 \, dx = 0$, which means that $\int_D \rho \, dx = 0$ by the equation (1.1a), (1.1d) and by Stokes formula. When $\int_D f_1 \, dx \neq 0$, taking $\varphi \in C_0^\infty(D)$ such that $\int_D \varphi(x) \, dx = 1$ and define the operators $N_j = N_j(\varphi, D)$ $(j = 1, 2, 3)$ from $X_q(D)$ into itself by the notations:

$$N_1 f = f - (N_D f) \cdot \varphi e_1$$

$$N_2 f = -(N_D f) \begin{pmatrix} 0 \\ \nabla\varphi \\ 0 \end{pmatrix} \text{for } f = {}^T\{f_1, f_2, f_3\} \in X_q(D), \tag{1.21}$$

$$N_3 f = (N_D f) \varphi \cdot e_1$$

where e_1 and $N_D f$ are the same symbols as in Corollary C. Then we can write $(\lambda + A)^{-1}$ as follows:

$$(1.22) \qquad (\lambda + A)^{-1} = (\lambda + A_D)^{-1} N_1 + \frac{\gamma}{\lambda}(\lambda + A_D)^{-1} N_2 + \frac{1}{\lambda} N_3.$$

Combining this and Theorem 1.5, we see that $-A$ is a closed linear operator in $X_q(D)$, $\rho(-A) \supset \Sigma'$ and the following properties are valid:

$$\begin{aligned} &|\lambda| \, \|(\lambda + A)^{-1} f\|_{X_q(D)} + \|P(\lambda + A)^{-1} f\|_{2,q,D} \\ &\leq C(\delta, q, D)\left\{ \|f\|_{X_q(D)} + \frac{1}{|\lambda|} \|f_1\|_{q,D} \right\} \end{aligned}$$

for any $\lambda \in \Sigma_\delta$ and any $f \in X_q(D)$.

§2. On the stationary problem in R^3

In this section, we shall show the basic estimations of solutions to the following stationary linearized equations in R^3 with a complex parameter λ:

$$(2.1) \qquad \begin{aligned} &\lambda\rho + \gamma \cdot \operatorname{div} v = f_1, \\ &\lambda v - \alpha\Delta v - \beta\nabla(\operatorname{div} v) + \gamma \cdot \nabla\rho + \omega \cdot \nabla\theta = f_2 \text{ in } R^3, \\ &\lambda\theta - \kappa\Delta\theta + \omega \cdot \operatorname{div} v = f_3. \end{aligned}$$

By taking Fourier transform on (2.1) we obtain

$$[\lambda \cdot I + \hat{A}(\xi)]\hat{u} = \hat{f},$$

where I is the identity, $\mathscr{F}(f) = \hat{f}$ stands for the Fourier transforms of f, $u = {}^T(\rho, v, \theta)$, $f = {}^T(f_1, f_2, f_3)$. Here $\hat{A}(\xi)$ is 5×5 symmetric matrix as follows:

$$\hat{A}(\xi) = \begin{pmatrix} 0 & i\gamma\xi_k & 0 \\ i\gamma\xi_j & \delta_{jk}\alpha|\xi|^2 + \beta\xi_j\xi_k & i\omega\xi_j \\ 0 & i\omega\xi_k & \kappa|\xi|^2 \end{pmatrix}$$

where $i = \sqrt{-1}$ and $\delta_{jk} = 0$ when $k \neq j$ and $= 1$ when $k = j$. Then we have

$$(2.2a) \qquad [\lambda \cdot I + \hat{A}(\xi)]^{-1} = \{\det[\lambda \cdot I + \hat{A}(\xi)]\}^{-1} \cdot \tilde{A}(\lambda; \xi),$$

$$(2.2b) \qquad \det[\lambda \cdot I + \hat{A}(\xi)] = (\lambda + \alpha|\xi|^2)^2 F(\lambda; |\xi|),$$

where

$$F(\lambda; |\xi|) = \lambda^3 + (\alpha+\beta+\kappa)|\xi|^2\lambda^2 + [(\alpha+\beta)\kappa|\xi|^2 + \gamma^2 + \omega^2]|\xi|^2\lambda + \gamma^2\kappa|\xi|^4, \tag{2.2c}$$

and $\tilde{A}(\lambda;\xi) = (\tilde{a}_{ij}(\lambda;\xi))$ is the 5×5 matrix and the components are

$$\begin{aligned}
\tilde{a}_{11} &= (\lambda+\alpha|\xi|^2)^2\{\lambda^2 + (\alpha+\beta+\kappa)|\xi|^2\lambda + [\omega^2 + (\alpha+\beta)\kappa|\xi|^2]\cdot|\xi|^2\},\\
\tilde{a}_{15} &= \tilde{a}_{51} = -\gamma\omega(\lambda+\alpha|\xi|^2)^2|\xi|^2,\\
\tilde{a}_{1,j} &= \tilde{a}_{j,1} = -i\gamma(\lambda+\alpha|\xi|^2)^2(\lambda+\kappa|\xi|^2)\xi_{j-1} \quad (j=2,3,4),\\
\tilde{a}_{5,j} &= \tilde{a}_{j,5} = -i\omega\lambda(\lambda+\alpha|\xi|^2)^2\xi_{j-1} \quad (j=2,3,4),\\
\tilde{a}_{55} &= (\lambda+\alpha|\xi|^2)^2\{\lambda^2 + (\alpha+\beta)|\xi|^2\lambda + \gamma^2|\xi|^2\},\\
\tilde{a}_{ij} &= (\lambda+\alpha|\xi|^2)\{\lambda(\lambda+\alpha|\xi|^2)(\lambda+\kappa|\xi|^2)\delta_{ij}\\
&\quad + (\delta_{ij}|\xi|^2 - \xi_{i-1}\xi_{j-1})(\beta\lambda^2 + [\beta\kappa|\xi|^2 + \omega^2 + \gamma^2]\lambda + \gamma^2\kappa|\xi|^2),\\
&\qquad (i,j = 2,3,4).
\end{aligned} \tag{2.2d}$$

From the spectral analysis of $\hat{A}(\xi)$ given by Matsumura and Nishida [12] (cf. Ponce [17]) we have

LEMMA 2.1. *Let* $\{\lambda_j(\xi)\}_{j=1}^5$ *be the roots of* $\det[\lambda\cdot I + \hat{A}(\xi)] = 0$, *where* $\lambda_4(\xi) = \lambda_5(\xi) = -\alpha|\xi|^2$. *Then it follows that*:

(i) $\lambda_j(\xi)$ *depends on* $|\xi|$ *only,* $\lambda_j(0) = 0$ *and* $\operatorname{Re}\lambda_j(\xi) < 0$ *for any* $|\xi| > 0$, $j = 1,\dots,5$.

(ii) $\lambda_j(\xi) \neq \lambda_k(\xi)$, $j\neq k$ *and* $j,k = 1,2,3,4$ *for all* $|\xi|$ *except at most four points of* $|\xi| > 0$.

(iii) *There exist positive constants* r_1 *such that* $\lambda_j(\xi)$ *has a Taylor series expansion for* $|\xi| < r_1$ *as follows*: $\lambda_1(\xi) = \overline{\lambda_2(\xi)}$ *is a complex number,* $\lambda_3(\xi)$ *is a real number and*

$$\lambda_1(\xi) = (\gamma^2+\omega^2)^{1/2}(i|\xi|) + \frac{(\gamma^2+\omega^2)(\alpha+\beta)+\omega^2\kappa}{2(\gamma^2+\omega^2)}(i|\xi|)^2 + \cdots,$$

$$\lambda_3(\xi) = \frac{\gamma^2\kappa}{\gamma^2+\omega^2}(i|\xi|)^2 + \frac{\gamma^2\omega^2\kappa^2\{(\gamma^2+\omega^2)(\alpha+\beta)-\gamma^2\kappa\}}{(\gamma^2+\omega^2)^4}(i|\xi|)^4 + \cdots.$$

Similarly, there exist positive constants $r_2 > r_1$ *such that* $\lambda_j(\xi)$ *has a Laurent series* *expansion for* $|\xi| > r_2$ *as follows: If* $\alpha+\beta \neq \kappa$, *then* $\lambda_j(\xi)$ *are real numbers and*

$$\lambda_1(\xi) = (\alpha+\beta)(i|\xi|)^2 - \frac{\gamma^2\kappa - (\gamma^2+\omega^2)(\alpha+\beta)}{(\alpha+\beta)(\alpha+\beta-\kappa)} + \cdots\cdots,$$

$$\lambda_2(\xi) = \kappa(i|\xi|)^2 + \frac{\omega^2}{\kappa-\alpha-\beta} + \cdots\cdots\cdots,$$

$$\lambda_3(\xi) = -\frac{\gamma^2}{\alpha+\beta} + \cdots\cdots\cdots.$$

If $\alpha+\beta=\kappa$, *then* $\lambda_1(\xi) = \overline{\lambda_2(\xi)}$ *is a complex number,* $\lambda_3(\xi)$ *is a real number and*

$$\lambda_1(\xi) = \kappa(i|\xi|)^2 + \sqrt{\omega}(i|\xi|) + \cdots\cdots,$$

$$\lambda_3(\xi) = -\frac{\gamma^2}{\kappa} + \cdots\cdots\cdots.$$

(iv) *rank*$[\lambda_1(\xi)\cdot I + \hat{A}(\xi)] = 3$ *for all* $|\xi|>0$ *except at most one point of* $|\xi|>0$.

(v) *The matrix exponential has the spectral resolution*

$$e^{-t\hat{A}(\xi)} = \sum_{j=1}^{5} e^{t\lambda_j}(\xi)P_j(\xi)$$

for all $|\xi|$ *except at most four points of* $|\xi|>0$.

(vi) *There exists a positive constants* $\beta_0, \beta_1, \beta_2$ *and* r_1 *such that* $-\beta_0|\xi|^2 \leq \mathrm{Re}\,\lambda_j(\xi) \leq -\beta_1|\xi|^2$ *for* $|\xi|<r_1$ *and* $\mathrm{Re}\,\lambda_j(\xi) < -\beta_2$ *for* $|\xi|>r_2$, $j=1,2,\dots,5$.

(v) $\|P_j(\xi)\| \leq C$ *for* $|\xi| \leq r_1$.

(vii) $\|e^{-t\hat{A}(\xi)}\| \leq C(1+t)^3 e^{-\beta t}$ *for* $|\xi|>r_1$ *and a positive constant* β.

Now we set for $f \in X_q(R^3)$, $f = {}^T\{f_j\}_{j=1}^5$

$$R_0(\lambda)f(x) = \mathscr{F}^{-1}\{[\lambda\cdot I + \hat{A}(\xi)]^{-1}\hat{f}(\xi)\}(x) \tag{2.3}$$
$$= {}^T\left\{\sum_{i=1}^{5} R_{ji}(\lambda)f_i(x)\right\}_{j=1}^{5},$$

where $R_{ij}(\lambda) = \mathscr{F}^{-1}\{\det[\lambda\cdot + \hat{A}(\xi)]^{-1}\tilde{a}_{ij}(\lambda;\xi)\mathscr{F}\}$. When $f = {}^T\{f_1, \boldsymbol{f}_2, f_5\}$ where $\boldsymbol{f}_2 = (f_2, f_3, f_4)$ we shall use the representation as follows:

$$R_0(\lambda)f(x) = {}^T\{R_{0,\rho}(\lambda)f(x), R_{0,v}(\lambda)f(x), R_{0,\theta}(\lambda)f(x)\}. \tag{2.4}$$

Then we shall have the following estimates of $R_0(\lambda)f$ which is the core of our argument.

THEOREM 2.2. *Let $1 < q < \infty$, b be a positive number and $X_{q,b}(R^3)$ be the same symbol as in (0.7). Then for any $f \in X_{q,b}(R^3)$ any $\lambda \in \{\lambda \in C; Re\,\lambda \geq 0, 0 < |\lambda| \leq 1\}$*

$$\|R_0(\lambda)f\|_{X_q(B_b)} + \|PR_0(\lambda)f\|_{2,q,B_b} \leq C\|f\|_{X_q(R^3)},$$

$$\left\|\left(\frac{d}{d\lambda}\right)^k R_0(\lambda)f\right\|_{X_q(B_b)} + \left\|\left(\frac{d}{d\lambda}\right)^k PR_0(\lambda)f\right\|_{2,q,B_b} \leq C|\lambda|^{1/2-k}\|f\|_{X_q(R^3)},$$

where k are integers ≥ 1 and $C = C(q,b,k)$ is a constant.

PROOF. First we note that since it follows from (2.2b), (2.2c) and Lemma 2.1 that $F(\lambda;|\xi|) = (\lambda - \lambda_1(\xi))(\lambda - \lambda_2(\xi))(\lambda - \lambda_3(\xi))$, we have

$$\begin{aligned} F(\lambda;|\xi|)^{-1} = &\frac{1}{\lambda_1(\xi) - \lambda_2(\xi)} \cdot \frac{1}{\lambda_1(\xi) - \lambda_3(\xi)} \cdot \frac{1}{\lambda - \lambda_1(\xi)} \\ &+ \frac{1}{\lambda_2(\xi) - \lambda_3(\xi)} \cdot \frac{1}{\lambda_2(\xi) - \lambda_1(\xi)} \cdot \frac{1}{\lambda - \lambda_2(\xi)} \\ &+ \frac{1}{\lambda_3(\xi) - \lambda_1(\xi)} \cdot \frac{1}{\lambda_3(\xi) - \lambda_2(\xi)} \cdot \frac{1}{\lambda - \lambda_3(\xi)}. \end{aligned}$$

Combining this equation and Lemma 2.1 (iii) means that

$$(2.5) \qquad |F(\lambda;|\xi|)^{-1}| \leq C_\varepsilon|\lambda|^{-2\varepsilon}|\xi|^{-4+2\varepsilon} \quad \text{for } \mathrm{Re}\,\lambda \geq 0, \xi \in R^3 \text{ and } 0 \leq \varepsilon \leq 1,$$

and which implies that

$$(2.6) \qquad |\det[\lambda + \hat{A}(\xi)]|^{-1} \leq C|\xi|^{-8} \quad \text{for } \mathrm{Re}\,\lambda \geq 0 \text{ and } \xi \in R^3,$$

since $|\lambda + \alpha|\xi|^2| \geq \alpha|\xi|^2$ for $\mathrm{Re}\,\lambda \geq 0$ and $\xi \in R^3$.

Now let $f = {}^T\{f_j\}_{j=1}^5$. Choosing $\chi(r) \in C_0^\infty(R)$ so that $\chi(r) = 1$ if $|r| \leq 1$ and $= 0$ if $|r| \geq 2$, put

$$\begin{aligned} (2.7) \qquad R_{ij}(\lambda)f_j(x) &= \mathscr{F}^{-1}\{\chi(|\xi|)\det[\lambda \cdot I + \hat{A}(\xi)]^{-1}\tilde{a}_{ij}(\lambda;\xi)\hat{f}_j(\xi)\}(x) \\ &\quad + \mathscr{F}^{-1}\{(1 - \chi(|\xi|))\det[\lambda \cdot I + \hat{A}(\xi)]^{-1}\tilde{a}_{ij}(\lambda;\xi)\hat{f}_j(\xi)\}(x) \\ &= T_{1,ij}(\lambda)f_j(x) + T_{2,ij}(\lambda)f_j(x). \end{aligned}$$

Using Theorem 7.9.5 of [6] concerning the L_q-estimate of the Fourier multiplier, it follows from (2.2a), (2.2d), (2.6) and (2.7) that

$$\sum_{j=1}^{5}\left\|\left(\frac{d}{d\lambda}\right)^{k} T_{2,1j}(\lambda)f_j\right\|_{1,q,R^3}+\sum_{j=1}^{5}\sum_{i=2}^{5}\left\{\left\|\left(\frac{d}{d\lambda}\right)^{k} T_{2,ij}(\lambda)f_j\right\|_{2,q,R^3}\right\} \le C\{\|f_1\|_{1,q,R^3}+\|f_2\|_{q,R^3}+\|f_3\|_{q,R^3}\}, \tag{2.8}$$

where k are integers ≥ 0 and C is a constant independent of $|\lambda| \leq 1$. Using a polar coordinate system, we can write as follows: for multi-index α_i $(i = 1,\dots,5)$: $|\alpha_1| \leq 1, |\alpha_i| \leq 2$ $(i = 2,\dots,5)$

$$\begin{aligned}&\left(\frac{d}{d\lambda}\right)^{k}(\partial_x)^{\alpha_i}T_{1,ij}(\lambda)f_j(x) \\ &= \frac{1}{(2\pi)^{3/2}}\int_{R^3}(\imath\xi)^{\alpha_i}e^{\imath x\cdot\xi}\chi(|\xi|)\left(\frac{d}{d\lambda}\right)^{k}\{(\det[\lambda\cdot I+\hat{A}(\xi)])^{-1}\tilde{a}_{ij}(\lambda;\xi)\}\hat{f}_j(\xi)d\xi \\ &= \frac{1}{(2\pi)^{3/2}}\int_0^2\left(\frac{d}{d\lambda}\right)^{k} r^{|\alpha_i|+2}\{(\det[\lambda\cdot I+\hat{A}(r)])^{-1}\tilde{a}_{ij}(\lambda;r\omega)\} \\ &\quad\cdot\int_{|\omega|=1}(\imath\omega)^{\alpha_i}e^{\imath(x\cdot\omega)r}\chi(r)\hat{f}_j(r\omega)\,dr\,dS_\omega,\end{aligned} \tag{2.9}$$

where dS_ω denote the surface element on the unit surface. By Taylor series expansion, we have

$$e^{\imath(x\cdot\omega)r}\chi(r)\hat{f}_j(r\omega) = \hat{f}(0)+\sum_{\ell=1}^{m-1}g_\ell(x,\omega)r^\ell+\int_0^1 H_m(x,\omega,s,r)\,dsr^m \tag{2.10}$$

where

$$\begin{aligned}g_\ell(x,\omega) &= \frac{1}{\ell!}\left(\frac{\partial}{\partial r}\right)^{\ell} e^{\imath(x\cdot\omega)r}\chi(r)\hat{f}_j(r\omega)\bigg|_{r=0}, \ell \geq 1, \\ H_m(x,\omega,s,r) &= \frac{(1-s)^{m-1}}{(m-1)!}\left(\frac{\partial}{\partial\sigma}\right)^{k} e^{\imath(x\cdot\omega)\sigma}\chi(\sigma)\hat{f}_j(\sigma\omega)\bigg|_{\sigma=sr}.\end{aligned}$$

Note that since $f_j \in L_{q,b}(R^3)$, we have

$$\begin{aligned}&|\hat{f}_j(0)| \leq C(b)\|f_j\|_{q,R^3}, \\ &|g_\ell(x,\omega)| \leq C(b,\ell)(1+|x|)^\ell\|f_j\|_{q,R^3}, \\ &\int_0^1|H_k(x,\omega,s,r)|\,ds \leq C(b,k)(1+|x|)^k\|f_j\|_{q,R^3}.\end{aligned} \tag{2.11}$$

In view of (2.2d), putting

$$\tilde{a}_{ij}(\lambda; r\omega) = \sum_{\beta} \tilde{a}_{\beta,ij}(\lambda; r) b_{\beta,ij}(\omega),$$

it follows from (2.9), (2.10) and (2.11) that

$$\begin{aligned}
(2.12)\quad & \left|\left(\frac{d}{d\lambda}\right)^k (\partial_x)^{\alpha_i} T_{1,ij}(\lambda) f_j(x)\right| \\
& \leq C(1+|x|)^m \|f_j\|_{q,R^3} \cdot \\
& \left\{\sum_{\beta}\sum_{\ell=0}^{m-1}\left|\int_0^1 \left(\frac{d}{d\lambda}\right)^k \{(\det[\lambda\cdot I + \hat{A}(r)])^{-1}\tilde{a}_{\beta,ij}(\lambda; r)\} r^{|\alpha_i|+2+\ell} dr\right|\right. \\
& \left. + \sum_{\beta}\int_0^1 \left|\left(\frac{d}{d\lambda}\right)^k \{(\det[\lambda\cdot I + \hat{A}(r)])^{-1}\tilde{a}_{\beta,ij}(\lambda; r)\} r^{|\alpha_i|+2+m}\right| dr\right\}.
\end{aligned}$$

In order to show that the rest of assertions in Theorem 2.2 holds, we need the following lemma.

Lemma 2.3. *Let $m \geq 0$, $M \geq 1$ be integers. Put*

$$I_{1,m,M}(\lambda) = \int_0^1 \frac{r^m}{F(\lambda; r)^M} dr, \quad I_{2,m,M}(\lambda) = \int_0^1 \frac{r^m}{(\lambda + \alpha r^2)^M F(\lambda; r)^M} dr$$

for $\mathrm{Re}\,\lambda \geq 0$, $|\lambda| \leq 1$. *Then the following facts hold.*

(i) $|I_{1,m,M}(\lambda)| \leq C(m, M)$ *if* $m \geq 4M$, $|I_{2,m,M}(\lambda)| \leq C(m, M)$ *if* $m \geq 6M$.

(ii) *If* $0 \leq m < 4M$, *then*

$$\begin{aligned}
|I_{1,m,M}(\lambda)| &\leq C(m, M)\max\{|\lambda|^{m/2-2M+1/2}, |\lambda|^{m-3M+1}\} \text{ when } m \text{ is even,} \\
&\leq C(m, M)\max\{|\lambda|^{m/2-2M+1/2}, |\lambda|^{m-3M+1}\}|\mathrm{Log}\,\lambda| \text{ when } m \text{ is odd.}
\end{aligned}$$

If $0 \leq m < 6M$ *and if* $\dfrac{1}{\alpha} \neq \dfrac{1}{\kappa}\left(1 + \dfrac{\omega^2}{\gamma^2}\right)$, *then*

$$\begin{aligned}
|I_{2,m,M}(\lambda)| &\leq C(m, M)\max\{|\lambda|^{m/2-3M+1/2}, |\lambda|^{m-4M+1}\} \text{ when } m \text{ is even,} \\
&\leq C(m, M)\max\{|\lambda|^{m/2-3M+1/2}, |\lambda|^{m-4M+1}\}|\mathrm{Log}\,\lambda| \text{ when } m \text{ is odd.}
\end{aligned}$$

(iii) *Let* $M \geq m$ *and* $\ell \geq 1$ *an integer. Then* $\mathrm{Re}\,\lambda \geq 0$, $|\lambda| \leq 1$

$$\left|\int_0^1 \frac{r^{2M-2m}}{(\lambda + \alpha r^2)^\ell F(\lambda; r)^M} dr\right| \leq c(m, \ell, M)\max\{|\lambda|^{-M-\ell-m}, |\lambda|^{-M-\ell-2m+1}\}.$$

PROOF OF LEMMA 2.3. (i) It follows from (2.5) and the inequality $|\lambda+\alpha r^2|\geq \alpha r^2$ when $\operatorname{Re}\lambda\geq 0$ that (i) holds.

(ii) We shall show (ii) by using decomposition into partial fractions. We can write $F(\lambda;r)$ as follows:

$$F(\lambda;r)=\kappa(\gamma^2+(\alpha+\beta)\lambda)(r^2-a_+(\lambda))(r^2-a_-(\lambda))$$

where

$$a_\pm(\lambda)=-\frac{\lambda}{2\kappa}\left\{1+\frac{\omega^2+\kappa\lambda}{\gamma^2+(\alpha+\beta)\lambda}\pm\left[\left(1+\frac{\omega^2+\kappa\lambda}{\gamma^2+(\alpha+\beta)\lambda}\right)^2-\frac{4\kappa\lambda}{\gamma^2+(\alpha+\beta)\lambda}\right]^{1/2}\right\}.$$

Then we have the following estimates

$$\text{(2.13a)}\qquad (a_+(\lambda)-a_-(\lambda)),\quad a_+(\lambda)=0(\lambda)\quad\text{and}\quad a_-(\lambda)=0(\lambda^2)\quad\text{as }\lambda\to 0,$$

$$\text{(2.13b)}\qquad \left(a_+(\lambda)+\frac{\lambda}{\alpha}\right)=0(\lambda)\quad\text{as }\lambda\to 0\quad\text{if }\frac{1}{\alpha}\neq\frac{1}{\kappa}\left(1+\frac{\omega^2}{\gamma^2}\right),$$

which implies that

$$\text{(2.14a)}\qquad \frac{x^m}{(x-a_+(\lambda))^M(x-a_-(\lambda))^M}=\sum_{j=1}^{M}\{A_j(\lambda)(x-a_+(\lambda))^{-j}+B_j(\lambda)(x-a_-(\lambda))^{-j}\}$$

$$\text{(2.14b)}\qquad |A_j(\lambda)|\leq C|\lambda|^{m-2M+j},\quad |B_j(\lambda)|\leq C|\lambda|^{2m-3M+2j}\quad\text{for }|\lambda|\leq 1.$$

Also we have by (2.13)

$$\text{(2.14c)}\qquad \frac{x^m}{(x+\frac{\lambda}{\alpha})^M(x-a_+(\lambda))^M(x-a_-(\lambda))^M}=\sum_{j=1}^{M}\left\{C_j(\lambda)\left(x+\frac{\lambda}{\alpha}\right)^{-j}+D_j(\lambda)(x-a_+(\lambda))^{-j}+E_j(\lambda)(x-a_-(\lambda))^{-j}\right\},$$

$$\text{(2.14d)}\qquad |C_j(\lambda)|,\quad |D_j(\lambda)|\leq C|\lambda|^{m-3M+j}\quad\text{for }|\lambda|\leq 1\quad\text{if }\frac{1}{\alpha}\neq\frac{1}{\kappa}\left(1+\frac{\omega^2}{\gamma^2}\right),$$

$$|E_j(\lambda)|\leq C|\lambda|^{2m-4M+2j}\quad\text{for }|\lambda|\leq 1\quad\text{if }\frac{1}{\alpha}\neq\frac{1}{\kappa}\left(1+\frac{\omega^2}{\gamma^2}\right).$$

Moreover, putting $a(\lambda) = -\frac{\lambda}{\alpha}$, $a_\pm(\lambda)$, we have by elementary calculus,

(2.15a)
$$\int_0^1 \frac{ds}{s - a(\lambda)} = C_1 \log|a(\lambda)| + C_2,$$

(2.15b)
$$\int_0^1 \frac{ds}{(s - a(\lambda))^{k+1}} = C_3 a(\lambda)^{-k} + C_4,$$

(2.15c)
$$\int_0^1 \frac{dr}{(r^2 - a(\lambda))^k} = C_5 a(\lambda)^{1/2-k},$$

where k are positive integers, C_j $(j = 1, 3, 5)$ complex constants depending only on k and C_j $(j = 2, 4) C^\infty(\{\lambda \in C; \operatorname{Re}\lambda \geq 0$ and $|\lambda| \leq 1\})$-functions depending also essentially on k. Combining (2.13), (2.14) and (2.15) shall reach to the statement.

(iii) Noting that

$$\frac{r^{2M-2m}}{(\lambda + \alpha r^2)^\ell F(\lambda; r)^M}$$
$$= \frac{1}{\lambda^{M+\ell+m}} \sum_{k=0}^{M+\ell+m} \binom{M+\ell+m}{k} (-\alpha)^{M+\ell+m-k} (\lambda + \alpha r^2)^{k-\ell} \cdot \frac{r^{2(2M+\ell-k)}}{F(\lambda; r)^M},$$

it follows from (ii) that

$$\left| \int_0^1 \frac{1}{\lambda^{M+\ell+m}} \sum_{k=\ell}^{M+\ell+m} \binom{M+\ell+m}{k} (-\alpha)^{M+\ell+m-k} (\lambda + \alpha r^2)^{k-\ell} \cdot \frac{r^{2(2M+\ell-k)}}{F(\lambda; r)^M} \, dr \right|$$
$$= \left| \sum_{k=\ell}^{M+\ell+m} \sum_{n=0}^{k-\ell} \binom{M+\ell+m}{k} \binom{k-\ell}{n} (-\alpha)^{M+\ell+m-k+n} \right.$$
$$\left. \cdot \lambda^{-M-2\ell-m+k-n} \int_0^1 \frac{r^{2(2M+\ell-k+n)}}{F(\lambda; r)^M} \, dr \right|$$
$$\leq C(m, \ell, M) \max\{|\lambda|^{1/2-M-\ell-m}, |\lambda|^{-M-\ell-2m+1}\},$$

and it follows from (2.5) that

$$\left| \int_0^1 \frac{1}{\lambda^{M+\ell+m}} \sum_{k=0}^{\ell-1} \binom{M+\ell+m}{k} (-\alpha)^{M+\ell+m-k} (\lambda + \alpha r^2)^{k-\ell} \cdot \frac{r^{2(2M+\ell-k)}}{F(\lambda; r)^M} \, dr \right|$$
$$= \left| \frac{1}{\lambda^{M+\ell+m}} \sum_{k=0}^{\ell-1} \binom{M+\ell+m}{k} (-\alpha)^{M+\ell+m-k} \int_0^1 \frac{r^{2\ell-2k}}{(\lambda + \alpha r^2)^{\ell-k}} \cdot \frac{r^{4M}}{F(\lambda; r)^M} \, dr \right|$$
$$\leq C(m, \ell, M) |\lambda|^{-M-\ell-m}.$$

This completes the proof of Lemma 2.3.

NOW WE RETURN TO THE PROOF OF THEOREM 2.2. By direct calculation we have

$$F(\lambda;r)^k=\sum_{\ell=0}^{k}\sum_{n=0}^{k-\ell}\binom{k}{\ell}\binom{k-\ell}{n}\{(\alpha+\beta+\kappa)\lambda+(\gamma^2+\omega^2)\}^\ell \cdot\{(\alpha+\beta)\kappa\lambda+\gamma^2\kappa\}^n\lambda^{3k-2\ell-3n}r^{2\ell+4n}, \tag{2.16}$$

$$\left\{\frac{d}{d\lambda}F(\lambda;r)\right\}^k=\sum_{\ell=0}^{k}\sum_{n=0}^{k-\ell}\binom{k}{\ell}\binom{k-\ell}{n}\{2(\alpha+\beta+\kappa)\lambda+\gamma^2+\omega^2\}^\ell \cdot 3^{k-\ell-n}(\alpha+\beta)^n\kappa^n\lambda^{2k-2n-2\ell}r^{2\ell+4n}, \tag{2.17}$$

$$\left\{\left(\frac{d}{d\lambda}\right)^2F(\lambda;r)\right\}^k=\sum_{\ell=0}^{k}\binom{k}{\ell}2^k3^{k-\ell}(\alpha+\beta+\kappa)^\ell\lambda^{k-\ell}r^{2\ell}, \tag{2.18a}$$

$$(\lambda+\alpha r^2)^k=\sum_{\ell=0}^{k}\binom{k}{\ell}\alpha^{2\ell}\lambda^{k-\ell}r^{2\ell}. \tag{2.18b}$$

First when $\frac{1}{\alpha}\neq\frac{1}{\kappa}\left(1+\frac{\omega^2}{\gamma^2}\right)$, setting

$$J_1(\lambda;r)=r^4,\lambda r^2,r^2,\lambda r \text{ or } r^3,$$

$$J_2(\lambda;r)=\lambda^2r^2,\lambda r^4,\lambda r^2 \text{ or } r^4,$$

$$G(\lambda;r)=(\lambda+\alpha r^2)F(\lambda;r),$$

it follows from (2.16), (2.17), (2.18), Appendix 1 and Lemma 2.3 that

$$\begin{aligned}&\left|\int_0^1\left(\frac{d}{d\lambda}\right)^n\{F(\lambda;r)^{-1}J_1(\lambda;r)\}r^{|\alpha_i|+2}\,dr\right| \\ &=\left|\int_0^1\left\{\sum_{k=0}^{2}\binom{n}{k}\left(\frac{d}{d\lambda}\right)^{n-k}F(\lambda;r)^{-1}\left(\frac{d}{d\lambda}\right)^kJ_1(\lambda;r)\right\}r^{|\alpha_i|+2}\,dr\right| \\ &\leq C\max\{1,|\lambda|^{1/2-n}\},\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\left|\int_0^1\left(\frac{d}{d\lambda}\right)^n\{G(\lambda;r)^{-1}J_2(\lambda;r)\}r^{|\alpha_i|+2}\,dr\right| \\ &=\left|\int_0^1\left\{\sum_{k=0}^{2}\binom{n}{k}\left(\frac{d}{d\lambda}\right)^{n-k}G(\lambda;r)^{-1}\left(\frac{d}{d\lambda}\right)^kJ_2(\lambda;r)\right\}r^{|\alpha_i|+2}\,dr\right| \\ &\leq C\max\{1,|\lambda|^{1/2-n}\}.\end{aligned} \tag{2.20}$$

Also when $\dfrac{1}{\alpha}=\dfrac{1}{\kappa}\left(1+\dfrac{\omega^2}{\gamma^2}\right)$, noting that by (2.2d) we have

$$\beta\lambda^2+[\beta\kappa|\xi|^2+\omega^2+\gamma^2]\lambda+\gamma^2\kappa|\xi|^2=\beta\lambda(\lambda+\kappa|\xi|^2)+(\omega^2+\gamma^2)(\lambda+\alpha|\xi|^2),$$

in view of (2.19), our task is to show that

$$\left|\int_0^1\left(\frac{d}{d\lambda}\right)^n\{G(\lambda;r)^{-1}J_3(\lambda;r)\}r^{|\alpha_i|+2}\,dr\right|\le C\max\{1,|\lambda|^{1/2-n}\} \tag{2.21}$$

where $J_3(\lambda;r)=\lambda^2r^2$ or λr^6. It follows from Lemma 2.3 (iii), (2.17), (2.18a) and Appendix 1 that (2.21) holds. Hence it follows from (2.2), (2.13), (2.19), (2.20) and (2.21) that

$$\begin{aligned}\sum_{j=1}^5\left\|\left(\frac{d}{d\lambda}\right)^kT_{1,1j}(\lambda)f_j\right\|_{1,q,B_b}&+\sum_{j=1}^5\sum_{i=2}^5\left\{\left\|\left(\frac{d}{d\lambda}\right)^kT_{1,ij}(\lambda)f_j\right\|_{2,q,B_b}\right\}\\&\le C\max\{1,|\lambda|^{1/2-k}\}\cdot\{\|f_1\|_{1,q,R^3}+\|f_2\|_{q,R^3}+\|f_3\|_{q,R^3}\},\end{aligned}$$

where k are integers ≥ 0 and C is a constant independent of $|\lambda|\le 1$ and $\mathrm{Re}\,\lambda\ge 0$, and combining this with (2.8) implies that the statement of this theorem holds.

Finally in this section, we shall investigate the continuity as $\lambda\to 0$ for the operator $\boldsymbol{R}_0(\lambda)$ and the properties for $\boldsymbol{R}_0(0)$.

Lemma 2.4. *Let $1<q<\infty$, b be a positive number and let $\boldsymbol{f}\in X_{q,b}(R^3)$. Then ${}^T\boldsymbol{R}_0(0)\boldsymbol{f}\in W^1_{q,\mathrm{loc}}(R^3)\times W^2_{q,\mathrm{loc}}(R^3)\times W^2_{q,\mathrm{loc}}(R^3)$ and*

$$\lim_{R\to\infty}R^{-3}\int_{R<|x|<2R}|\boldsymbol{R}_0(0)\boldsymbol{f}(x)|^q\,dx=0. \tag{2.22}$$

Moreover, for any $a>0$ and $0<\varepsilon<1/2$ the following estimates are valid:

$$\begin{aligned}&\|{}^T\boldsymbol{R}_0(\lambda)\boldsymbol{f}-{}^T\boldsymbol{R}_0(0)\boldsymbol{f}\|_{W^1_q(B_a)\times W^2_q(B_a)\times W^2_q(B_a)}\\&\quad\le C(q,a,b,\varepsilon)|\lambda|^{\varepsilon}\|\boldsymbol{f}\|_{X_q(R^3)}\end{aligned} \tag{2.23}$$

for $\mathrm{Re}\,\lambda\ge 0$, $|\lambda|\le 1$ and $\boldsymbol{f}\in X_{q,b}(R^3)$, where $C(q,a,b,\varepsilon)$ is a constant independent of $\mathrm{Re}\,\lambda\ge 0$, $|\lambda|\le 1$ and $\boldsymbol{f}\in X_{q,b}(R^3)$.

Proof. Noting that when $\lambda=0$

$$\hat{A}(\xi)^{-1}=\frac{1}{\gamma^2\kappa|\xi|^4}\begin{pmatrix}\{\omega^2+(\alpha+\beta)\kappa|\xi|^2\}|\xi|^2 & -i\gamma\kappa|\xi|^2\xi_k & -\gamma\omega|\xi|^2\\ -i\gamma\kappa|\xi|^2\xi_j & \{\delta_{jk}|\xi|^2-\xi_j\xi_k\}\alpha^{-1}\gamma^2\kappa & 0\\ -\gamma\omega|\xi|^2 & 0 & \gamma^2|\xi|^2\end{pmatrix},$$

since the kernels of Fourier integral operators in $R_0(0)$ are the same as those of the Stokes system and the system Δ, we have (2.22) by Lemma 2.2 and 2.3 in Iwashita [9]. Hence our task is to show (2.23). Choosing $\chi(r) \in C_0^\infty(R)$ so that $\chi(r) = 1$ if $|r| \le 1$ and $= 0$ if $|r| \ge 2$, using the notations defined in (2.3) and (2.4), we have

$$
\begin{aligned}
(2.24)\quad & R_{ij}(\lambda)f_j(x) - R_{ij}(0)f_j(x) \\
&= \mathscr{F}^{-1}\left\{\chi(|\xi|)\left\{\frac{\tilde{a}_{ij}(\lambda;\xi)}{\det[\lambda\cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{ij}(0;\xi)}{\det \hat{A}(\xi)}\right\}\hat{f}_j(\xi)\right\}(x) \\
&\quad + \mathscr{F}^{-1}\left\{(1-\chi(|\xi|))\left\{\frac{\tilde{a}_{ij}(\lambda;\xi)}{\det[\lambda\cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{ij}(0;\xi)}{\det\hat{A}(\xi)}\right\}\hat{f}_j(\xi)\right\}(x) \\
&= \{T_{1,ij}(\lambda) - T_{1,ij}(0)\}f_j(x) + \{T_{2,ij}(\lambda) - T_{2ij}(0)\}f_j(x).
\end{aligned}
$$

Since it follows from (2.2a), (2.5) and (2.6) that

$$
\left|\xi^\eta\partial_\xi^\eta\left[\{1-\chi(|\xi|)\}\left\{\frac{\tilde{a}_{11}(\lambda;\xi)}{\det[\lambda\cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{11}(0;\xi)}{\det\hat{A}(\xi)}\right\}\right]\right| \le C|\lambda|,
$$

$$
\left|\xi^\eta\partial_\xi^\eta\left[\{1-\chi(|\xi|)\}\left\{\frac{\tilde{a}_{1j}(\lambda;\xi)}{\det[\lambda\cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{1j}(0;\xi)}{\det\hat{A}(\xi)}\right\}\right]\right| \le C\frac{|\lambda|}{|\xi|} \quad (j = 2, \dots, 5),
$$

and

$$
\left|\xi^\eta\partial_\xi^\eta\left[\{1-\chi(|\xi|)\}\left\{\frac{\tilde{a}_{ij}(\lambda;\xi)}{\det[\lambda\cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{ij}(0;\xi)}{\det\hat{A}(\xi)}\right\}\right]\right| \le C\frac{|\lambda|}{|\xi|^2} \quad (i \ne 1, j \ne 1),
$$

for $|\eta| \le 2$, $\mathrm{Re}\,\lambda \ge 0$, $|\lambda| \le 1$ and $\xi \in R^3$, by using Theorem 7.9.5 of [6] concerning the L_q-estimate of Fourier multiplier we obtain that

$$
\begin{aligned}
(2.25)\quad & \sum_{j=1}^{5}\|\{T_{2,1j}(\lambda) - T_{2,1j}(0)\}f_j\|_{W_q^1(R^3)} + \sum_{j=1}^{5}\sum_{i=2}^{5}\|\{T_{2,ij}(\lambda) - T_{2,ij}(0)\}f_j\|_{W_q^2(R^3)} \\
&\le C|\lambda|\,\|\boldsymbol{f}\|_{X_q(R^3)}.
\end{aligned}
$$

Also since it follows from (2.2a), (2.5) and (2.6) that

$$
\left|\chi(|\xi|)\left\{\frac{a_{ij}(\lambda;\xi)}{\det[\lambda\cdot I + \hat{A}(\xi)]} - \frac{a_{ij}(0;\xi)}{\det\hat{A}(\xi)}\right\}\right| \le C|\lambda|^\varepsilon|\xi|^{-2-2\varepsilon}
$$

for $0 < \varepsilon < \frac{1}{2}$, $\mathrm{Re}\,\lambda \ge 0$, $|\lambda| \le 1$ and $\xi \in R^3$, we obtain that for $|\alpha_1| \le 1$,

$|\alpha_i| \le 2$ $(i \ne 1)$

$$(2.26) \quad |\partial_x^{\alpha_i}\{T_{1,ij}(\lambda) - T_{1,ij}(0)\}f_j(x)|$$

$$\le C(q,b)\|\chi(|\xi|)(i\xi)^{\alpha_i}\left\{\frac{a_{ij}(\lambda;\xi)}{\det[\lambda\cdot I + \hat{A}(\xi)]} - \frac{a_{ij}(0;\xi)}{\det \hat{A}(\xi)}\right\}\|_{L_1(R^3)}\|f\|_{L_q(R^3)}$$

$$\le C(q,b)|\lambda|^{\varepsilon}\|f\|_{X_q(R^3)} \quad \text{for } f \in X_{q,b}(R^3).$$

Thus it follows from (2.25), (2.26) and (2.24) that (2.23). This completes the proof.

§3. The resolvent set of $-A$

In this section, we shall prove Theorem A. To prove this theorem we need the following lemma concerning the uniqueness, which is a key in our argument. First note that by Lemma 2.1 (iii)

$$\det[\lambda + \hat{A}(\xi)] \ne 0 \quad \text{for } \lambda \in \Sigma'' = \{\lambda \in \mathrm{C}; C_1 \operatorname{Re}\lambda + (\operatorname{Im}\lambda)^2 > 0\}$$

where C_1 is a constant depending only on $\alpha, \beta, \gamma, \kappa$, and ω. In the view of this and Theorem 1.5, taking a constant C in the parabolic region

$$\Sigma = \{\lambda \in \mathrm{C}; C \operatorname{Re}\lambda + (\operatorname{Im}\lambda)^2 > 0\}$$

so that $\Sigma \subset \Sigma' \cap \Sigma''$, we have

Lemma 3.1. *Let $1 < q < \infty$. If $\lambda \in \Sigma$, then*

$$\operatorname{Ker}(\lambda + A) = \{0\}.$$

Proof. Let $(\lambda + A)\boldsymbol{u} = 0$. In view of the proof of Lemma 1.8, by bootstrap argument, we see that ${}^T\boldsymbol{u} \in W_q^{\ell+1}(\Omega) \times W_q^{\ell+2}(\Omega) \times W_q^{\ell+2}(\Omega)$ for any integer $\ell \ge 1$. We fix an integer ℓ such that $\ell = 0$ when $2 \le q < \infty$ and $\ell \ge 3(1/q - 1/2)$ when $1 < q < 2$. Let ${}^T\boldsymbol{v} \in W_q^{\ell+1}(R^3) \times W_q^{\ell+2}(R^3) \times W_q^{\ell+2}(R^3)$ be functions such that $\boldsymbol{v} = \boldsymbol{u}$ in Ω. Put $\boldsymbol{f} = (\lambda + A)\boldsymbol{v}$, then since $(\lambda + A)\boldsymbol{u} = 0$ in Ω, we see that $\operatorname{supp}\boldsymbol{f}$ is compact, and moreover $\boldsymbol{f} \in X_q^{l+1}(R^3)$. Since $\operatorname{supp}\boldsymbol{f}$ is compact, $\boldsymbol{f} \in X_2^1(\Omega)$ when $2 \le q < \infty$. When $1 < q < 2$, since $\ell \ge 3(1/q - 1/2)$, by Sobolev's imbedding theorem we have $\boldsymbol{f} \in X_2^1(\Omega)$ too. Put $\boldsymbol{w} = \boldsymbol{R}_0(\lambda)\boldsymbol{f}$ where the symbols are the same as in (2.4). Since $\det[\lambda + \hat{A}(\xi)] \ne 0$ for any $\xi \in R^3$ and $\lambda \in \Sigma$, by Parseval's formula we know that ${}^T\boldsymbol{w} \in W_2^1(R^3) \times W_2^2(R^3) \times W_2^2(R^3)$. Since $(\lambda + A)\{\boldsymbol{v} - \boldsymbol{R}_0(\lambda)\boldsymbol{f}\} = 0$ in R^3, by Fourier transform we have $\{\lambda + \hat{A}(\xi)\}$

$\{v(\xi)-\hat{w}(\xi)\}=0$, which implies that $v=w$ in R^3 because $\det[\lambda+A(\xi)]\neq 0$. Thus employing the same argument as in the proof of Lemma 1.8, we have $u=0$. This completes the proof.

A PROOF OF THEOREM A. In view of Lemma 1.6, we only show (0.4). Now we shall construct parametrix to (1.1) in Ω. Let $\partial\Omega\subset B_{R_0}$, b be a fixed constant $b>R_0+3$ and let $\Omega_b=\Omega\cap B_b$. Given $\lambda\in\Sigma$ and $g\in X_q(\Omega_b)$, let $w\in W_q^1(\Omega_b)\times W_q^2(\Omega_b)\times W_q^2(\Omega_b)$ be solutions to the problem:

$$(\lambda+A)w=g \text{ in } \Omega_b,$$
$$Pw=0 \text{ on } \partial\Omega_b.$$

The existence of such w is guaranteed by Remark 1.9. In terms of w, let us define the operator $L(\lambda)$ by relations:

$$\begin{aligned}w&=L(\lambda)g\\&=\{L_\rho(\lambda)g, L_v(\lambda)g, L_\theta(\lambda)g\}.\end{aligned} \tag{3.1}$$

Here and hereafter, for $f\in X_q(\Omega)$, we put $f_0(x)=f(x)$ for $x\in\Omega$ and $=0$ for $x\in R^3\backslash\Omega$, $\Pi_b f$ stands for the restriction of f to Ω_b. By Remark 1.9 and (3.1) we have

$$\begin{aligned}&\|L(\lambda)\Pi_b f\|_{X_q(\Omega_b)}+\|PL(\lambda)\Pi_b f\|_{2,q,\Omega_b}\\&\quad\le C(q,b,\lambda)\|f\|_{X_q(\Omega)}\quad\text{for any } f\in X_q(\Omega).\end{aligned} \tag{3.2}$$

Let $R_0(\lambda), R_{0,\rho}(\lambda), R_{0,v}(\lambda)$ and $R_{0,\theta}(\lambda)$ be the same symbol as in (2.3) and (2.4). Since $\det[\lambda+\hat{A}(\xi)]\neq 0$ whenever $\xi\in R^3$ and $\lambda\in\Sigma$, by Theorem 7.9.5 of [6], we see that

$$\begin{aligned}&\|R_0(\lambda)f_0\|_{X_q(R^3)}+\|PR_0(\lambda)f_0\|_{2,q,R^3}\\&\quad\le C(q,\lambda)\|f\|_{X_q(\Omega)}\quad\text{for any } f\in X_q(\Omega).\end{aligned} \tag{3.3}$$

Let $\varphi\in C^\infty(R^3)$ such that $\varphi(x)=0$ for $|x|\le b-2$ and $=1$ for $|x|\ge b-1$. We introduce the operator $Q_1(\lambda)$ by the relations:

$$\begin{aligned}Q_1(\lambda)f&={}^T\{Q_{1,\rho}(\lambda)f, Q_{1,v}(\lambda)f, Q_{1,\theta}(\lambda)f\}\\&:=\varphi R_0(\lambda)(f_0)+(1-\varphi)L(\lambda)\Pi_b f\quad\text{for any } f\in X_q(\Omega),\end{aligned} \tag{3.4}$$

Then by (3.2) and (3.3) we have

$$(3.5)\qquad {}^T\boldsymbol{Q}_1(\lambda)\boldsymbol{f} \in W_q^1(\Omega) \times W_q^2(\Omega) \times W_q^2(\Omega) \quad \text{for any } \boldsymbol{f} \in X_q(\Omega),$$

$$(3.6)\qquad \|\boldsymbol{Q}_1(\lambda)\boldsymbol{f}\|_{X_q(\Omega)} + \|\boldsymbol{P}\boldsymbol{Q}_1(\lambda)\boldsymbol{f}\|_{2,q,\Omega} \le C(q, \lambda, b)\|\boldsymbol{f}\|_{X_q(\Omega)} \quad \text{for any } \boldsymbol{f} \in X_q(\Omega),$$

and

$$(3.7\text{a})\qquad (\lambda + \boldsymbol{A})\boldsymbol{Q}_1(\lambda)\boldsymbol{f} = \boldsymbol{f} + \boldsymbol{V}(\lambda)\boldsymbol{f} \text{ in } \Omega,$$

$$(3.7\text{b})\qquad \boldsymbol{P}\boldsymbol{Q}_1(\lambda)\boldsymbol{f} = 0 \text{ on } \partial\Omega,$$

where $\boldsymbol{V}(\lambda)\boldsymbol{f} = {}^T\{V_\rho(\lambda)\boldsymbol{f}, V_v(\lambda)\boldsymbol{f}, V_\theta(\lambda)\boldsymbol{f}\}$ and

$$(3.8\text{a})\qquad V_\rho(\lambda)\boldsymbol{f} = \gamma\nabla\varphi[R_{0,v}(\lambda)(\boldsymbol{f}_0) - L_v(\lambda)\Pi_b\boldsymbol{f}],$$

$$\begin{aligned}(3.8\text{b})\qquad V_v(\lambda)\boldsymbol{f} = &-\alpha[\Delta\varphi + 2(\partial_j\varphi)\partial_j][R_{0,v}(\lambda)(\boldsymbol{f}_0) - L_v(\lambda)\Pi_b\boldsymbol{f}] \\ &- \beta\nabla\{\partial_j\varphi[R_{0,v}(\lambda)(\boldsymbol{f}_0) - L_v(\lambda)\Pi_b\boldsymbol{f}]_j\} \\ &- \beta\nabla\varphi\{\mathrm{div}[R_{0,v}(\lambda)(\boldsymbol{f}_0) - L_v(\lambda)\Pi_b\boldsymbol{f}]\} \\ &+ \gamma\nabla\varphi[R_{0,\rho}(\lambda)(\boldsymbol{f}_0) - L_\rho(\lambda)\Pi_b\boldsymbol{f}] \\ &+ \omega\partial_j\varphi[R_{0,\theta}(\lambda)(\boldsymbol{f}_0) - L_\theta(\lambda)\Pi_b\boldsymbol{f}]_j,\end{aligned}$$

$$\begin{aligned}(3.8\text{c})\qquad V_\theta(\lambda)\boldsymbol{f} = &-\kappa[\Delta\varphi + 2\partial_j\varphi\partial_j][R_{0,\theta}(\lambda)(\boldsymbol{f}_0) - L_\theta(\lambda)\Pi_b\boldsymbol{f}] \\ &+ \omega\partial_j\varphi[R_{0,v}(\lambda)(\boldsymbol{f}_0) - L_v(\lambda)\Pi_b\boldsymbol{f}]_j.\end{aligned}$$

Our task is to prove that $I + \boldsymbol{V}(\lambda)$ has the bounded inverse from $X_q(\Omega)$ onto itself. It follows from (3.2), (3.3) and (3.8) that ${}^T\boldsymbol{V}(\lambda) \in \mathscr{B}(X_q(\Omega), W_q^2(\Omega) \times W_q^1(\Omega) \times W_q^1(\Omega))$ for each $\lambda \in \Sigma$. Since $\mathrm{supp}\, \boldsymbol{V}(\lambda)\boldsymbol{f} \subset D_{b-1} = \{x \in R^3; b-2 < |x| < b-1\}$, by Rellich's compactness theorem $\boldsymbol{V}(\lambda)$ is a compact operator from $X_q(\Omega)$ onto itself. Thus by Fredholm's alternative theorem, it suffices to show that $I + \boldsymbol{V}(\lambda)$ is injective in $X_q(\Omega)$ in order to prove that $I + \boldsymbol{V}(\lambda)$ has the bounded inverse. Let $(I + \boldsymbol{V}(\lambda))\boldsymbol{f} = 0$ in Ω, $\boldsymbol{f} \in X_q(\Omega)$. Then it follows from (3.5), (3.7) and Lemma 3.1 that

$$\boldsymbol{Q}_1(\lambda)\boldsymbol{f} = 0 \text{ in } \Omega,$$
$$\boldsymbol{P}\boldsymbol{Q}_1(\lambda)\boldsymbol{f} = 0 \text{ on } \partial\Omega,$$

which together with (3.4) implies that

(3.9a) $$R_0(\lambda)(f_0) = 0 \quad \text{for } |x| \geq b-1,$$

(3.9b) $$L(\lambda)\Pi_b f = 0 \quad \text{for } |x| \leq b-2.$$

Put $z = \Pi_b R_0(\lambda)(f_0) - w$ where $w = L(\lambda)\Pi_b f$ in Ω_b and $= 0$ in $R^3 \backslash \Omega$. By (3.9b) we know that ${}^T w \in W_q^1(B_b) \times W_q^2(B_b) \times W_q^2(B_b)$ and

$$(\lambda + A)w = \Pi_b^0 f_0 \text{ in } B_b, Pw = 0 \text{ on } |x| = b,$$

where $\Pi_b^0 f_0$ stands for the restriction of f_0 to B_b, and hence we see that

$$(\lambda + A)z = 0 \text{ in } B_b, Pz = 0 \text{ on } |x| = b,$$

which with the help of Theorem 1.5 means that $z = 0$ in B_b. As a result, we have

(3.10) $$R_0(\lambda)(f_0) = L(\lambda)\Pi_b f \quad \text{in } \Omega_b.$$

Combining (3.4) and (3.10), we see that

(3.11) $$\begin{aligned} R_0(\lambda)(f_0) &= \varphi\{R_0(\lambda)(f_0) - L(\lambda)\Pi_b f\} + R_0(\lambda)(f_0) \\ &= Q_1(\lambda)f = 0 \text{ in } \Omega_b. \end{aligned}$$

It follows from (3.9) and (3.11) that $R_0(\lambda)(f_0) = 0$ in Ω, which together with (2.1) implies that $f_0 = f = 0$ in Ω. Therefore, we have proved that $(I + V(\lambda))$ has the bounded inverse $(I + V(\lambda))^{-1}$ from $X_q(\Omega)$ onto itself. Given $f \in X_q(\Omega)$, if we put $u = Q_1(\lambda)(I + V(\lambda))^{-1}$, by (3.7) and (3.6) we see that $(\lambda + A)u = f$ in $X_q(\Omega)$ and $u \in \mathscr{D}(A)$, which means that the inverse $(\lambda + A)^{-1}$ of $(\lambda + A)$ exists, and it is bounded, that is by (3.6)

$$\begin{aligned} &\|(\lambda + A)^{-1} f\|_{X_q(\Omega)} + \|P(\lambda + A)^{-1} f\|_{2,q,\Omega} \\ &\leq C(q, b, \lambda)\|(I + V(\lambda))^{-1}\|_{\mathscr{B}(X_q(\Omega))}\|f\|_{X_q(\Omega)} \end{aligned}$$

for any $f \in X_q(\Omega)$, which completes the proof.

§4. Behaviour of $(\lambda + A)^{-1}$ near $\lambda = 0$

In this section we shall discuss behaviour of $(\lambda + A)^{-1}$ near $\lambda = 0$. Our goal of this section is to prove the following theorem.

Let $Y_q(\Omega)$ and $Y_{q,b}(\Omega)$ be the same symbols as in (1.6) and (0.5), respectively.

THEOREM 4.1. *Let* $1 < q < \infty$, b_0 *a number such that* $B_{b_0} \supset R^3 \backslash \Omega$ *and let* $b > b_0$. *Put* $D_\varepsilon = \{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \geq 0,\ 0 < |\lambda| \leq \varepsilon\}$, $\mathscr{Y} = \mathscr{B}(Y_{q,b}(\Omega);\ \mathscr{D}(A))$ *and* $\mathscr{A}(D_\varepsilon; \mathscr{Y})$ *is the set of all* $\mathscr{Y}$*-valued holomorphic functions in* D_ε. *Then, there exists a positive number* ε *and* $\tilde{\boldsymbol{R}}(\lambda) \in \mathscr{A}(D_\varepsilon; \mathscr{Y})$ *such that*

$$\tilde{\boldsymbol{R}}(\lambda)\boldsymbol{f} = (\lambda + A)^{-1}\boldsymbol{f}, \tag{4.1}$$

$$\|\tilde{\boldsymbol{R}}(\lambda)\boldsymbol{f}\|_{X_q(\Omega_b)} + \|P\tilde{\boldsymbol{R}}(\lambda)\boldsymbol{f}\|_{2,q,\Omega_b} \leq C(q,b,\varepsilon)\|\boldsymbol{f}\|_{X_q(\Omega)}, \tag{4.2a}$$

$$\left\|\left(\frac{d}{d\lambda}\right)^k \tilde{\boldsymbol{R}}(\lambda)\boldsymbol{f}\right\|_{X_q(\Omega_b)} + \left\|\left(\frac{d}{d\lambda}\right)^k P\tilde{\boldsymbol{R}}(\lambda)\boldsymbol{f}\right\|_{2,q,\Omega_b} \leq C(q,b,k,\varepsilon)|\lambda|^{(1/2)-k}\|\boldsymbol{f}\|_{X_q(\Omega)}, \tag{4.2b}$$

for any $\lambda \in D_\varepsilon$, $\boldsymbol{f} \in Y_{q,b}(\Omega_b)$ *and* $k \geq 1$ *integers.*

In Theorem 4.1, in view of proof of Remark 1,9, taking $\psi \in C_0^\infty(\Omega_b)$ such that $\int_{\Omega_b} \psi(x)\,dx = 1$, we have the following corollary:

COROLLARY 4.2. *Let* $1 < q < \infty$, b_0 *be a number such that* $B_{b_0} \supset R^3 \backslash \Omega$ *and let* $b > b_0$. *Put* $\mathscr{X} = \mathscr{B}(X_{q,b}(\Omega); \mathscr{D}(A))$. *Then, there exists a positive number* ε *and* $\boldsymbol{R}(\lambda) \in \mathscr{A}(D_\varepsilon; \mathscr{X})$ *such that* $\boldsymbol{R}(\lambda)\boldsymbol{f} = (\lambda + A)^{-1}\boldsymbol{f}$,

$$\|\boldsymbol{R}(\lambda)\boldsymbol{f}\|_{X_q(\Omega_b)} + \|P\boldsymbol{R}(\lambda)\boldsymbol{f}\|_{2,q,\Omega_b} \leq C(q,b,\varepsilon)\{\|\boldsymbol{f}\|_{X_q(\Omega)} + |\lambda|^{-1}\|f_1\|_{q,\Omega}\},$$

and

$$\left\|\left(\frac{d}{d\lambda}\right)^k \boldsymbol{R}(\lambda)\boldsymbol{f}\right\|_{X_q(\Omega_b)} + \left\|\left(\frac{d}{d\lambda}\right)^k P\boldsymbol{R}(\lambda)\boldsymbol{f}\right\|_{2,q,\Omega_b} \leq C(q,b,k,\varepsilon)|\lambda|^{(1/2)-k}\{\|\boldsymbol{f}\|_{X_q(\Omega)} + |\lambda|^{-1}\|f_1\|_{q,\Omega}\},$$

for any $\lambda \in D_\varepsilon$, $\boldsymbol{f} = {}^T\{f_1, f_2, f_3\} \in X_{q,b}(\Omega)$ *and* $k \geq 1$ *integers. Moreover,*

$$\boldsymbol{R}(\lambda) = \tilde{\boldsymbol{R}}(\lambda)N_1 + \frac{\gamma}{\lambda}\tilde{\boldsymbol{R}}(\lambda)N_2 + \frac{1}{\lambda}N_3$$

where $N_j = N_j(\psi, \Omega_b)$ $(j = 1,2,3)$, are the same symbols as in (1.21).

To prove Theorem 4.1, in the same way to the proof of Theorem A we shall construct a parametrix near $\lambda = 0$. The following proposition concerning the uniqueness is a key in our argument, which was proved by Iwashita [9].

For an integer $m \geq 0$ and real numbers τ, q with $1 < q < \infty$, we set

$$W_q^{m,\tau}(\Omega) = \{u; (1+|x|^2)^{\tau/2}\partial_x^\alpha u \in L_q(\Omega), |\alpha| \leq m\},$$

$$\hat{W}_q^m(\Omega) = \text{the completion of } C_0^\infty(\bar{\Omega}) \text{ by } \sum_{|\alpha|=m} \|\partial_x^\alpha \cdot\|_{q,\Omega}.$$

PROPOSITION 4.3. *Let $1 < q < \infty$. Suppose that $\boldsymbol{u} \in \hat{W}_q^2(\Omega) \cap W_q^{1,\tau}(\Omega)$ and $p \in \hat{W}_q^1(\Omega) \cap L_q^{\tau'}(\Omega)$ with some τ, $\tau' \in R$ satisfy*

$$-\Delta \boldsymbol{u} + \nabla p = 0, \quad \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,$$

$$\boldsymbol{u}|_{\partial\Omega} = 0 \text{ on } \partial\Omega,$$

and

$$\lim_{R\to\infty} \frac{1}{R^3}\int_{R<|x|<2R} |\boldsymbol{u}(x)|^q\, dx = \lim_{R\to\infty} \frac{1}{R^3}\int_{R<|x|<2R} |p(x)|^q\, dx = 0.$$

Then, $\boldsymbol{u} = 0$ and $p = 0$ in Ω.

REMARK 4.4. In view of proof of Proposition 4.3, we can replace $\hat{W}_q^2(\Omega) \cap W_q^{1,\tau}(\Omega)$ by $W_{q,E}^2(\Omega)$, $\hat{W}_q^1(\Omega) \cap L_q^\tau(\Omega)$ by $W_{q,E}^1(\Omega)$, where

$$W_{q,E}^m(\Omega) = \{u; \text{there exists a } U \in W_{q,\mathrm{loc}}^m(R^3) \text{ such that } u = U \text{ in } \Omega\}.$$

Moreover, we can show the same uniqueness theorem for the system

$$-\Delta \boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u}|_{\partial\Omega} = 0 \text{ on } \partial\Omega,$$

as Proposition 4.3.

Now we shall show the following results on uniqueness for (1.1).

LEMMA 4.5. *Let $1 < q < \infty$. Suppose that ${}^T\{\rho, v, \theta\} \in W_{q,E}^1(\Omega) \times W_{q,E}^2(\Omega) \times W_{q,E}^2(\Omega)$ satisfies the homogeneous equation:*

$$\begin{aligned} &\gamma \operatorname{div} v = 0, \\ &-\alpha\Delta v - \beta\nabla \operatorname{div} v + \gamma\nabla p + \omega\nabla\theta = 0 \text{ in } \Omega, \\ &-\kappa\Delta\theta + \omega \operatorname{div} v = 0, \\ &v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.3}$$

and satisfies

$$
(4.4)\qquad
\begin{aligned}
&\lim_{R\to\infty}\frac{1}{R^3}\int_{R<|x|<2R}|\rho(x)|^q\,dx=0,\\
&\lim_{R\to\infty}\frac{1}{R^3}\int_{R<|x|<2R}|v(x)|^q\,dx=0,\\
&\lim_{R\to\infty}\frac{1}{R^3}\int_{R<|x|<2R}|\theta(x)|^q\,dx=0.
\end{aligned}
$$

Then $\rho=0$, $v=0$ *and* $\theta=0$ *in* Ω.

PROOF. By (4.3), we have

$$
(4.5)\qquad -\kappa\Delta\theta=0 \text{ in } \Omega, \theta|_{\partial\Omega}=0 \text{ on } \partial\Omega,
$$

and

$$
(4.6)\qquad
\begin{aligned}
-\alpha\Delta v+\gamma\nabla\rho&=\omega\nabla\theta \text{ in } \Omega,\\
\operatorname{div} v=0 \text{ in } \Omega,\ v|_{\partial\Omega}&=0 \text{ on } \partial\Omega.
\end{aligned}
$$

In view of Remark 4.4, applying Proposition 4.3 to the system (4.5) with (4.4), we have $\theta=0$ in Ω, which implies $\rho=0$ and $v=0$ in Ω by applying Proposition 4.3 to the system (4.6) with (4.4). This completes the proof.

A PROOF OF THEOREM 4.1. To prove Theorem 4.1, we shall use the symbols in the proof of Theorem A. For any $\boldsymbol{g}\in Y_{q,b}(\Omega)$, $\boldsymbol{w}=\boldsymbol{L}(0)\boldsymbol{g}$ satisfies the following relations:

$$
(4.7a)\qquad \boldsymbol{A}\boldsymbol{w}=\boldsymbol{g} \text{ in } \Omega_b, \boldsymbol{P}\boldsymbol{w}=0 \text{ on } \partial\Omega_b.
$$

$$
(4.7b)\qquad \|\boldsymbol{w}\|_{Y_q(\Omega_b)}+\|\boldsymbol{P}\boldsymbol{w}\|_{2,q,\Omega_b}\leq C(q,b)\|\boldsymbol{g}\|_{Y_q(\Omega_b)}.
$$

Choosing φ in $C^\infty(R^3)$ so that $\varphi(x)=1$ for $|x|\geq b-1$ and $=0$ if $|x|\leq b-2$, we define the operator $\boldsymbol{R}_1(\lambda)$ by the relations:

$$
(4.8a)\qquad
\begin{aligned}
\boldsymbol{R}_1(\lambda)\boldsymbol{f}&={}^T\{R_{1,\rho}(\lambda)\boldsymbol{f},R_{1,v}(\lambda)\boldsymbol{f},R_{1,\theta}(\lambda)\boldsymbol{f}\}\\
&=\varphi\boldsymbol{R}_0(\lambda)(\boldsymbol{f}_0)+(1-\varphi)\boldsymbol{L}(0)\boldsymbol{f},
\end{aligned}
$$

for $\boldsymbol{f}\in Y_{q,b}(\Omega)$ and $\lambda\in D_\varepsilon\cup\{0\}$. Here, note that ${}^T\{\rho,v,\theta\}=\boldsymbol{L}(0)\boldsymbol{f}$ satisfies the equations (1.11) and (1.13), and which implies that $\rho=L_\rho(0)\boldsymbol{f}$ is unique up to an additive constant by Proposition 1.1. Hence, $L_\rho(0)$ is chosen in such a way that

$$\int_{\Omega_b}(1-\varphi)L_\rho(0)\boldsymbol{f}\,dx=\int_{B_b}R_{0,\rho}(0)\boldsymbol{f}_0\,dx-\int_{\Omega_b}\varphi R_{0,\rho}(0)\boldsymbol{f}_0\,dx. \tag{4.8b}$$

Then by (4.7b), Theorem 2.2 and Lemma 2.4 we have

$$\boldsymbol{R}_1(\lambda)\in\mathscr{A}(D_\varepsilon;\mathscr{Y}), \tag{4.9a}$$

$${}^T\boldsymbol{R}_1(0)\in\mathscr{B}(Y_{q,b}(\Omega),W^1_{q,E}(\Omega)\times W^2_{q,E}(\Omega)\times W^2_{q,E}(\Omega)), \tag{4.9b}$$

$$(\lambda+A)\boldsymbol{R}_1(\lambda)\boldsymbol{f}=\boldsymbol{f}+\boldsymbol{S}_1(\lambda)\boldsymbol{f}\text{ in }\Omega,\ \boldsymbol{P}\boldsymbol{R}_1(\lambda)\boldsymbol{f}=0\text{ on }\partial\Omega, \tag{4.9c}$$

where

$$\boldsymbol{S}_1(\lambda)\boldsymbol{f}={}^T\{S_{1,\rho}(\lambda)\boldsymbol{f},S_{1,v}(\lambda)\boldsymbol{f},S_{1,\theta}(\lambda)\boldsymbol{f}\}, \tag{4.10a}$$

and

$$S_{1,\rho}(\lambda)\boldsymbol{f}=\lambda(1-\varphi)L_\rho(0)\boldsymbol{f}+\gamma\nabla\varphi[R_{0,v}(\lambda)(\boldsymbol{f}_0)-L_v(0)\boldsymbol{f}], \tag{4.10b}$$

$$\begin{aligned}S_{1,v}(\lambda)\boldsymbol{f}=&\,\lambda(1-\varphi)L_v(0)\boldsymbol{f}\\&-\alpha[\Delta\varphi+2(\partial_j\varphi)\partial_j][R_{0,v}(\lambda)(\boldsymbol{f}_0)-L_v(0)\boldsymbol{f}]\\&-\beta\nabla\{\partial_j\varphi[R_{0,v}(\lambda)(\boldsymbol{f}_0)-L_v(0)\boldsymbol{f}]_j\}\\&-\beta\nabla\varphi\{\mathrm{div}\,[R_{0,v}(\lambda)(\boldsymbol{f}_0)-L_v(0)\boldsymbol{f}]\}\\&+\gamma\nabla\varphi[R_{0,\rho}(\lambda)(\boldsymbol{f}_0)-L_\rho(0)\boldsymbol{f}]\\&+\omega\partial_j\varphi[R_{0,\theta}(\lambda)(\boldsymbol{f}_0)-L_\theta(0)\boldsymbol{f}]_j,\end{aligned} \tag{4.10c}$$

$$\begin{aligned}S_{1,\theta}(\lambda)\boldsymbol{f}=&\,\lambda(1-\varphi)L_\theta(0)\boldsymbol{f}\\&-\kappa[\Delta\varphi+2\partial_j\varphi\partial_j][R_{0,\theta}(\lambda)(\boldsymbol{f}_0)-L_\theta(0)\boldsymbol{f}]\\&+\omega\partial_j\varphi[R_{0,v}(\lambda)(\boldsymbol{f}_0)-L_v(0)\boldsymbol{f}]_j.\end{aligned} \tag{4.10d}$$

It follows from (4.10), (4.9b), Theorem 2.2 and Lemma 2.4 that

$${}^T\boldsymbol{S}_1(\lambda)\in\mathscr{B}(Y_{q,b}(\Omega),W^1_q(\Omega)\times W^1_q(\Omega)\times W^1_q(\Omega))\text{ for any }\lambda\in D_\varepsilon, \tag{4.11a}$$

$$\boldsymbol{S}_1(0)\in\mathscr{B}(Y_{q,b}(\Omega),X^1_q(\Omega)). \tag{4.11b}$$

Noting that the Stokes formula implies that

$$(4.12)\quad \int_{\Omega_b} S_{1,\rho}(\lambda)f\,dx$$
$$= \lambda\int_{\Omega_b}(1-\varphi)L_\rho(0)f\,dx + \int_{B_b}\gamma\,\mathrm{div}\,R_{0,v}(\lambda)f_0\,dx$$
$$- \int_{\Omega_b}\varphi\gamma\,\mathrm{div}[R_{0,v}(\lambda)f_0 - L_v(0)f]\,dx$$
$$= \lambda\left\{\int_{\Omega_b}(1-\varphi)L_\rho(0)f\,dx - \int_{B_b}R_{0,\rho}(\lambda)f_0\,dx + \int_{\Omega_b}\varphi R_{0,\rho}(\lambda)f_0\,dx\right\},$$

we have to modify $S_1(\lambda)$ such that total integral over Ω_b is zero because $S_1(\lambda)f$ does not belong to $Y_{q,b}(\Omega)$ when $\lambda \neq 0$. To do this, choosing $\psi \in C_0^\infty(\Omega_b)$ so that $\int_{\Omega_b}\psi(x)\,dx = 1$ and set

$$(4.13a)\qquad R_2(0) = R_1(0),$$

$$(4.13b)\qquad R_2(\lambda)f = {}^T\{R_{2,\rho}(\lambda)f, R_{2,v}(\lambda)f, R_{2,\theta}(\lambda)f \quad \text{for } \lambda \in D_\varepsilon,$$

where $R_{2,v}(\lambda) = R_{1,v}(\lambda), R_{2,\theta}(\lambda) = R_{1,\theta}(\lambda)$ and

$$(4.13c)\qquad R_{2,\rho}(\lambda)f = R_{1,\rho}(\lambda)f - \frac{1}{\lambda}\int_{\Omega_b}S_{1,\rho}(\lambda)f\,dx\,\psi.$$

Also, put

$$(4.14a)\qquad S_2(0) = S_1(0),$$

$$(4.14b)\qquad S_2(\lambda)f = {}^T\{S_{2,\rho}(\lambda)f, S_{2,v}(\lambda)f, S_{2,\theta}(\lambda)f\} \quad \text{for } \lambda \in D_\varepsilon,$$

where $S_{2,\theta}(\lambda) = S_{1,\theta}(\lambda)$,

$$(4.14c)\qquad S_{2,\rho}(\lambda)f = S_{1,\rho}(\lambda)f - \int_{\Omega_b}S_{1,\rho}(\lambda)f\,dx\,\psi,$$

and

$$(4.14d)\qquad S_{2,v}(\lambda)f = S_{1,v}(\lambda)f - \frac{\gamma}{\lambda}\int_{\Omega_b}S_{1,\rho}(\lambda)f\,dx\nabla\psi.$$

Then, it follows from (4.9), (4.10), (4.13) and (4.14) that

$$(4.15a)\qquad R_2(\lambda) \in \mathscr{A}(D_\varepsilon; \mathscr{Y}),$$

$$(4.15b)\qquad (\lambda + A)R_2(\lambda)f = f + S_2(\lambda)f \text{ in } \Omega, PR_2(\lambda)f = 0 \text{ on } \partial\Omega,$$

and by (4.10), (4.11) and (4.14) we have

(4.16a) $\quad {}^{T}S_2(\lambda) \in \mathscr{B}(Y_{q,b}(\Omega), W_q^1(\Omega) \times W_q^1(\Omega) \times W_q^1(\Omega))$ for any $\lambda \in D_\varepsilon$,

moreover, noting (4.8b) and (4.12), it follows from Lemma 2.4 that

$$\int_{\Omega_b} S_{2,p}(\lambda) f \, dx = 0 \quad \text{for } \lambda \in D_\varepsilon \cup \{0\}, \tag{4.16b}$$

$$\|S_2(\lambda) - S_2(0)\|_{\mathscr{B}(Y_{q,b}(\Omega), Y_{q,b}(\Omega))} \le C(q,b,\delta)|\lambda|^\delta \tag{4.16c}$$

for $\mathrm{Re}\,\lambda \ge 0$, $|\lambda| \le 1$, where $0 < \delta < 1/2$. Then, we shall show the following Lemma.

LEMMA 4.6. *Let $1 < q < \infty$. Then, $I + S_2(0) \in \mathscr{B}(Y_{q,b}(\Omega))$ has the bounded inverse $(I + S_2(0))^{-1}$.*

PROOF. Since $\mathrm{supp}\, S_2(0)f$ is contained in Ω_b, it follows from (4.11b), (4.14a), (4.16b) and Rellich's compactness theorem, $S_2(0)$ is a compact operator from $Y_{q,b}(\Omega)$ into itself. Thus, to prove this Lemma, by Fredholm's alternative theorem, it suffices to show that $I + S_2(0)$ is injective. Let $(I + S_2(0))f = 0$ in Ω, $f \in Y_{q,b}(\Omega)$. Our task is to prove that $f = 0$. It follows from (4.7b), (4.9b), (4.13a) and (4.15b) that ${}^{T}R_2(0)f \in W^1_{q,E}(\Omega) \times W^2_{q,E}(\Omega) \times W^2_{q,E}(\Omega)$ and satisfies

$$AR_2(0)f = 0 \text{ in } \Omega, PR_2(0)f = 0 \text{ on } \partial\Omega. \tag{4.17}$$

Since $R_2(0)f = R_0(0)(f_0)$ for $|x| \ge b - 1$ it follows from Lemma 2.4 that

$$\lim_{R \to \infty} \frac{1}{R^3} \int_{R < |x| < 2R} |(R_2(0)f)(x)|^q dx = 0.$$

Hence by (4.17) and Lemma 4.5 we have

$$R_2(0)f = 0 \text{ in } \Omega, \tag{4.18}$$

and it follows from (4.8a), (4.13a) and (4.18) that

$$R_0(0)(f_0) = 0 \text{ for } |x| \ge b - 1, \tag{4.19a}$$

$$L(0)f = 0 \text{ for } x \in \Omega_{b-2}. \tag{4.19b}$$

Let us define w by the relations: $w(x) = L(0)f(x)$ for $x \in \Omega_b$ and $= 0$ for $x \in R^3 \backslash \Omega_b$, and then by (4.19) we see that $z = \pi_b^0 R_0(0)(f_0) - w$ possess the following properties: ${}^T z \in W_q^1(B_b) \times W_q^2(B_b) \times W_q^2(B_b)$ and

$$Az = 0 \text{ in } B_b, Pz = 0 \text{ on } S_b,$$

where $\pi_b^0 v$ is the restriction of v to B_b, and hence by Lemma 1.8 we know that $z = 0$ in Ω_b, which means that

$$R_0(0)(f_0) = L(0)f \text{ in } \Omega_b. \tag{4.20}$$

Therefore, employing the same argument as in the proof of Theorem 3.1, by (4.19) and (4.20) we have $f = 0$, which completes the proof of this Lemma.

We return to the proof of Theorem 4.1. In view of Lemma 4.6, $(I + S_2(0))^{-1} \in \mathscr{B}(Y_{q,b}(\Omega))$, and then put

$$M = \|(I + S_2(0))^{-1}\|,$$

where $\|\cdot\|$ stands for the operation norm. By (4.16c) and Neumann series expansion, there exists an $\varepsilon > 0$ such that $I + S_2(\lambda)$ also has the bounded inverse $(I + S_2(\lambda))^{-1}$ from $Y_{q,b}(\Omega)$ onto itself whenever $\lambda \in D_\varepsilon$, and moreover

$$\|(I + S_2(\lambda))^{-1}\| \leq 2M \quad \text{for } \lambda \in D_\varepsilon. \tag{4.21}$$

If we look at (4.13) with (4.8) and (4.10), by Theorem 2.2 we have

$$\|R_2(\lambda)f\|_{X_q(\Omega_b)} + \|PR_2(\lambda)\|_{2,q,\Omega_b} \leq C(\varepsilon, b)\|f\|_{X_q(\Omega)}, \tag{4.22a}$$

$$\left\|\left(\frac{d}{d\lambda}\right)^k R_2(\lambda)f\right\|_{X_q(\Omega_b)} + \left\|\left(\frac{d}{d\lambda}\right)^k PR_2(\lambda)\right\|_{2,q,\Omega_b} \leq C(\varepsilon, b)|\lambda|^{1/2-k}\|f\|_{X_q(\Omega)}, \quad k \geq 1, \tag{4.22b}$$

for $f \in Y_{q,b}(\Omega)$ and $\lambda \in D_\varepsilon$. Put

$$\tilde{R}(\lambda) = R_2(\lambda)(I + S_2(\lambda))^{-1},$$

and then by (4.15) we see that $\tilde{R}(\lambda)f \in \mathscr{D}(A)$ and

$$(\lambda + A)\tilde{R}(\lambda)f = f \text{ in } \Omega \tag{4.23}$$

for any $\lambda \in D_\varepsilon$ and $f \in Y_{q,b}(\Omega)$. In particular, when $f \in Y_{q,b}(\Omega)$, by (4.23) and Lemma 3.1 we have $\tilde{R}(\lambda)f = (\lambda + A)^{-1}f$ for $\lambda \in D_\varepsilon$ and $f \in Y_{q,b}(\Omega)$. Combining (4.21), (4.22) we have (4.1) and (4.2), which completes the proof of Theorem 4.1.

§5. Proofs of Theorem B and Corollary C

In this section, we shall prove Theorem B and Corollary. C. To do this we prepare the following lemma, which was proved by Shibata. (see Theorem 3.2 and 3.7 of [18])

LEMMA 5.1. *Let X be a Banach space with norm $|\cdot|_X$. Let $f(\tau)$ be a function of $C^\infty(R-\{0\};X)$ such that $f(\tau)=0$, $|\tau|\geq a$ with some $a>0$. Assume that there exists a constant $C(f)$ depending on f such that for any $0<|\tau|\leq a$,*

$$\left|\left(\frac{d}{d\tau}\right)^k f(\tau)\right|_X \leq C(f)|\tau|^{-1/2-k}, k=0,1.$$

Put $g(t)=\int_{-\infty}^{\infty} f(\tau)e^{-it\tau}\,d\tau$. Then

$$|g(t)|_X \leq C(1+t)^{-1/2}C(f).$$

Now we shall prove Theorem B. In view of the facts that when $0<t\leq 1$ by Theorem A we have

$$\begin{aligned}&\|\partial_t^M e^{-tA}\boldsymbol{u}\|_{X_q(\Omega)}+\|P\partial_t^M e^{-tA}\boldsymbol{u}\|_{2,q,\Omega}\\ &\leq C\|(1+A)^{M+N}e^{-tA}\boldsymbol{u}\|_{X_q(\Omega)}\leq Ct^{-N-M}\|\boldsymbol{u}\|_{X_q(\Omega)}\end{aligned}$$

for any $\boldsymbol{u}\in X_q(\Omega)$ and any integers $N\geq 1, M\geq 0$, we have only to show the case $t\geq 1$. Note that by Corollary 7.5 of [16, Chapter 1] we can write

$$\begin{aligned}e^{-tA}\boldsymbol{u}&=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{t\lambda}(\lambda+A)^{-1}\boldsymbol{u}\,d\lambda\\ &=-\frac{1}{2\pi i t}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{t\lambda}\frac{d}{d\lambda}(\lambda+A)^{-1}\boldsymbol{u}\,d\lambda\end{aligned}\tag{5.1}$$

for all $\boldsymbol{u}\in\mathscr{D}(A^2)$, because

$$\left\|\frac{d}{d\lambda}(\lambda+A)^{-1}\boldsymbol{u}\right\|_{X_q(\Omega)}\leq\frac{C(\varepsilon)}{1+|\lambda|^2}\|\boldsymbol{u}\|_{X_q(\Omega)}\quad\text{for any }\operatorname{Re}\lambda\geq\varepsilon>0\tag{5.2}$$

by Theorem A. Since $\mathscr{D}(A^2)$ is dense in $X_q(\Omega)$, the equation (5.1) holds in $X_q(\Omega)$.

Let $\boldsymbol{u}\in Y_{q,b}(\Omega), b>b_0$ and let $\psi\in C_0^\infty(R^3)$ such that $\psi(x)=1$ for $|x|\leq b$ and $=0$ for $|x|\geq b+1$. Since we can move the path in the following integral to the imaginary axis by Theorem 4.1, (5.1) and (5.2), we have

$$\partial_X^\alpha \psi e^{-tA}\boldsymbol{u} = \frac{-1}{2\pi i t} D_X^\alpha \left\{ \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{t\lambda} \psi \frac{d}{d\lambda} (\lambda + A)^{-1} \boldsymbol{u}\, d\lambda \right\}$$
$$= \frac{-1}{2\pi t} D_X^\alpha \left\{ \int_{-\infty}^{\infty} e^{its} \psi \frac{d}{ds} (is + A)^{-1} \boldsymbol{u}\, ds \right\}$$

for any $\boldsymbol{u} \in Y_{q,b}(\Omega)$ and multi-index α_i $(i = 1, 2, 3) : |\alpha_1| \le 1, |\alpha_i| \le 2$ $(i = 2, 3)$ where $D_X^\alpha = {}^T\{(\partial_X)^{\alpha_1}, (\partial_X)^{\alpha_2}, (\partial_X)^{\alpha_3}\}$. Taking $\eta(s) \in C^\infty(R)$ so that $\eta(s) = 1$ for $|s| \le 1/4$ and $= 0$ for $|s| \ge 1/2$ we have

$$D_X^\alpha \psi e^{-tA}\boldsymbol{u} = \boldsymbol{J}_0(t)\boldsymbol{u} + \boldsymbol{J}_\infty(t)\boldsymbol{u} \tag{5.3}$$

where

$$\boldsymbol{J}_0(t)\boldsymbol{u} = \frac{-1}{2\pi t} D_X^\alpha \Big(\psi \int_{-\infty}^{\infty} e^{its} \eta(s) \frac{d}{ds} (is + A)^{-1} \boldsymbol{u}\, ds\Big),$$
$$\boldsymbol{J}_\infty(t)\boldsymbol{u} = \frac{-1}{2\pi t} D_X^\alpha \Big(\psi \int_{-\infty}^{\infty} e^{its} (1 - \eta(s)) \frac{d}{ds} (is + A)^{-1} \boldsymbol{u}\, ds\Big).$$

By Theorem A we have

$$\begin{aligned} &\|D_X^\alpha (1 - \eta(s)) \left(\frac{d}{ds}\right)^N (is + A)^{-1} \boldsymbol{u}\|_{q,\Omega} \\ &\le (1 - \eta(s))\{\|(is + A)^{-N-1}\boldsymbol{u}\|_{X_q(\Omega)} + \|\boldsymbol{P}(is + A)^{-N-1}\boldsymbol{u}\|_{2,q,\Omega}\} \\ &\le C(N)(1 + |s|)^{-N} \|\boldsymbol{u}\|_{X_q(\Omega)}, \end{aligned} \tag{5.4}$$

and hence by the relation $(1/t) \cdot (d/d\lambda) e^{t\lambda} = e^{t\lambda}$, we have

$$\|\partial_t^M \boldsymbol{J}_\infty(t)\boldsymbol{u}\|_{q,\Omega} \le C(N, M, \alpha) t^{-N} \|\boldsymbol{u}\|_{X_q(\Omega)} \tag{5.5}$$

for any integers $N \ge 2, M \ge 0$. On the other hand, noting that

$$\partial_t^M \boldsymbol{J}_0(t)\boldsymbol{f} = \frac{-1}{2\pi} \sum_{n=0}^{M} \binom{M}{n} \partial_t^{M-n} t^{-1} D_X^\alpha \left\{ \psi \int_{-\infty}^{\infty} e^{ist} \eta(s) (is)^n \frac{d}{ds} \tilde{\boldsymbol{R}}(is) \boldsymbol{f}\, ds \right\}$$
$$= -t^{-(M+1)} \sum_{n=0}^{M} c(n) D_X^\alpha \left\{ \psi \int_{-\infty}^{\infty} e^{ist} \left(\frac{d}{ds}\right)^n \{\eta(s)(is)^n \frac{d}{ds} \tilde{\boldsymbol{R}}(is)\boldsymbol{f}\}\, ds \right\},$$

it follows from Theorem 4.1 and Lemma 5.1 that

$$\|\partial_t^M \boldsymbol{J}_0(t)\boldsymbol{u}\|_{q,\Omega} \le C(M, b, q)(1 + t)^{-(M+3/2)} \|\boldsymbol{u}\|_{X_q(\Omega)} \tag{5.6}$$

for any $u \in Y_{q,b}(\Omega)$, integer $M \geq 0$ and $t \geq 1$. Combining (5.3), (5.5) and (5.6) we have for any $u \in Y_{q,b}(\Omega)$, integer $M \geq 0$ and $t \geq 1$

$$\|\partial_t^M e^{-tA} u\|_{Y_{q,b}(\Omega)} + \|\partial_t^M P e^{-tA} u\|_{2,q,\Omega_b} \leq C(1+t)^{-3/2-M} \|u\|_{Y_{q,b}(\Omega)}. \tag{5.7}$$

This completes the proof of Theorem B.

Next we shall prove Corollary C. Let $u \in X_{q,b}(\Omega)$. Taking $\phi \in C_0^\infty(\Omega_b)$, such that $\int_{\Omega_b} \phi(x)\,dx = 1$, in view of Remark 1.9, we have

$$(\lambda + A)^{-1} u = (\lambda + A)^{-1} N_1 u + \frac{\gamma}{\lambda}(\lambda + A)^{-1} N_2 u + \frac{1}{\lambda} N_3 u \quad \text{for } u \in X_{q,b}(\Omega)$$

where $N_j = N_j(\phi, \Omega_b)$ $(j = 1, 2, 3)$ be the same symbol as in (1.21). Combining this and (5.1), we have

$$\begin{aligned} e^{-tA} u = {} & \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda} (\lambda + A)^{-1} N_1 u \, d\lambda \\ & + \frac{\gamma}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda} (\lambda + A)^{-1} N_2 u \frac{d\lambda}{\lambda} \\ & + \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{1}{\lambda} e^{t\lambda} N_3 u \, d\lambda. \end{aligned} \tag{5.8}$$

Putting $T_1(b, \phi, t) u = e^{-tA} N_1 u$ and $T_2(b, \phi, t) u = \gamma \int_0^t e^{-sA} N_2 u \, ds + N_3 u$, since $\frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta - i\infty} \frac{1}{\lambda} e^{t\lambda} u \, d\lambda = u$ for any $u \in X_q(\Omega)$, and since by Theorem 7.4 of [16, Chapter 1] we have

$$\int_0^t e^{-sA} u \, ds = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda} (\lambda + A)^{-1} u \frac{d\lambda}{\lambda} \quad \text{for } u \in \mathscr{D}(A) \text{ and } t > 0,$$

it follows from (5.1) and (5.8) that the relation (0.8) holds. Moreover, nothing that $N_1 u, N_2 u \in Y_{q,b}(\Omega)$, since by (5.1) and (5.8) we have

$$\begin{aligned} \partial_t e^{-tA} u = {} & \partial_t \left\{ \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda} (\lambda + A)^{-1} N_1 u \, d\lambda \right\} \\ & + \frac{-\gamma}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda} (\lambda + A)^{-1} N_2 u \, d\lambda, \end{aligned} \tag{5.9}$$

it follows from (5.7), (5.8) and (5.9) that the estimates (0.9) and (0.10) hold. This completes the proof of Corollary C.

APPENDIX 1. Let n be an integer ≥ 0 and let

$$F(\lambda;r)=\lambda^3+(\alpha+\beta+\kappa)r^2\lambda^2+\{(\alpha+\beta)\kappa r^2+\gamma^2+\omega^2\}r^2\lambda+\gamma^2\kappa r^4.$$

Then

$$\text{(App1)}\quad \left(\frac{d}{d\lambda}\right)^n F(\lambda;r)^{-1}=\sum_{0\leq\ell\leq[(n-\ell)/2]}\ \sum_{k=0}^{[(n-\ell)/2]-\ell} C(k,\ell,n)\{F(\lambda;r)^{-n-1+2\ell+k} \times\left\{\frac{d}{d\lambda}F(\lambda;r)\right\}^{n-3\ell-2k}\left\{\left(\frac{d}{d\lambda}\right)^2F(\lambda;r)\right\}^k.$$

Moreover, set $G(\lambda;r)=(\lambda+\alpha r^2)F(\lambda;r)$, then

$$\text{(App2)}\quad \left(\frac{d}{d\lambda}\right)^n G(\lambda;r)^{-1}$$
$$=\sum_{m=0}^{n}\ \sum_{0\leq\ell\leq[(m-\ell)/2]}\ \sum_{k=0}^{[(m-\ell)/2]-\ell} C(m,k,\ell,n)\Bigg\{G(\lambda;r)^{-n-1}F(\lambda;r)^{n-m+2\ell+k}$$
$$\times\left\{\frac{d}{d\lambda}F(\lambda;r)\right\}^{m-3\ell-2k}\left\{\left(\frac{d}{d\lambda}\right)^2F(\lambda;r)\right\}^k\Bigg\}(\lambda+\alpha r^2)^m.$$

PROOF. Since it directly follows from (App1) and Leibniz rule that (App2) holds, our task is to show (App1). Now we shall show (App1) by induction on n. When $n=0$, obviously (App1) holds. Assume that $n\geq 1$ and that (App1) and that (App1) is valid for smaller values of n. Noting that $(d/d\lambda)^3F(\lambda;r)=6$, we have

(App3)

$$\frac{d}{d\lambda}I_n(\lambda;r)$$
$$=\frac{d}{d\lambda}\left\{\sum_{k=0}^{[(n-\ell)/2]-\ell} C(k,\ell,n)F(\lambda;r)^{-n-1+2\ell+k}\left\{\frac{d}{d\lambda}F(\lambda;r)\right\}^{n-3\ell-2k}\left\{\left(\frac{d}{d\lambda}\right)^2F(\lambda;r)\right\}^k\right\}$$
$$=\sum_{k=0}^{[(n-\ell)/2]-\ell} C(k,\ell,n)F(\lambda;r)^{-n-2+2\ell+k}\left\{\frac{d}{d\lambda}F(\lambda;r)\right\}^{n+1-3\ell-2k}\left\{\left(\frac{d}{d\lambda}\right)^2F(\lambda;r)\right\}^k$$
$$+\sum_{k=1}^{[(n-\ell)/2]-\ell+1} C(k,\ell,n)F(\lambda;r)^{-n-2+2\ell+k}\left\{\frac{d}{d\lambda}F(\lambda;r)\right\}^{n+1-3\ell-2k}\left\{\left(\frac{d}{d\lambda}\right)^2F(\lambda;r)\right\}^k$$
$$+\sum_{k=0}^{[(n-\ell)/2]-\ell-1} C(k,\ell,n)F(\lambda;r)^{-n+2\ell+k}\left\{\frac{d}{d\lambda}F(\lambda;r)\right\}^{n-2-3\ell-2k}\left\{\left(\frac{d}{d\lambda}\right)^2F(\lambda;r)\right\}^k$$
$$=I_{n,1}(\lambda;r)+I_{n,2}(\lambda;r)+I_{n,3}(\lambda;r).$$

Since $[(n-\ell)/2]-\ell=[(n+1-\ell)/2]-\ell$ when both n and ℓ are even or odd, and since $[(n-\ell)/2]-\ell=[(n+1-\ell)/2]-\ell-1$ when n (resp. ℓ) is even and ℓ (resp. n) is odd, we have

$$I_{n,1}(\lambda;r)=I_{n+1}(\lambda;r). \tag{App4}$$

Also since $n-3\ell-2([(n-\ell)/2]-\ell)=0$ when both n and ℓ are even or odd, and since $n-3\ell-2([(n-\ell)/2]-\ell)=1$ when n (resp. ℓ) is even and ℓ (resp. n) is odd, we have

$$I_{n,2}(\lambda;r)=I_{n+1}(\lambda;r). \tag{App5}$$

Note that $0\le\ell\le m$ if $0\le\ell\le[(n-\ell)/2]$ and $n=3m+k$ $(k=0,1,2)$. When $n=3m$, $3m+1$, since $0\le\ell\le m$ if $0\le\ell\le[(n+1-\ell)/2]$, it follows from (App3), (App4), (App5) and the induction assumption that

$$\begin{aligned}\left(\frac{d}{d\lambda}\right)^{n+1}F(\lambda;r)^{-1}&=\sum_{\ell=0}^{m}\frac{d}{d\lambda}I_n(\lambda;r)\\&=\sum_{\ell=0}^{m}I_{n+1}(\lambda;r)+\sum_{\ell=0}^{m-1}I_{n,3}(\lambda;r)\\&=\sum_{\ell=0}^{m}I_{n+1}(\lambda;r)\\&=\sum_{0\le\ell\le[(n+1-\ell)/2]}I_{n+1}(\lambda;r).\end{aligned}$$

Similarly, when $n=3m+2$, since $0\le\ell\le m+1$ if $0\le\ell\le[(n+1-\ell)/2]$, and since $[(n-\ell)/2]-\ell-1=0$ if $\ell=m$, it follows from (App3), (App4), (App5) and the induction assumption that

$$\begin{aligned}\left(\frac{d}{d\lambda}\right)^{n+1}F(\lambda;r)^{-1}&=\sum_{\ell=0}^{m}\frac{d}{d\lambda}I_n(\lambda;r)\\&=\sum_{\ell=0}^{m}I_{n+1}(\lambda;r)+\sum_{\ell=0}^{m}I_{n,3}(\lambda;r)\\&=\sum_{\ell=0}^{m}I_{n+1}(\lambda;r)+\sum_{\ell=1}^{m+1}I_{n+1}(\lambda;r)\\&=\sum_{0\le\ell\le[(n+1-\ell)/2]}I_{n+1}(\lambda;r).\end{aligned}$$

This completes the proof.

Acknowledgement

The author is very grateful to Professor Tosinobu Muramatu and Professor Yoshihiro Shibata for valuable advice and discussions. The author is also grateful to the referee for pointing out Ströhmer's paper and many valuable comments and helpful suggestions.

References

[1] Agmon, S., Douglis, A. and Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math., **12** (1959), 623–727. II. ibib, **17** (1964) 35–92.

[2] Cattabriga, L., Su un problema al contorno relativo al sistema di equazioni di Stokes, Rend. Mat. Sem. Univ. Padova **31** (1961), 308–340.

[3] Deckelnick, K., Decay estimates for the compressible Navier-Stokes equations in unbounded domains, Math. Z. **209** (1992), 115–130.

[4] Deckelnick, K., L^2-decay for the compressible Navier-Stokes equations in unbounded domains, Commun. in Partial Differential Equations, **18** (9&10) (1993), 1445–1476.

[5] Edmunds, D. E. and Evans W.D., Spectral Theory and Differential Operators. Oxford: Oxford University Press, 1987.

[6] Hörmander, L., The analysis of linear partial differential operators I, Grund. math. Wiss. 256, Springer-Verlag, Berlin at al, 1983.

[7] Itaya, N., On the Caucy problem for the system of fundamental equations describing the movement of compressible viscous fluid, Kōdai Math. Sem. Rep., **23** (1971), 60–120.

[8] Itaya, N., On the initial value problem of the motion of compressible viscous fluid, especially on the problem of uniqueness, J. Math., Kyoto Univ., **16** (1976) 413–427.

[9] Iwashita, H., L_q–L_r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces, Math. Ann. **285** (1989), 265–288.

[10] Iwashita, H. and Shibata, Y., On the analyticity of spectral functions for some exterior boundary value problem, Glasnik Matematicki **23** (1988), 291–313.

[11] Kobayashi, T. and Shibata, Y., On the Oseen equation in the three dimensional exterior domains, Math. Annal. **309** (1997) (to appear).

[12] Matsumura, A. and Nishida, T., The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Jpn. Acad. Ser. A, **55** (1979), 337–342.

[13] Matsumura, A. and Nishida, T., The initial value problems for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. **20–1** (1980), 67–104.

[14] Matsumura, A. and Nishida, T., Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, Commun. Math. Phys. **89** (1983), 445–464.

[15] Nash, J., Le problème de Cauchy pour les équations différentielles d'un fluid général, Bull. Soc. Math. France, **90** (1962), 487–497.

[16] Pazy, A., Semigroups of linear operators and applications to partial differential equations, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

[17] Ponce, G., Global existence of small solutions to a class of nonlinear evolution equations, Nonlinear Anal. TMA **9** (1985), 399–418.

[18] Shibata, Y., On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain, Tsukuba J. Math. **7** (1983), 1–68.

[19] Shibata, Y., On an exterior initial boundary value problem for Navier-stokes equations, preprint.
[20] Ströhmer, G., About the resolvent of an operator from fluid dynamics, Math. Z., **194** (1987), 183–191.
[21] Ströhmer, G., About compressible viscous fluid flow in a bounded region, Pacific J. Math., **143** (1990), 359–375.
[22] Tani, A., On the first initial-boundary problem of compressible viscous fluid motion, Publ. RIMS, Kyoto Univ., **13** (1977), 193–253.

Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki 305