

ON THE STOKES MATRIX FOR A FAMILY OF INFINITELY DEGENERATE OPERATORS OF SECOND ORDER

By

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0. Introduction

In [17] hypoellipticity and local non-solvability were studied for a special family of infinitely degenerate operators of second order. The family of operators was chosen such that for $\varphi \in (-\pi/2, \pi/2)$ the operators P_φ are degenerate elliptic besides for $\varphi = \pm \frac{\pi}{2}$. In these cases the operators are weakly hyperbolic. More precisely, the family of operators $\{P_\varphi\}_{\varphi \in [0, \pi/2)}$,

$$(0.1) \quad P_\varphi = D_t^2 + e^{2i\varphi} t^{-4} \exp(-2|t|^{-1}) D_x^2 - b(t) e^{i\varphi} t^{-4} \exp(-|t|^{-1}) D_x$$

was considered, while

$$(0.2) \quad b(t) = \begin{cases} b_- \in \mathbf{C} & \text{when } t < 0, \\ b_+ \in \mathbf{C} & \text{when } t \geq 0, \end{cases}$$

is a piecewise constant function. In [10] hypoellipticity and non-hypoellipticity for P_0 were studied in the cases $b_+ = b_-$ and $b_+ = -b_-$. In both papers [10], [17] a connection between hypoellipticity for degenerate elliptic and effect of branching of singularities for weakly hyperbolic operator was mentioned. The full classification for hypoellipticity in [17] and for branching properties for the corresponding weakly hyperbolic operator in [1] makes it possible to conclude an interesting connection.

“The weakly hyperbolic equation $P_{\pi/2}u = 0$ has a solution whose wave front set coincides with a simple ray passes through the origin $(0, 0)$ and is completely reflected by it if and only if the operator P_0 is not hypoelliptic at the origin.”

This observation seems to be new and very interesting in the theory of degenerate partial differential operators. Is this observation a chance for the special family of operators (0.1), (0.2)? May be not, because one can find the same connection for a special family of finitely degenerate operators. It follows

from results of [7], [14], [22]. In the present paper we deal with the family of infinitely degenerate operators $\{L_\varphi\}_{\varphi \in (-\pi/2, \pi/2)}$,

$$(0.3) \quad L_\varphi := D_t^2 u + \lambda^2(t) e^{2i\varphi} D_x^2 u - b(t) e^{i\varphi} D_x u,$$

where the coefficients $\lambda(t)$, $b(t)$ satisfy the assumptions which are cited in section 1.

Hypoellipticity and local solvability for L_0 were investigated in several papers. Firstly, let us mention the case that $\lambda = \lambda(t)$ has a zero of finite order in $t = 0$. In [8] hypoellipticity was proved if $b(t) = 0$. Some examples of operators L_0 with term of lower order were studied in [7], [14]. Hypoellipticity for general operator L_0 was obtained in [26], [16].

Secondly, we cite some papers for the case that $\lambda = \lambda(t)$ has a zero of infinite order (infinite degeneracy). The result from [8] was generalized in [5]. The papers [9], [21] are devoted even to the case of operator L_0 with infinite degeneracy including term of lower order. The Levi conditions were used there are more restrictive than the ones are formulated later (see (1.5)).

May be that one cannot expect this interesting connection in the general case $\{L_\varphi\}$ of (0.3) because it implies connection between elliptic and hyperbolic theory. But for $\{P_\varphi\}_{\varphi \in [0, \pi/2)}$ some other properties could be proved in [17]:

- (A1) Assume that neither $b_+ = -2l - 1, b_- = -2n - 1$ nor $b_+ = 2l + 1, b_- = 2n + 1$, where n and l are non-negative integers. Then every operator $P_\varphi, \varphi \in [0, \pi/2)$, is hypoelliptic as well as locally solvable at $(0, 0)$.
- (A2) Assume that either $b_+ = -2l - 1, b_- = -2n - 1$ or $b_+ = 2l + 1, b_- = 2n + 1$, where n and l are non-negative integers. Then every operator $P_\varphi, \varphi \in [0, \pi/2)$, is non-hypoelliptic as well as locally non-solvable at $(0, 0)$.

The goal of the present paper is to derive corresponding results for the family $\{L_\varphi\}_{\varphi \in (-\pi/2, \pi/2)}$.

An important tool for describing the properties of $\{L_\varphi\}_{\varphi \in (-\pi/2, \pi/2)}$ is the so-called Stokes matrix $(T_{ij}(b, \varphi, \xi))_{i, j=1}^2$ (see section 1). In [2], [19], [20] this Stokes matrix is used for precise description of branching of singularities for weakly hyperbolic operators with finite degeneracy.

Among other results will be proved in the present paper we mention only one which follows from Theorem 4.7.

THEOREM 0.1. *If one of the Stokes coefficients $T_{11}(\pm b, \varphi_0, \xi), \varphi_0 \in (-\pi/2, \pi/2)$, vanishes for all sufficiently large positive ξ , then all operators $L_\varphi, \varphi \in (-\pi/2, \pi/2)$, are non-hypoelliptic at $(0, 0)$.*

1. Assumptions, tools and philosophy of approach

At first we want to describe the class of operators will be studied in this paper. With an arbitrary $\varphi \in (-\pi/2, \pi/2)$ let us consider (0.3). The real-valued function $\lambda \in C^\infty([-1, 1])$ vanishes at $t = 0$ ($\lambda(0) = \lambda'(0) = 0$), while $|\lambda'(t)| > 0$ for $t \neq 0$. In the following λ' means $d\lambda/dt$. Moreover, defining $\Lambda(t) = \int_0^t \lambda(r) dr$ we assume

$$(1.1) \quad \lambda^2 \Lambda^{-1} \in C^\infty([-1, 1]),$$

$$(1.2) \quad C_0 |\lambda(t)/\Lambda(t)| \leq |\lambda'(t)/\lambda(t)| \leq C_1 |\lambda(t)/\Lambda(t)|,$$

$$(1.3) \quad |\lambda^{(k)}(t)| \leq c_k |\lambda'(t)/\lambda(t)|^{k-1} |\lambda'(t)|,$$

$$(1.4) \quad C_2 |\ln \lambda(t)| \leq |\lambda'(t)/\lambda(t)| \leq C_3 |\ln \lambda(t)|^d$$

for all $t \in [-1, 1] \setminus \{0\}$, $k = 1, 2, \dots$, where d, c_k , and C_i are non-negative constants, $C_0 > 1/2$. It is easy to see that the condition (1.4) implies degeneracy of infinite order. Moreover, (1.4) implies additionally

$$\lambda(t) \leq \exp\{-\varepsilon_0 |t|^{-\varepsilon_1}\} \quad \text{for all } t \in [-1, 1] \setminus \{0\}$$

with some positive constants ε_0 and ε_1 .

The condition $|\lambda'(t)| > 0$ excludes difficulties which arise by oscillations of the principal symbol near $t = 0$. Indeed, the result of [4] shows that in the case of weakly hyperbolic operator without terms of lower order and which has infinitely number of oscillations near $t = 0$ one cannot expect results of non-degenerate theory.

If the equation $L_\varphi u = 0$ has lower order terms, then it is reasonable to suppose Levi conditions. For L_φ the Levi conditions are the following [24]:

$$(1.5) \quad \left| \left(\frac{d}{dt} \right)^k b(t) \right| \leq C_k \frac{\lambda^2(t)}{|\Lambda(t)|} \left(\frac{\lambda'(t)}{\lambda(t)} \right)^k$$

for all $t \in [-1, 1] \setminus \{0\}$, $k = 0, 1, \dots$, where C_k are positive constants.

To study $L_\varphi u = f$ we transform L_φ to an ordinary differential equation with a turning point of infinite order $t = 0$. Following the approach of [25] two linear independently solutions u_1 and u_2 of

$$(1.6) \quad (d/dt)^2 u - \lambda^2(t) e^{2i\varphi} \xi^2 u + b(t) e^{i\varphi} \xi u = 0$$

of the form

$$u_1(t, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t)} a_1(t, \varphi, \xi), \quad u_2(t, \varphi, \xi) = e^{-\xi e^{i\varphi} \Lambda(t)} a_2(t, \varphi, \xi)$$

are constructed in section 2, where a_1 and a_2 are amplitude functions having relations to suitable symbol classes (see Theorem 2.1).

We restrict ourselves to the case $\xi > 0$. The case $\xi < 0$ can be transformed to the first one. One has only to study

$$(d/dt)^2 u - \lambda^2(t)e^{2i\varphi}(-\xi)^2 u - b(t)e^{i\varphi}(-\xi)u = 0.$$

For the construction of these two solutions u_1, u_2 we use the subsets $Z_{ext}(M, N)$ and $Z_{int}(M, N)$ of $\{(t, \xi) \in [0, 1] \times \overline{\mathbf{R}_M^+}\}$. Here $\overline{\mathbf{R}_M^+} := \{\xi \in \mathbf{R} \mid \xi \geq M > 0\}$. Firstly, we consider the exterior zone $Z_{ext}(M, N)$ defined for some positive numbers M and N as

$$Z_{ext}(M, N) = \{(t, \xi) \in [0, 1] \times \overline{\mathbf{R}_M^+} \mid \langle \xi \rangle \geq M, \Lambda(t)\langle \xi \rangle \geq N \ln \langle \xi \rangle\}.$$

Let $t_\xi > 0$ be the root of

$$(1.7) \quad \Lambda(t)\langle \xi \rangle = N \ln \langle \xi \rangle.$$

We carry out these constructions for all $(\varphi, \xi) \in Z_e(M, N, \delta)$ is defined as

$$(1.8) \quad Z_e(M, N, \delta) = \{(\varphi, \xi) \in (-\pi/2, \pi/2) \times \overline{\mathbf{R}_M^+} \mid \ln \langle \xi \rangle \cos \varphi \geq \delta\}$$

with suitable positive constants M, N, δ . This is enough for the consideration of the degenerate elliptic operator L_φ in section 4.

A main tool to describe the construction of a_1 are the symbol classes

$$\begin{aligned} S\{m_1, m_2\}_{M,N} &= \{a(t, \varphi, \xi) \in C^\infty(Z_{e,ext}(M, N, \delta)) \mid |D_t^l D_\xi^p a(t, \varphi, \xi)| \\ &\leq C_{l,p} \langle \xi \rangle^{m_1-p} (\lambda(t)/\Lambda(t))^l (\ln \langle \xi \rangle)^{m_2}\}. \end{aligned}$$

Here

$$\begin{aligned} Z_{e,ext}(M, N, \delta) &= \{(t, \varphi, \xi) \in [0, 1] \times (-\pi/2, \pi/2) \times \overline{\mathbf{R}_M^+} \mid \\ &(t, \xi) \in Z_{ext}(M, N) \text{ and } (\varphi, \xi) \in Z_e(M, N, \delta)\}. \end{aligned}$$

The amplitude function a_1 has a relation to $S\{m_1, m_2\}_{M,N}$.

To construct a_2 we shall use the symbol classes (compare with [23])

$$\begin{aligned} S\{m_1, m_2, m_3\}_{M,N} &= \{a(t, \varphi, \xi) \in C^\infty(Z_{e,ext}(M, N, \delta)) \mid |D_t^l D_\xi^p a(t, \varphi, \xi)| \\ &\leq C_{l,p} \langle \xi \rangle^{m_1-p} \lambda(t)^{m_2} (\lambda(t)/\Lambda(t))^{m_3+l}\}. \end{aligned}$$

The amplitude function a_2 has a relation to $S\{m_1, m_2, m_3\}_{M,N}$. For better understanding of this symbol class we note that $a \in S\{-l, -l, l\}_{M,N}$ satisfies

$$|a(t, \varphi, \xi)| \leq C_{0,0}(\langle \xi \rangle \Lambda(t))^{-l} \leq C_{0,0}(N \ln \langle \xi \rangle)^{-l}.$$

For further calculations we need

LEMMA 1.1. *The function t_ξ is as a smooth function of ξ defined on $\overline{\mathbf{R}_M^+}$. For its derivatives one has*

$$(1.9) \quad \frac{\partial t_\xi}{\partial \xi} = \frac{N(1 - \ln \xi)}{\xi^2 \lambda(t_\xi)},$$

while for every integer $k, k \geq 1$, the following estimate holds:

$$(1.10) \quad \left| \frac{\partial^{(k+1)} t_\xi}{\partial \xi^{(k+1)}} \right| \leq C_k \xi^{-k} \left| \frac{\partial t_\xi}{\partial \xi} \right|.$$

PROOF. The first formula is obvious, while the estimates for derivatives can be proved by induction.

Taking account of $(t_\xi, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$ we have data $u_i(t_\xi, \varphi, \xi)$ and $\partial_t u_i(t_\xi, \varphi, \xi)$ on $t = t_\xi$. One can show that these solutions u_1, u_2 can be continued with respect to t to the interval $[0, t_\xi]$. Consequently, $(t, \varphi, \xi) \in Z_{e,int}(M, N, \delta)$. Here

$$Z_{e,int}(M, N, \delta) = \{(t, \varphi, \xi) \in [0, 1] \times (-\pi/2, \pi/2) \times \overline{\mathbf{R}_M^+} | \\ (t, \xi) \in Z_{int}(M, N) \text{ and } (\varphi, \xi) \in Z_e(M, N, \delta)\},$$

where, $Z_{int}(M, N)$ is defined as

$$Z_{int}(M, N) = \{(t, \xi) \in [0, 1] \times \overline{\mathbf{R}_M^+} | \langle \xi \rangle \geq M, \Lambda(t) \langle \xi \rangle \leq N \ln \langle \xi \rangle\}.$$

The amplitude functions are satisfying estimates from Theorem 2.1.

In section 4.1 we follow the approach of [14], [17]. The construction of Green's function is carried out. An estimate which leads to hypoellipticity will be given in Corollary 4.1. In [17] we were able to express conditions for hypoellipticity and non-hypoellipticity through the behaviour of $b = b(t)$ (see (0.2), (A1) and (A2) from introduction). This seems to be impossible for the general differential operator L_φ .

An effective tool for the description of sufficient conditions for hypoellipticity, local solvability, local non-solvability and non-hypoellipticity is the so-called Stokes matrix $(T_{ij}(b, \varphi, \xi))_{i,j=1}^2$. Let us define this matrix.

Suppose that we have constructed solutions u_1, u_2 of (1.6) with coefficient $b = b(t)$. Let us denote these by

$$(1.11) \quad u_1^+(t, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t)} a_1^+(t, \varphi, \xi), \quad u_2^+(t, \varphi, \xi) = e^{-\xi e^{i\varphi} \Lambda(t)} a_2^+(t, \varphi, \xi)$$

for $t \in [0, 1]$, and

$$(1.12) \quad u_1^-(t, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t)} a_1^-(t, \varphi, \xi), \quad u_2^-(t, \varphi, \xi) = e^{-\xi e^{i\varphi} \Lambda(t)} a_2^-(t, \varphi, \xi)$$

for $t \in [-1, 0]$. Then the solutions u_1^- and u_2^- can be continued as solutions of ordinary differential equations for $t \in [0, 1]$. If we know that u_1^+, u_2^+ are linear independent solutions for $t \in [0, 1]$, then it is clear that the continuations u_1^{-c}, u_2^{-c} of u_1^-, u_2^- respectively, are linear combinations of u_1^+, u_2^+ . Especially, it holds for $t = 0$ and $i = 1, 2$

$$(1.13) \quad u_i^-(0, \varphi, \xi) = T_{i1}(b, \varphi, \xi) u_1^+(+0, \varphi, \xi) + T_{i2}(b, \varphi, \xi) u_2^+(+0, \varphi, \xi),$$

$$(1.14) \quad \partial_t u_i^-(0, \varphi, \xi) = T_{i1}(b, \varphi, \xi) \partial_t u_1^+(+0, \varphi, \xi) + T_{i2}(b, \varphi, \xi) \partial_t u_2^+(+0, \varphi, \xi).$$

The elements $T_{ij}(b, \varphi, \xi)$ form the Stokes matrix. By the aid of some elements of Stokes matrix sufficient conditions for hypoellipticity, local solvability, local non-solvability and non-hypoellipticity will be given in Theorem 4.2, 4.4, 4.6 and Corollary 4.2, respectively. A uniqueness property of $T_{11}(b, \varphi, \xi)$ will be proved in section 4.2.

It seems to be interesting to formulate the conditions for hypoellipticity, local solvability, local non-solvability and non-hypoellipticity from [7], [14] by the aid of the Stokes matrix.

2. Equations with a turning point of infinite order depending on parameter

We consider the linear ordinary equation

$$(2.1) \quad (d/dt)^2 u - \lambda^2(t) e^{2i\varphi} \xi^2 u + b(t) e^{i\varphi} \xi u = 0$$

with parameters $(\varphi, \xi) \in Z_e(M, N, \delta)$. Here $t = 0$ is a turning point of infinite order [15], [25].

THEOREM 2.1. *Under the assumptions (1.1)–(1.5) there exists for all $\varphi \in (-\pi/2, \pi/2)$ solutions $u_1(t, \varphi, \xi), u_2(t, \varphi, \xi)$ having representations*

$$(2.2) \quad u_1(t, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t)} a_1(t, \varphi, \xi), \quad u_2(t, \varphi, \xi) = e^{-\xi e^{i\varphi} \Lambda(t)} a_2(t, \varphi, \xi).$$

The functions a_i are amplitude functions satisfying with suitable constants m_i

$$(2.3) \quad \left| \left(\frac{\partial}{\partial t} \right)^l \left(\frac{\partial}{\partial \xi} \right)^p a_i(t, \varphi, \xi) \right| \leq C_{l,p,i} \langle \xi \rangle^{m_i - p}$$

for all $t \in [0, 1]$ and $(\varphi, \xi) \in Z_e(M, N, \delta)$.

A proof of this result is deferred to the following subsections.

2.1. Construction of formal asymptotic solutions in $Z_{e,ext}(M, N, \delta)$

Firstly, we are looking for formal asymptotic solutions

$$(2.4) \quad y_1(t, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t)} a_1(t, \varphi, \xi), \quad y_2(t, \varphi, \xi) = e^{-\xi e^{i\varphi} \Lambda(t)} a_2(t, \varphi, \xi)$$

with amplitude functions $a_1(t, \varphi, \xi), a_2(t, \varphi, \xi)$ developing in finite expansions

$$(2.5) \quad a_1 = \sum_{l=0}^L a_{1,l}, \quad a_2 = \sum_{l=0}^L a_{2,l}.$$

We seek for $a_{i,j}$ as solutions of the following equations:

$$\begin{aligned} (a_{1,0})_t + K_1(t)a_{1,0} &= 0, & (a_{2,0})_t + K_2(t)a_{2,0} &= 0, \\ (a_{1,k+1})_t + K_1(t)a_{1,k+1} &= -\frac{1}{2\xi\lambda(t)e^{i\varphi}}(a_{1,k})_{tt}, \\ (a_{2,k+1})_t + K_2(t)a_{2,k+1} &= \frac{1}{2\xi\lambda(t)e^{i\varphi}}(a_{2,k})_{tt}, \end{aligned}$$

where we have denoted

$$K_1(t) = \frac{\lambda'(t) + b(t)}{2\lambda(t)}, \quad K_2(t) = \frac{\lambda'(t) - b(t)}{2\lambda(t)}.$$

Moreover we prescribe the data

$$a_{1,0}(t_\xi, \varphi, \xi) = a_{2,0}(T, \varphi, \xi) = 1, \quad a_{1,k+1}(t_\xi, \varphi, \xi) = a_{2,k+1}(T, \varphi, \xi) = 0.$$

Hence we conclude for $k = 0, 1, \dots$ the representations

$$\begin{aligned} a_{1,0}(t, \varphi, \xi) &= e^{-\int_{t_\xi}^t K_1(s) ds}, & a_{2,0}(t, \varphi, \xi) &= e^{\int_t^T K_2(s) ds}, \\ a_{1,k+1}(t, \varphi, \xi) &= -\int_{t_\xi}^t \frac{1}{2\xi\lambda(t_1)e^{i\varphi}} (a_{1,k}(t_1, \varphi, \xi))_{t_1 t_1} e^{-\int_{t_1}^t K_1(s) ds} dt_1, \\ a_{2,k+1}(t, \varphi, \xi) &= -\int_t^T \frac{1}{2\xi\lambda(t_1)e^{i\varphi}} (a_{2,k}(t_1, \varphi, \xi))_{t_1 t_1} e^{\int_{t_1}^t K_2(s) ds} dt_1. \end{aligned}$$

Moreover, the functions $b_{1,k}(t, \varphi, \xi)$ are defined by

$$(2.6) \quad b_{1,k}(t, \varphi, \xi) = a_{1,k}(t, \varphi, \xi) \exp\left(\int_{t_\xi}^t K_1(s) ds\right)$$

fulfil for $k = 0, 1, 2, \dots$

$$(2.7) \quad (b_{1,k+1}(t, \varphi, \xi))_t = -\frac{1}{2\xi e^{i\varphi} \lambda(t)} \{ (b_{1,k}(t, \varphi, \xi))_{tt} - 2K_1(t)(b_{1,k}(t, \varphi, \xi))_t - ((K_1(t))_t + K_1^2(t))b_{1,k}(t, \varphi, \xi) \},$$

$$(2.8) \quad b_{1,0}(t, \varphi, \xi) = 1, \quad b_{1,k+1}(t_\xi, \varphi, \xi) = 0.$$

Hence we have in $Z_{e,ext}(M, N, \delta)$ for $k = 1, 2, \dots$, the representations

$$(2.9) \quad b_{1,k}(t, \varphi, \xi) = -\frac{1}{2\xi e^{i\varphi}} \int_{t_\xi}^t \frac{1}{\lambda(s)} \{ (b_{1,k-1}(s, \varphi, \xi))_{ss} - 2K_1(s)(b_{1,k-1}(s, \varphi, \xi))_s - ((K_1(s))_s + K_1^2(s))b_{1,k-1}(s, \varphi, \xi) \} ds.$$

LEMMA 2.1. *The functions $b_{1,k}(t, \varphi, \xi)$ have the following properties:*

For every k , and any l, p there exist constants $C_{k,l}, C_{k,p}, C_{k,l,p}$ such that the inequalities

$$(2.10) \quad \left| \left(\frac{\partial}{\partial t} \right)^l b_{1,k}(t, \varphi, \xi) \right| \leq C_{k,l} \left(\frac{\lambda(t)}{\Lambda(t)} \right)^l \rho_k(t, \xi),$$

$$(2.11) \quad \left| \left(\frac{\partial}{\partial \xi} \right)^p b_{1,k}(t, \varphi, \xi) \right| \leq C_{k,p} \langle \xi \rangle^{-p} (\rho_k(t, \xi) + (\ln \langle \xi \rangle)^{-k}),$$

$$(2.12) \quad \left| \left(\frac{\partial}{\partial t} \right)^l \left(\frac{\partial}{\partial \xi} \right)^p b_{1,k}(t, \varphi, \xi) \right| \leq C_{k,l,p} \langle \xi \rangle^{-p} \left(\frac{\lambda(t)}{\Lambda(t)} \right)^l \rho_k(t, \xi), \quad l \neq 0,$$

hold for all $(t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$, where $\rho_0(t, \xi) = 1$ and

$$\rho_k(t, \xi) = \langle \xi \rangle^{-k} \left\{ \frac{1}{\Lambda^k(t)} + \sum_{i=0}^{k-1} \int_{t_\xi}^t \frac{\lambda(s)}{\Lambda^{k+1-i}(s)} \left(\int_{t_\xi}^s \frac{\lambda(\tau)}{\Lambda^2(\tau)} d\tau \right)^i ds \right\}, \quad \text{when } k = 1, 2, \dots$$

PROOF. This follows, immediately, by induction if we use (2.7) to (2.9).

COROLLARY 2.1. *There are positive constants M, N, C_0, δ_0 such that*

$$(2.13) \quad 1) \text{ the function } a_1(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) \text{ belongs to } S\{0, 0\}_{M,N},$$

$$2) |a_1(t, \varphi, \xi)| \geq C_0(1 + \xi)^{-\delta_0} \quad \text{for all } (t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta).$$

LEMMA 2.2. 1) Let P be a given positive integer number. Let us set $y_1(t, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t)} a_1(t, \varphi, \xi)$. Then for a sufficiently large L there exists an amplitude function $h_{1,L}(t, \varphi, \xi)$ satisfying

$$\left(\frac{d^2}{dt^2} - \lambda^2(t) e^{2i\varphi} \xi^2 + b(t) e^{i\varphi} \xi \right) y_1 = e^{\xi e^{i\varphi} \Lambda(t)} h_{1,L}(t, \varphi, \xi),$$

where $h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right)$ belongs to $S\{0, -P\}_{M,N}$.

2) Moreover, for every k and l and sufficiently large $L = L(k, l)$

$$\left| \left(\frac{\partial}{\partial t} \right)^l \left(h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) \right) \right| \leq C_{k,l} \rho_k(t, \xi) \quad \text{for all } (t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta).$$

PROOF. With the operator

$$Q \left(t, \xi, \frac{\partial}{\partial t} \right) = \frac{\partial}{\partial t} + K_1(t) + \frac{1}{2\xi e^{i\varphi} \lambda(t)} \frac{\partial^2}{\partial t^2}$$

we have

$$\begin{aligned} & \left(\frac{d^2}{dt^2} - \lambda^2(t) e^{2i\varphi} \xi^2 + b(t) e^{i\varphi} \xi \right) y_1 \\ &= 2\xi e^{i\varphi} \lambda(t) e^{\xi e^{i\varphi} \Lambda(t)} Q \left(t, \xi, \frac{\partial}{\partial t} \right) a_1(t, \varphi, \xi) \\ &= 2\xi e^{i\varphi} \lambda(t) e^{\xi e^{i\varphi} \Lambda(t)} \frac{1}{2\xi e^{i\varphi} \lambda(t)} (a_{1,L}(t, \varphi, \xi))_{ii} \end{aligned}$$

if we use $a_1(t, \varphi, \xi) = \sum_{l=0}^L a_{1,l}(t, \varphi, \xi)$. Thus

$$h_{1,L}(t, \varphi, \xi) = (a_{1,L}(t, \varphi, \xi))_{ii} = \left(b_{1,L}(t, \varphi, \xi) \exp \left(- \int_{t_\xi}^t K_1(s) ds \right) \right)_{ii}.$$

On the other side

$$\begin{aligned} h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) &= \{ (b_{1,L}(t, \varphi, \xi))_{ii} - 2K_1(t) (b_{1,L}(t, \varphi, \xi))_i \\ &\quad - ((K_1(t))_i + K_1^2(t)) b_{1,L}(t, \varphi, \xi) \}. \end{aligned}$$

Using Lemma 2.1 we conclude with a constant $C = C_{l,L}$

$$\begin{aligned} & \left| \partial_t^l \left(h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) \right) \right| \\ & \leq C_{l,L} \left(\frac{\lambda(t)}{\Lambda(t)} \right)^{l+2} \rho_L(t, \xi) \leq C_{l,L} \langle \xi \rangle^{-L} \left(\frac{\lambda(t)}{\Lambda(t)} \right)^{l+2} \\ & \quad \times \left\{ \frac{1}{\Lambda^L(t)} + \sum_{i=0}^{L-1} \int_{t_\xi}^t \frac{\lambda(s)}{\Lambda^{L+1-i}(s)} \left(\int_{t_\xi}^s \frac{\lambda(\tau)}{\Lambda^2(\tau)} d\tau \right)^i ds \right\} \\ & \leq C_{l,L} \langle \xi \rangle^{-L} \left(\frac{\lambda(t)}{\Lambda(t)} \right)^{l+2} \left\{ \frac{1}{\Lambda^k(t) \Lambda^{L-k}(t_\xi)} \right. \\ & \quad + \frac{1}{\Lambda^{L-k}(t_\xi)} \sum_{i=0}^{k-1} \int_{t_\xi}^t \frac{\lambda(s)}{\Lambda^{k+1-i}(s)} \left(\int_{t_\xi}^s \frac{\lambda(\tau)}{\Lambda^2(\tau)} d\tau \right)^i \\ & \quad \left. + \sum_{i=k}^{L-1} \frac{1}{\Lambda^{L-1-i}(t_\xi)} \left(\int_{t_\xi}^s \frac{\lambda(\tau)}{\Lambda^2(\tau)} d\tau \right)^{i-k+1} \int_{t_\xi}^t \frac{\lambda(s)}{\Lambda^2(s)} \left(\int_{t_\xi}^s \frac{\lambda(\tau)}{\Lambda^2(\tau)} d\tau \right)^{k-1} ds \right\}. \end{aligned}$$

Using $\int_{t_\xi}^s (\lambda(\tau)/\Lambda^2(\tau)) d\tau \leq (1/\Lambda(t_\xi))$ it follows

$$\left| \partial_t^l \left(h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) \right) \right| \leq C_{k,l,L} (\langle \xi \rangle \Lambda(t_\xi))^{-L+k} \left(\frac{\lambda(t)}{\Lambda(t)} \right)^{l+2} \rho_k(t, \xi).$$

Finally, the condition (1.4) and definition of t_ξ from (1.7) imply

$$\left| \partial_t^l \left(h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) \right) \right| \leq C_{k,l,L} (\ln \langle \xi \rangle)^{(l+2)d-L+k} \rho_k(t, \xi).$$

Then a sufficiently large $L = L(k, l)$ gives the second statement for $(t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$.

To prove the first statement we have to show

$$\left| \partial_t^l \partial_\xi^p \left(h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) \right) \right| \leq C_{l,L,p} \langle \xi \rangle^{-p} \left(\frac{\lambda(t)}{\Lambda(t)} \right)^l (\ln \langle \xi \rangle)^{-p}.$$

Using (2.11), (2.12) from Lemma 2.1 we conclude for $l \geq 1$

$$\begin{aligned} & \left| \partial_t^l \partial_\xi^p \left(h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) \right) \right| \leq C_{l,L,p} \langle \xi \rangle^{-p} \left(\frac{\lambda(t)}{\Lambda(t)} \right)^{l+2} \rho_L(t, \xi), \\ & \left| \partial_\xi^p \left(h_{1,L}(t, \varphi, \xi) \exp \left(\int_{t_\xi}^t K_1(s) ds \right) \right) \right| \leq C_{L,p} \langle \xi \rangle^{-p} (\rho_L(t, \xi) + (\ln \langle \xi \rangle)^{-L}) \frac{\lambda^2(t)}{\Lambda^2(t)}. \end{aligned}$$

Together with condition (1.4) and $\rho_L(t, \xi) \leq (\ln\langle \xi \rangle)^{-L}$ we arrive at the first statement if L is large.

Starting from the construction of $a_{2,k}$ one can prove corresponding results, where we have to define

$$(2.14) \quad b_{2,k}(t, \varphi, \xi) = a_{2,k}(t, \varphi, \xi) e^{-\int_t^T K_2(s) ds}.$$

LEMMA 2.3. 1) *The function $b_{2,k}(t, \varphi, \xi)$ belongs to symbol class $S\{-k, -k, k\}_{M,N}$ for $k = 0, 1, \dots$*

2) *Setting*

$$(2.15) \quad a_2(t, \varphi, \xi) = \sum_{l=0}^L b_{2,l}(t, \varphi, \xi) e^{\int_t^T K_2(s) ds}$$

then there exist constants M, N, C_0, δ_0 such that

$$(2.16) \quad |a_2(t, \varphi, \xi)| \geq C_0(1 + \xi)^{-\delta_0} \quad \text{for all } (t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta).$$

(3) *There exists to a given L a symbol $h_{2,L}(t, \varphi, \xi)$ satisfying*

$$(2.17) \quad \left(\frac{d^2}{dt^2} - \lambda^2(t) e^{2i\varphi} \xi^2 + b(t) e^{i\varphi} \xi \right) y_2 = e^{-\xi e^{i\varphi} \Lambda(t)} h_{2,L}(t, \varphi, \xi),$$

where $h_{2,L}(t, \varphi, \xi) \exp\left(-\int_t^T K_2(s) ds\right)$ belongs to $S\{-L, -L, L+2\}_{M,N}$ for $(t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$.

2.2. Construction of exact solution in $Z_{e,ext}(M, N, \delta)$

Let us look for exact solutions $u_1(t, \varphi, \xi), u_2(t, \varphi, \xi)$ of (2.1) for $(t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$ in the following form

$$u_1(t, \varphi, \xi) = y_1(t, \varphi, \xi) v_1(t, \varphi, \xi), \quad u_2(t, \varphi, \xi) = y_2(t, \varphi, \xi) v_2(t, \varphi, \xi).$$

Then we have

$$\begin{aligned} \left(\frac{d^2}{dt^2} - \lambda^2(t) e^{2i\varphi} \xi^2 + b(t) e^{i\varphi} \xi \right) u_1(t, \varphi, \xi) &= y_1(t, \varphi, \xi) \partial_t^2 v_1(t, \varphi, \xi) \\ &+ 2\partial_t y_1(t, \varphi, \xi) \partial_t v_1(t, \varphi, \xi) + e^{\xi e^{i\varphi} \Lambda(t)} h_{1,L}(t, \varphi, \xi) v_1(t, \varphi, \xi). \end{aligned}$$

This equation can be transformed to

$$\frac{d^2 v_1}{dt^2} + 2 \left(\frac{1}{y_1} \frac{dy_1}{dt} \right) \frac{dv_1}{dt} = \frac{1}{y_1^2} \frac{d}{dt} \left(y_1^2 \frac{dv_1}{dt} \right) = - \frac{e^{\xi e^{i\varphi} \Lambda(t)} h_{1,L} v_1}{y_1},$$

respectively,

$$y_1^2 \frac{dv_1}{dt} = - \int_{t_\xi}^t e^{\xi e^{i\varphi} \Lambda(\tau)} h_{1,L}(\tau, \varphi, \xi) v_1(\tau, \varphi, \xi) y_1(\tau, \varphi, \xi) d\tau,$$

where we use here the data $v_1(t_\xi, \varphi, \xi) = 1$, $\partial_t v_1(t_\xi, \varphi, \xi) = 0$. Using (2.13) of Corollary 2.1 it is possible to divide by y_1^2 . Consequently,

$$(2.18) \quad v_1(t, \varphi, \xi) + \int_{t_\xi}^t a_1(s, \varphi, \xi) h_{1,L}(s, \varphi, \xi) \left(\int_s^t \frac{e^{2\xi e^{i\varphi} (\Lambda(s) - \Lambda(\tau))}}{a_1^2(\tau, \varphi, \xi)} d\tau \right) v_1(s, \varphi, \xi) ds = 1.$$

In an analog way one can derive

$$(2.19) \quad v_2(t, \varphi, \xi) + \int_T^t a_2(s, \varphi, \xi) h_{2,L}(s, \varphi, \xi) \left(\int_s^t \frac{e^{-2\xi e^{i\varphi} (\Lambda(s) - \Lambda(\tau))}}{a_2^2(\tau, \varphi, \xi)} d\tau \right) v_2(s, \varphi, \xi) ds \\ = 1 - g(\varphi, \xi) \int_t^T y_2^{-2}(s, \varphi, \xi) ds.$$

Here we have chosen data $v_2(T, \varphi, \xi) = 1$, $\partial_t v_2(T, \varphi, \xi) = g(\varphi, \xi)$.

Thus let us consider both Volterra integral equations. We prefer to study instead of these equations the following ones:

$$(2.20) \quad w_1(t, \varphi, \xi) + \int_{t_\xi}^t P_1(t, s, \varphi, \xi) w_1(s, \varphi, \xi) ds = f_1(t, \varphi, \xi),$$

$$(2.21) \quad w_2(t, \varphi, \xi) + \int_T^t P_2(t, s, \varphi, \xi) w_2(s, \varphi, \xi) ds = f_2(t, \varphi, \xi)$$

for the unknown functions $w_i = v_i - 1$ and with right hand sides

$$f_1(t, \varphi, \xi) = - \int_{t_\xi}^t P_1(t, s, \varphi, \xi) ds,$$

$$f_2(t, \varphi, \xi) = - \int_t^T (P_2(t, s, \varphi, \xi) - g(\varphi, \xi) y_2(s, \varphi, \xi)^{-2}) ds.$$

Here we introduced the notations

$$P_1(t, s, \varphi, \xi) = a_1(s, \varphi, \xi) h_{1,L}(s, \varphi, \xi) \int_s^t \frac{e^{2\xi e^{i\varphi} (\Lambda(s) - \Lambda(\tau))}}{a_1^2(\tau, \varphi, \xi)} d\tau,$$

$$P_2(t, s, \varphi, \xi) = a_2(s, \varphi, \xi) h_{2,L}(s, \varphi, \xi) \int_s^t \frac{e^{-2\xi e^{i\varphi} (\Lambda(s) - \Lambda(\tau))}}{a_2^2(\tau, \varphi, \xi)} d\tau.$$

LEMMA 2.4. For every non-negative integer r there exists a constant C_r such that

$$(2.22) \quad |P_i(t, s, \varphi, \xi)| \leq C_r (\ln \xi)^{-r}, \quad i = 1, 2,$$

for all admissible s and $(t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$.

PROOF. For P_1 we have due to (2.5) and (2.6)

$$P_1(t, s, \varphi, \xi) = \left(\sum_{l=0}^L b_{1,L}(s, \varphi, \xi) \right) (h_{1,L}(s, \varphi, \xi) e^{\int_{t_0}^s K_1(s_1) ds_1}) \\ \times \int_s^t \left(\sum_{l=0}^L b_{1,L}(\tau, \varphi, \xi) \right)^{-2} e^{2 \int_s^\tau K_1(s_1) ds_1 + 2\xi e^{i\varphi} (\Lambda(s) - \Lambda(\tau))} d\tau.$$

For $(t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$ we have

$$\operatorname{Re} \left(\int_s^\tau 2K_1(s_1) ds_1 + 2\xi e^{i\varphi} (\Lambda(s) - \Lambda(\tau)) \right) \\ = -2 \operatorname{Re} \int_s^\tau (\xi e^{i\varphi} \lambda(s_1) - K_1(s_1)) ds_1 \leq -2 \operatorname{Re} \int_s^\tau (\xi e^{i\varphi} \lambda(s_1) - |K_1(s_1)|) ds_1 \leq 0$$

for a corresponding δ .

Using Lemma 2.1 and 2.2 we obtain the desired estimate after a suitable choice of $L = L(r)$.

In the same way one gets the estimate for P_2 . The lemma is proved.

The right hand sides f_1 and f_2 of (2.20), (2.21) are continuous in their arguments. Consequently, the solutions of (2.20), (2.21) can be written in the form

$$w_1(t, \varphi, \xi) = f_1(t, \varphi, \xi) + \int_{t_0}^t \omega_1(t, s, \varphi, \xi) f_1(s, \varphi, \xi) ds, \\ w_2(t, \varphi, \xi) = f_2(t, \varphi, \xi) + \int_T^t \omega_2(t, s, \varphi, \xi) f_2(s, \varphi, \xi) ds$$

with kernels $\omega_i = \sum_{v=1}^\infty \omega_{v,i}$, where

$$\omega_{i,1}(t, s, \varphi, \xi) = -P_i(t, s, \varphi, \xi), \quad \omega_{i,v+1}(t, s, \varphi, \xi) = \int_s^t \omega_{i,1}(t, \theta, \varphi, \xi) \omega_{i,v}(\theta, s, \varphi, \xi) d\theta.$$

But (2.22) implies $|w_1(t, \varphi, \xi)| \leq C \max_{s \in [t_0, t]} |f_1(s, \varphi, \xi)|$, $|w_2(t, \varphi, \xi)| \leq C \max_{s \in [t, T]} |f_2(s, \varphi, \xi)|$. If we use the function $g = g(\varphi, \xi)$ as a symbol of suitable negative order, then $|f_1(s, \varphi, \xi)| \leq C_r (\ln \xi)^{-r}$.

To estimate derivatives of v_i we have due to Lemma 2.1 and 2.2 for all $(t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$

$$|\partial_t^l \partial_\xi^p v_i(t, \varphi, \xi)| \leq C_{l,p,r} \langle \xi \rangle^{-p} (\ln \xi)^{-r}, \quad l+p \geq 1.$$

Thus we have proved the following

LEMMA 2.5. *There are solutions $u_i(t, \varphi, \xi), i = 1, 2$, of (2.1) having for $(t, \varphi, \xi) \in Z_{e,ext}(M, N, \delta)$ the representations*

$$u_1(t, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t)} a_1(t, \varphi, \xi), \quad u_2(t, \varphi, \xi) = e^{-\xi e^{i\varphi} \Lambda(t)} a_2(t, \varphi, \xi).$$

The amplitude functions are satisfying with suitable constants m_i

$$|\partial_t^l \partial_\xi^p a_i(t, \varphi, \xi)| \leq C_{l,p,i} \langle \xi \rangle^{m_i-p}$$

for all $l, p \geq 0, i = 1, 2$.

2.3. Construction of exact solution into $Z_{e,int}(M, N, \delta)$

In the first two steps we have constructed exact solutions u_1, u_2 in $Z_{e,ext}(M, N, \delta)$. Using Lemma 2.5 and the definition of $Z_{e,ext}(M, N, \delta)$ we conclude that $u_1(t_\xi, \varphi, \xi), u_2(t_\xi, \varphi, \xi), \partial_t u_1(t_\xi, \varphi, \xi), \partial_t u_2(t_\xi, \varphi, \xi)$ are symbols. For continuation into $Z_{e,int}(M, N, \delta)$ we can use straightforward approach.

Let $\rho(t, \xi)$ be the positive root of the equation

$$\rho^2 - 1 - \langle \xi \rangle \lambda^2(t) \Lambda^{-1}(t) \ln \langle \xi \rangle = 0.$$

Then one can prove by (1.4) that

$$(2.23) \quad |\rho(t, \xi)| \leq C_\varepsilon \langle \xi \rangle^\varepsilon, \quad \left| \left(\frac{\partial}{\partial t} \right)^l \left(\frac{\partial}{\partial \xi} \right)^p \rho(t, \xi) \right| \leq C_{l,p,\varepsilon} \langle \xi \rangle^{l\varepsilon-p} \rho(t, \xi)$$

for all $(t, \xi) \in Z_{int}(M, N)$ and every positive ε . Moreover,

$$(2.24) \quad \int_0^{t_\xi} \left(\rho(t, \xi) + \frac{\rho_t(t, \xi)}{\rho(t, \xi)} \right) dt \leq K \ln \langle \xi \rangle.$$

Furthermore, by means of matrix-valued functions

$$H(t, \xi) = \begin{pmatrix} \rho(t, \xi) & , & 0 \\ 0 & , & 1 \end{pmatrix}, \quad H^{-1}(t, \xi) = \begin{pmatrix} \rho^{-1}(t, \xi) & , & 0 \\ 0 & , & 1 \end{pmatrix}$$

and vector $\mathcal{U} := {}^t(\mathcal{U}_1, \mathcal{U}_2) := H(t)^t(u, d_t u)$ the equation (2.1) can be trans-

formed to

$$(2.25) \quad \frac{d}{dt} \mathcal{U} = \mathcal{A}(t, \varphi, \xi) \mathcal{U},$$

where

$$(2.26) \quad \mathcal{A} = \begin{pmatrix} \rho_t(t, \xi) / \rho(t, \xi) & , & \rho(t, \xi) \\ \lambda^2(t) e^{2i\varphi} \xi^2 \rho^{-1}(t, \xi) - b(t) e^{i\varphi} \xi \rho^{-1}(t, \xi) & , & 0 \end{pmatrix}.$$

Every solution $\mathcal{U}(t, \varphi, \xi)$ of (2.25) can be represented by the following explicit formula [6]:

$$(2.27) \quad \begin{aligned} \mathcal{U}(t, \varphi, \xi) = & \mathcal{U}(t_\xi, \xi) + \left(\sum_{l=1}^{\infty} \int_{t_\xi}^t ds_1 \int_{t_\xi}^{s_1} ds_2 \dots \right. \\ & \left. \dots \int_{t_\xi}^{s_{l-1}} ds_l \mathcal{A}(s_1) \dots \mathcal{A}(s_l) \right) \mathcal{U}(t_\xi, \xi). \end{aligned}$$

Using the operator $(Ir)(t) = \int_{t_\xi}^t r(s) ds$ one can write (2.27) in the form

$$\mathcal{U}(t, \varphi, \xi) = \mathcal{U}(t_\xi, \xi) + \sum_{l=1}^{\infty} \underbrace{I \mathcal{A} I \mathcal{A} \dots I \mathcal{A} I \mathcal{A}}_l \mathcal{U}(t_\xi, \xi).$$

LEMMA 2.6. *There exist constants m_i ($i = 1, 2$) such that for every p*

$$(2.28) \quad \left| \left(\frac{\partial}{\partial \xi} \right)^p \mathcal{U}_i(t_\xi, \varphi, \xi) \right| \leq C_p \langle \xi \rangle^{m_i - p}$$

hold for all $(\varphi, \xi) \in Z_e(M, N, \delta)$.

PROOF. The estimates follow immediately from Lemma 2.5, (1.9) and (1.10).

LEMMA 2.7. *Let $\mathcal{V}_i(t, \varphi, \xi)$ be solutions of (2.25) which are smooth continuations into $Z_{e,int}(M, N, \delta)$ of $H(t, \xi)^l (u_i(t, \varphi, \xi), \partial_t u_i(t, \varphi, \xi))$, $i = 1, 2$. Then there exist positive constants m_i such that*

$$\left| \left(\frac{\partial}{\partial t} \right)^l \left(\frac{\partial}{\partial \xi} \right)^p \mathcal{V}_i(t, \varphi, \xi) \right| \leq C_{p,l} \langle \xi \rangle^{m_i - p}$$

holds for all $(t, \varphi, \xi) \in Z_{e,int}(M, N, \delta)$ and $p, l \geq 0$.

PROOF. Let us consider the matrix-valued function

$$\mathcal{E}(t, \varphi, \xi) := I + \sum_{l=1}^{\infty} \underbrace{I \mathcal{A} I \mathcal{A} \cdots I \mathcal{A} I \mathcal{A}}_l.$$

Taking account of Lemma 2.6 and (2.27) it is enough to estimate $\mathcal{E}(t, \varphi, \xi)$ and its derivatives with respect to ξ only. As a consequence of (2.23) and (2.26) we have

$$(2.29) \quad \left| \left(\frac{\partial}{\partial \xi} \right)^k \mathcal{A}(t, \varphi, \xi) \right| \leq C_k \langle \xi \rangle^{-k} g(t, \xi) \quad \text{for all } (t, \varphi, \xi) \in Z_{e, \text{int}}(M, N, \delta),$$

where the notation $g(t, \xi) = \rho(t, \xi) + \frac{\rho_t(t, \xi)}{\rho(t, \xi)}$ is used. Then it is clear that by (2.24)

$$\begin{aligned} \|\mathcal{E}(t, \varphi, \xi)\| &\leq \sum_{l=0}^{\infty} \frac{1}{l!} \left(C_0 \int_t^{t_\xi} g(s, \xi) ds \right)^l \\ &\leq \sum_{l=0}^{\infty} \frac{1}{l!} (C_0 K \ln \langle \xi \rangle)^l = \langle \xi \rangle^{C_0 K}. \end{aligned}$$

Further, for the derivative $(\partial/\partial \xi)\mathcal{E}(t, \varphi, \xi)$ we have

$$\begin{aligned} &\left\| \frac{\partial}{\partial \xi} \mathcal{E}(t, \varphi, \xi) \right\| \\ &\leq \sum_{l=1}^{\infty} \left\| I \underbrace{\frac{\partial \mathcal{A}}{\partial \xi} I \mathcal{A} \cdots I \mathcal{A} I \mathcal{A}}_{l-1} \right\| + \cdots + \sum_{l=1}^{\infty} \left\| \underbrace{i \mathcal{A} I \mathcal{A} \cdots I \mathcal{A} I}_{l-1} \frac{\partial \mathcal{A}}{\partial \xi} I \mathcal{A} \right\| \\ &\quad + \sum_{l=1}^{\infty} \left\| \underbrace{I \mathcal{A} I \mathcal{A} \cdots I \mathcal{A} I \mathcal{A}}_{l-1} \frac{\partial \mathcal{A}}{\partial \xi} \right\| + \sum_{l=1}^{\infty} \left\| \underbrace{I \mathcal{A} I \mathcal{A} \cdots I \mathcal{A} I \mathcal{A}}_{l-1} \frac{\partial t_\xi}{\partial \xi} \mathcal{A}(t_\xi, \xi) \right\| \\ &\leq C \langle \xi \rangle^{-1} \sum_{l=1}^{\infty} \frac{l}{l!} \left(C_0 \int_t^{t_\xi} g(s, \xi) ds \right)^{l-1} \left(C_1 \int_t^{t_\xi} g(s, \xi) ds \right) \\ &\quad + g(t_\xi, \xi) \left| \frac{\partial t_\xi}{\partial \xi} \right| C \sum_{l=0}^{\infty} \frac{1}{l!} \left(C_0 \int_t^{t_\xi} g(s, \xi) ds \right)^l. \end{aligned}$$

Then one has due to (1.9) from Lemma 1.1 and (2.23)

$$g(t_\xi, \xi) \left| \frac{\partial t_\xi}{\partial \xi} \right| \leq C \langle \xi \rangle^{-1}.$$

Consequently,

$$\begin{aligned} \left\| \frac{\partial}{\partial \xi} \mathcal{E}(t, \varphi, \xi) \right\| &\leq C \left(C_1 \left(C_1 \int_t^{t_\xi} g(s, \xi) ds \right) + 1 \right) \langle \xi \rangle^{-1} \sum_{l=0}^{\infty} \frac{1}{l!} \left(\int_t^{t_\xi} g(s, \xi) ds \right)^l \\ &\leq C \langle \xi \rangle^{C_0 K - 1} \ln \langle \xi \rangle. \end{aligned}$$

All other derivatives can be estimated in a similar way. We obtain $\|\partial_\xi^p \mathcal{E}(t, \varphi, \xi)\| \leq C_p \langle \xi \rangle^{C_0 K - p} (\ln \langle \xi \rangle)^p$. Consequently, $\|\partial_\xi^p \mathcal{E}(t, \varphi, \xi)\| \leq C_p \langle \xi \rangle^{m-p}$ in $Z_{e, \text{int}}(M, N, \delta)$ if $m > C_0 K$. Together with Lemma 2.6 we derive the estimates for derivatives with respect to ξ .

To obtain the estimates for derivatives with respect to t one has to use (1.4), (2.23) and (2.25). The lemma is proved.

Setting now $u_i(t, \varphi, \xi) = \mathcal{V}_i(t, \varphi, \xi) / \rho(t, \xi)$ then u_i can be represented in the form

$$u_1(t, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t)} a_1(t, \varphi, \xi), \quad u_2(t, \varphi, \xi) = e^{-\xi e^{i\varphi} \Lambda(t)} a_2(t, \varphi, \xi)$$

in $Z_{e, \text{int}}(M, N, \delta)$.

Using the definition of $Z_{e, \text{int}}(M, N, \delta)$ and Lemma 2.7 then $a_i(t, \varphi, \xi)$ are amplitude functions satisfying $|\partial_t^l \partial_\xi^p a_i(t, \varphi, \xi)| \leq C_{l,p,i} \langle \xi \rangle^{m_i - p}$ with suitable constants m_i . Thus, the statements (2.2), (2.3) and consequently the Theorem 2.1 are proved.

COROLLARY 2.2. 1) *The constructed solution $u_1 = u_1(t, \varphi, \xi)$ depends analytically on φ in $Z_e(M, N, \delta) \times [0, T]$.*

2) *If $g = g(\varphi, \xi)$ depends holomorphically on $\xi = \xi e^{i\varphi}$ in $Z_e(M, N, \delta)$, then the constructed solution $u_{2,g}$ defined in $Z_e(M, N, \delta) \times [0, T]$ depends holomorphically on ξ , too.*

PROOF. TO 1). We have on $t = t_\xi$

$$\begin{aligned} u_1(t_\xi, \varphi, \xi) &= y_1(t_\xi, \varphi, \xi) v_1(t_\xi, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t_\xi)} a_1(t_\xi, \varphi, \xi) v_1(t_\xi, \varphi, \xi) = e^{\xi e^{i\varphi} \Lambda(t_\xi)}, \\ \partial_t u_1(t_\xi, \varphi, \xi) &= y_1(t_\xi, \varphi, \xi) \partial_t v_1(t_\xi, \varphi, \xi) + (\partial_t y_1(t_\xi, \varphi, \xi) v_1(t_\xi, \varphi, \xi) \\ &= e^{\xi e^{i\varphi} \Lambda(t_\xi)} \lambda(t_\xi) e^{i\varphi} \xi + e^{\xi e^{i\varphi} \Lambda(t_\xi)} K_1(t_\xi) \sum_{l=0}^L b_{1,l}(t_\xi, \varphi, \xi) \\ &\quad + e^{\xi e^{i\varphi} \Lambda(t_\xi)} \sum_{l=0}^L \partial_t b_{1,l}(t_\xi, \varphi, \xi) \\ &= e^{\xi e^{i\varphi} \Lambda(t_\xi)} \left(\lambda(t_\xi) e^{i\varphi} \xi + K_1(t_\xi) + \sum_{l=0}^L \partial_t b_{1,l}(t_\xi, \varphi, \xi) \right). \end{aligned}$$

From construction of $a_{1,k}(t, \varphi, \xi)$ we have analytic dependence on φ . The same holds for $b_{1,k}(t, \varphi, \xi)$ and derivatives with respect to φ . Hence, the data $u_1, \partial_t u_1$ depend analytically on φ on $t = t_\xi$. But the coefficients of equations are analytic in φ . This proves the first statement.

TO 2). The statement can be proved by the same reasoning.

The considerations in section 2.3 lead to an effective tool for further studies, namely to energy estimates in $Z_{e,int}(M, N, \delta)$.

COROLLARY 2.3. *If u is a solution of (2.1) in $Z_{e,int}(M, N, \delta)$, then we have the following energy estimate*

$$(2.30) \quad E(u)(t_1, \varphi, \xi) \leq C \langle \xi \rangle^m E(u)(t_2, \varphi, \xi)$$

for all $(t_1, \varphi, \xi), (t_2, \varphi, \xi) \in Z_{e,int}(M, N, \delta)$. Here

$$E(u)(t, \varphi, \xi) = |u(t, \varphi, \xi)|^2 + |\partial_t u(t, \varphi, \xi)|^2.$$

PROOF. Setting $\mathcal{U} = {}^t(U_1, U_2) = {}^t(\rho u, \partial_t u)$ then

$$\begin{aligned} E(u)(t, \varphi, \xi) &\leq E(\mathcal{U})(t, \varphi, \xi) = |U_1(t, \varphi, \xi)|^2 + |U_2(t, \varphi, \xi)|^2 \\ &\leq \rho^2(t, \xi) E(u)(t, \varphi, \xi) \leq \langle \xi \rangle^2 E(u)(t, \varphi, \xi). \end{aligned}$$

The explicit representation (2.27) for $\mathcal{U} = \mathcal{U}(t, \varphi, \xi)$ and estimates (2.28), (2.29) imply

$$E(\mathcal{U})(t_1, \varphi, \xi) \leq \langle \xi \rangle^{C_0 K + 1} E(\mathcal{U})(t_2, \varphi, \xi)$$

for all $(t_1, \varphi, \xi), (t_2, \varphi, \xi) \in Z_{e,int}(M, N, \delta)$. With the above inequalities we have (2.30), where $m = C_0 K + 3$.

3. About the Wronskian $W(u_1, u_2)$

In the previous section we have constructed two solutions u_1 and u_2 . Up to now we have no information about the linear independence of these solutions, especially no information about the Wronskian. Following the construction scheme with $g(\varphi, \xi) = 0$ then one concludes the next result.

LEMMA 3.1. *The asymptotical behaviour of $u_1, u_2, \partial_t u_1, \partial_t u_2$ is in $Z_{e,ext}(M, N, \delta)$ the following:*

$$\begin{aligned}
 u_1(t, \varphi, \xi) &\sim e^{\xi e^{i\varphi} \Lambda(t)} e^{\int_{t_\xi}^t \frac{\lambda'(s)+b(s)}{2\lambda(s)} ds} \left(1 + O\left(\frac{1}{\ln\langle \xi \rangle}\right) \right), \\
 u_2(t, \varphi, \xi) &\sim e^{-\xi e^{i\varphi} \Lambda(t)} e^{\int_t^T \frac{\lambda'(s)-b(s)}{2\lambda(s)} ds} \left(1 + O\left(\frac{1}{\xi \Lambda(t)}\right) \right), \\
 \partial_t u_1(t, \varphi, \xi) &\sim \left(\xi e^{i\varphi} \lambda(t) + \frac{\lambda'(t) + b(t)}{2\lambda(t)} \right) e^{\xi e^{i\varphi} \Lambda(t)} e^{\int_{t_\xi}^t \frac{\lambda'(s)+b(s)}{2\lambda(s)} ds} \left(1 + O\left(\frac{1}{\ln\langle \xi \rangle}\right) \right) \\
 &\quad + e^{\xi e^{i\varphi} \Lambda(t)} e^{\int_{t_\xi}^t \frac{\lambda'(s)+b(s)}{2\lambda(s)} ds} O\left(\frac{1}{\ln\langle \xi \rangle}\right), \\
 \partial_t u_2(t, \varphi, \xi) &\sim - \left(\xi e^{i\varphi} \lambda(t) + \frac{\lambda'(t) - b(t)}{2\lambda(t)} \right) e^{-\xi e^{i\varphi} \Lambda(t)} e^{\int_t^T \frac{\lambda'(s)-b(s)}{2\lambda(s)} ds} \left(1 + O\left(\frac{1}{\xi \Lambda(t)}\right) \right) \\
 &\quad + e^{-\xi e^{i\varphi} \Lambda(t)} e^{\int_t^T \frac{\lambda'(s)-b(s)}{2\lambda(s)} ds} O\left(\frac{1}{\xi \Lambda(t)}\right).
 \end{aligned}$$

LEMMA 3.2. *There exists a constant m such that for all $(t, \varphi, \xi) \in [0, 1] \times Z_e(M, N, \delta)$ the Wronskian $W(u_1, u_2)$ is an hypoelliptic symbol, that is*

$$(3.1) \quad |W(u_1, u_2)| \geq C \langle \xi \rangle^m.$$

PROOF. It holds

$$\begin{aligned}
 W(u_1, u_2)(t, \varphi, \xi) &= W(u_1, u_2)(T, \varphi, \xi) \\
 &= u_1(T, \varphi, \xi) \partial_t u_2(T, \varphi, \xi) - \partial_t u_1(T, \varphi, \xi) u_2(T, \varphi, \xi)
 \end{aligned}$$

and due to Lemma 3.1

$$\begin{aligned}
 W(u_1, u_2)(t, \varphi, \xi) &\sim - e^{\int_{t_\xi}^T \frac{\lambda'(s)+b(s)}{2\lambda(s)} ds} \left(\left(2\xi e^{i\varphi} \lambda(T) + \frac{\lambda'(T)}{\lambda(T)} \right) \left(1 + O\left(\frac{1}{\ln\langle \xi \rangle}\right) \right) \right. \\
 &\quad \left. \times \left(1 + O\left(\frac{1}{\xi \Lambda(T)}\right) \right) \right).
 \end{aligned}$$

Consequently, using $|2\xi e^{i\varphi} \lambda(T) + (\lambda'(T)/\lambda(T))| \geq |(\lambda'(T)/\lambda(T))|$ and (1.7) we get

$$|W(u_1, u_2)(t, \varphi, \xi)| \geq C \langle \xi \rangle e^{-K \int_{t_\xi}^T d_s \ln \Lambda(s) ds} \geq C \langle \xi \rangle e^{K \ln \Lambda(t_\xi)} \geq C \langle \xi \rangle^m.$$

This completes the proof.

Thus we have an estimate for $W(u_1, u_2)$ under the assumption that we suppose for the construction of $u_2(t, \varphi, \xi)$ the initial condition $\partial_t v_2(T, \varphi, \xi) =$

$g(\varphi, \xi) = 0$ in section 2.2. The statement of Lemma 3.2 is preserved if $g = g(\varphi, \xi)$ satisfies the estimate of the following corollary.

COROLLARY 3.1. *Let us suppose that for all $(\varphi, \xi) \in Z_e(M, N, \delta)$*

$$|g(\varphi, \xi)| \leq C\langle \xi \rangle^{m-m_1-1},$$

where m_1 and m are the constants from Theorem 2.1 and Lemma 3.2, respectively. Then the estimate (3.1) remains valid.

PROOF. Let $u_{2,g}$ be the solution of (2.1) was constructed in section 2 by the aid of initial condition $\partial_t v_2(T, \varphi, \xi) = g(\varphi, \xi)$ for v_2 in section 2.2, in particular $u_{2,0} = u_2$. Then $W(u_1, u_{2,g}) = u_1(T, \varphi, \xi)\partial_t u_{2,g}(T, \varphi, \xi) - \partial_t u_1(T, \varphi, \xi)u_{2,g}(T, \varphi, \xi) = W(u_1, u_2) + u_1(T, \varphi, \xi)v_2(T, \varphi, \xi)\partial_t v_2(T, \varphi, \xi) = W(u_1, u_2) + e^{\xi e^{i\varphi}\Lambda(T)}a_1(T, \varphi, \xi) \cdot e^{-\xi e^{i\varphi}\Lambda(T)}g(\varphi, \xi)$. The assumption concerning g and (2.3) from Theorem 2.1 imply

$$|W(u_1, u_{2,g})| \geq C\langle \xi \rangle^m - C_{0,0,1}\langle \xi \rangle^{m_1} C\langle \xi \rangle^{m-m_1-1} \geq C\langle \xi \rangle^m.$$

4. About hypoellipticity and non-hypoellipticity

4.1. Sufficient condition for hypoellipticity

Let us define Green's function $G(t, s, \varphi, \xi)$ of

$$(4.1) \quad (d/dt)^2 u - \lambda^2(t)e^{2i\varphi}\xi^2 u + b(t)e^{i\varphi}\xi u = 0$$

by the aid of two independent solutions $u(t, \varphi, \xi), v(t, \varphi, \xi)$ of equation (4.1) as follows:

$$(4.2) \quad G(t, s; \varphi, \xi) = \begin{cases} u(t, \varphi, \xi)v(s, \varphi, \xi) & \text{when } t \geq s, \\ v(t, \varphi, \xi)u(s, \varphi, \xi) & \text{when } t < s, \end{cases}$$

where $u(t, \varphi, \xi)$ and $v(s, \varphi, \xi)$ are satisfying

$$(4.3) \quad u(1, \varphi, \xi) = v(-1, \varphi, \xi) = 0 \quad \text{for all } \xi \in \mathbf{R}.$$

Moreover, we suppose for the Wronskian

$$(4.4) \quad W(u, v) := \partial_t u(t, \varphi, \xi)v(t, \varphi, \xi) - u(t, \varphi, \xi)\partial_t v(t, \varphi, \xi) = 1$$

for all $t \in [-1, 1], \xi \in \mathbf{R}$. By u_1^+, u_2^+ we denote the solutions u_1 and u_2 of Theorem 2.1, (1.11). In the same way we find solutions u_1^-, u_2^- for $t \leq 0$, (1.12). Pay attention that $\Lambda(t)$ and t change their sign simultaneously. For this reason u_1^-, u_2^- play the rule of u_2^+, u_1^+ , respectively. Using the linear independence of (u_1^+, u_2^+) ,

(u_1^-, u_2^-) respectively, the solutions u and v can be represented in $[-1, 1] \times Z_e(M, N, \delta)$ by the following way

$$(4.5) \quad u(t, \varphi, \xi) = \begin{cases} c_{1,-}^u(\varphi, \xi)u_1^-(t, \varphi, \xi) + c_{2,-}^u(\varphi, \xi)u_2^-(t, \varphi, \xi) & \text{when } t \leq 0, \\ c_{1,+}^u(\varphi, \xi)u_1^+(t, \varphi, \xi) + c_{2,+}^u(\varphi, \xi)u_2^+(t, \varphi, \xi) & \text{when } t \geq 0, \end{cases}$$

$$(4.6) \quad v(t, \varphi, \xi) = \begin{cases} c_{1,-}^v(\varphi, \xi)u_1^-(t, \varphi, \xi) + c_{2,-}^v(\varphi, \xi)u_2^-(t, \varphi, \xi) & \text{when } t \leq 0, \\ c_{1,+}^v(\varphi, \xi)u_1^+(t, \varphi, \xi) + c_{2,+}^v(\varphi, \xi)u_2^+(t, \varphi, \xi) & \text{when } t \geq 0. \end{cases}$$

Moreover, the functions $u(t, \varphi, \xi)$ and $v(t, \varphi, \xi)$ have to be continuously differentiable. Therefore,

$$\partial_t u(+0, \varphi, \xi)v(-0, \varphi, \xi) - u(+0, \varphi, \xi)\partial_t v(-0, \varphi, \xi) = 1.$$

Hence, we have seven conditions for the eight unknown coefficients. The constructions of section 2 guarantee that the one-sided limits of $u_1^+, u_2^+, u_1^-, u_2^-$ and their derivatives with respect to t exist in $t = 0$. The boundary conditions (4.3) imply

$$\begin{aligned} c_{1,+}^u(\varphi, \xi)a_1^+(1, \varphi, \xi)e^{2\xi e^{\varphi} \Lambda(1)} &= -c_{2,+}^u(\varphi, \xi)a_2^+(1, \varphi, \xi), \\ c_{1,-}^v(\varphi, \xi)a_1^-(-1, \varphi, \xi)e^{2\xi e^{\varphi} \Lambda(-1)} &= -c_{2,-}^v(\varphi, \xi)a_2^-(-1, \varphi, \xi). \end{aligned}$$

Taking account of $a_2^+(1, \varphi, \xi) = a_1^-(-1, \varphi, \xi) = 1$ gives

$$\begin{aligned} c_{2,+}^u(\varphi, \xi) &= -e^{2\xi e^{\varphi} \Lambda(1)} a_1^+(1, \varphi, \xi) c_{1,+}^u(\varphi, \xi), \\ c_{1,-}^v(\varphi, \xi) &= -e^{-2\xi e^{\varphi} \Lambda(-1)} a_2^-(-1, \varphi, \xi) c_{2,-}^v(\varphi, \xi). \end{aligned}$$

Using this together with continuously differentiability of u and v in $t = 0$ we can express $c_{1,-}^u(\varphi, \xi), c_{2,-}^u(\varphi, \xi)$ by the aid of $c_{1,+}^u(\varphi, \xi)$, namely,

$$\begin{aligned} c_{1,-}^u(\varphi, \xi) &= \frac{1}{W(u_1^-, u_2^-)} \{ c_{1,+}^u(\varphi, \xi) (a_1^+(+0, \varphi, \xi) \partial_t a_2^-(-0, \varphi, \xi) \\ &\quad - a_2^-(-0, \varphi, \xi) \partial_t a_1^+(+0, \varphi, \xi)) + c_{2,+}^u(\varphi, \xi) (a_2^+(+0, \varphi, \xi) \partial_t a_2^-(-0, \varphi, \xi) \\ &\quad - a_2^-(-0, \varphi, \xi) \partial_t a_2^+(+0, \varphi, \xi)) \}, \\ c_{2,-}^u(\varphi, \xi) &= \frac{1}{W(u_1^-, u_2^-)} \{ c_{1,+}^u(\varphi, \xi) (a_1^-(-0, \varphi, \xi) \partial_t a_1^+(+0, \varphi, \xi) \\ &\quad - a_1^+(+0, \varphi, \xi) \partial_t a_1^-(-0, \varphi, \xi)) + c_{2,+}^u(\varphi, \xi) (a_1^-(-0, \varphi, \xi) \partial_t a_2^+(+0, \varphi, \xi) \\ &\quad - a_2^+(+0, \varphi, \xi) \partial_t a_1^-(-0, \varphi, \xi)) \}. \end{aligned}$$

In the same way one can express $c_{1,+}^v(\varphi, \xi), c_{2,+}^v(\varphi, \xi)$ by $c_{2,-}^v(\varphi, \xi)$, where

the corresponding denominator is $W(u_1^+, u_2^+)$. In $[0, 1] \times Z_e(M, N, \delta)$, $[-1, 0] \times Z_e(M, N, \delta)$, respectively, we have the estimate (3.1) for $|W(u_1^-, u_2^-)|$ and $|W(u_1^+, u_2^+)|$. Setting the above conditions into condition (4.4) for Wronskian gives

(4.7)

$W(u, v)$

$$\begin{aligned} &= c_{1,+}^u(\varphi, \xi)c_{2,-}^v(\varphi, \xi)((a_1^+(+0, \varphi, \xi)\partial_t a_2^-(-0, \varphi, \xi) - a_2^-(-0, \varphi, \xi)\partial_t a_1^+(+0, \varphi, \xi)) \\ &\quad + e^{2\xi e^{i\varphi} \Lambda(1)} a_1^+(1, \varphi, \xi)(a_2^-(-0, \varphi, \xi)\partial_t a_2^+(+0, \varphi, \xi) - a_2^+(+0, \varphi, \xi)\partial_t a_2^-(-0, \varphi, \xi)) \\ &\quad + e^{-2\xi e^{i\varphi} \Lambda(-1)} a_2^-(-1, \varphi, \xi)(a_1^-(-0, \varphi, \xi)\partial_t a_1^+(+0, \varphi, \xi) - a_1^+(+0, \varphi, \xi)\partial_t a_1^-(-0, \varphi, \xi)) \\ &\quad + e^{2\xi e^{i\varphi} (\Lambda(1) - \Lambda(-1))} a_1^+(1, \varphi, \xi)a_2^-(-1, \varphi, \xi) \\ &\quad \times (a_2^+(+0, \varphi, \xi)\partial_t a_1^-(-0, \varphi, \xi) - a_1^-(-0, \varphi, \xi)\partial_t a_2^+(+0, \varphi, \xi)) = 1. \end{aligned}$$

In $Z_e(M, N, \delta)$ we have the estimates (2.13), (2.16) for $a_1^+(1, \varphi, \xi), a_2^-(-1, \varphi, \xi)$. Consequently, if

$$|a_2^+(+0, \varphi, \xi)\partial_t a_1^-(-0, \varphi, \xi) - a_1^-(-0, \varphi, \xi)\partial_t a_2^+(+0, \varphi, \xi)| \geq C\langle \xi \rangle^{N_0}$$

in $Z_e(M, N, \delta)$, N_0 real, then the term

$$u_1^+(1, \varphi, \xi)u_1^-(-1, \varphi, \xi)(a_2^+(+0, \varphi, \xi)\partial_t a_1^-(-0, \varphi, \xi) - a_1^-(-0, \varphi, \xi)\partial_t a_2^+(+0, \varphi, \xi))$$

in parenthesis of (4.7) dominates the others. It gives the asymptotic behaviour for the product $c_{1,+}^u(\varphi, \xi)c_{2,-}^v(\varphi, \xi) \sim e^{-2\xi e^{i\varphi} (\Lambda(1) - \Lambda(-1))}$ in $Z_e(M, N, \delta)$. Using the construction of Stokes coefficients from section 1 we have

$$a_2^+(+0, \varphi, \xi)\partial_t a_1^-(-0, \varphi, \xi) - a_1^-(-0, \varphi, \xi)\partial_t a_2^+(+0, \varphi, \xi) = T_{11}(b, \varphi, \xi)W(u_1^+, u_2^+).$$

But we can estimate $W(u_1^+, u_2^+)$ by (3.1). Thus the above condition is satisfied if and only if $|T_{11}(b, \varphi, \xi)| \geq C_\varphi \langle \xi \rangle^{m_0}$ in $Z_e(M, N, \delta)$. Hence, we are able to prove the following result.

THEOREM 4.1. *If for a given $\varphi_0 \in (-\pi/2, \pi/2)$ there exist constants C_{φ_0} and m_0 such that $|T_{11}(b, \varphi_0, \xi)| \geq C_{\varphi_0} \langle \xi \rangle^{m_0}$ is satisfied for all ξ with $(\varphi_0, \xi) \in Z_e(M, N, \delta)$, then*

$$(4.8) \quad \int_{-1}^1 \left| \left(\frac{\partial}{\partial t} \right)^l \left(\frac{\partial}{\partial s} \right)^k \left(\frac{\partial}{\partial \xi} \right)^p G(t, s, \varphi_0, \xi) \right| ds \leq C_{l,k,p} \langle \xi \rangle^{m_{\varphi_0} - p}$$

for all ξ with $(\varphi_0, \xi) \in Z_e(M, N, \delta)$ and all $t \in [-1, 1]$, where m_{φ_0} is a suitable real constant.

PROOF. We can follow the reasoning from [17]. In order to estimate $\int_{-1}^1 |G(t, s, \varphi_0, \xi)| ds$ we have to consider a lot of integrals. But the asymptotic behaviour of $c_{1,+}^u(\varphi_0, \xi)c_{2,-}^v(\varphi_0, \xi) \sim e^{-2\xi e^{\varphi_0}(\Lambda(1) - \Lambda(-1))}$, the property that $a_1^+, a_2^+, a_1^-, a_2^-$ and their one-sided derivatives with respect to t are amplitude functions satisfying the estimates (2.3) from Theorem 2.1 and the relations between $c_{1,+}^u(\varphi_0, \xi)$ and $c_{2,+}^u(\varphi_0, \xi), c_{1,-}^u(\varphi_0, \xi), c_{2,-}^u(\varphi_0, \xi)$, respectively, $c_{2,-}^v(\varphi_0, \xi)$ and $c_{1,-}^v(\varphi_0, \xi), c_{1,+}^v(\varphi_0, \xi), c_{2,+}^v(\varphi_0, \xi)$ imply that all the integrals $I_1 - I_{24}$ from [17] can be estimated by $C\langle \xi \rangle^{m_{\varphi_0}}$. Hence, the assertion for $\int_{-1}^1 |G(t, s, \varphi_0, \xi)| ds$ is proved.

The assertion for derivatives follows immediately if we use additionally to the behaviour of $u_1^+, u_2^+, u_1^-, u_2^-$ that of their derivatives with respect to t and ξ , too. The theorem is proved.

For study of hypoellipticity we need an estimate of $\int_{-1}^1 |G(t, s, \varphi_0, \xi)| ds$ for large negative ξ , too. Thus means, we have to formulate in (4.7) a condition for the first term of parenthesis

$$a_1^+(+0, \varphi, \xi)\partial_t a_2^-(-0, \varphi, \xi) - a_2^-(-0, \varphi, \xi)\partial_t a_1^+(+0, \varphi, \xi) = T_{22}(b, \varphi, \xi)W(u_1^+, u_2^+).$$

COROLLARY 4.1. Let $Z_e(M, N, \delta)$ be such a zone that the Stokes coefficient T_{11} satisfies for a given $\varphi_0 \in (-\pi/2, \pi/2)$ the estimate $|T_{11}(\pm b, \varphi_0, \xi)| \geq C_{\varphi_0} \langle \xi \rangle^{m_0}$ for all $(\varphi_0, \xi) \in Z_e(M, N, \delta)$. Then

$$(4.9) \quad \int_{-1}^1 \left| \left(\frac{\partial}{\partial t} \right)^l \left(\frac{\partial}{\partial s} \right)^k \left(\frac{\partial}{\partial \xi} \right)^p G(t, s, \varphi_0, \xi) \right| ds \leq C_{l,k,p} \langle \xi \rangle^{m_{\varphi_0} - p}$$

for all ξ with (φ_0, ξ) or $(\varphi_0, -\xi) \in Z_e(M, N, \delta)$ and all $t \in [-1, 1]$. Here m_{φ_0} is a suitable real constant.

Now we have all tools for proving our result about hypoellipticity.

THEOREM 4.2. Under the assumptions of Corollary 4.1 the operator

$$L_{\varphi_0} = D_t^2 + \lambda^2(t)e^{2i\varphi_0} D_x^2 - b(t)e^{i\varphi_0} D_x$$

is hypoelliptic at $(0, 0)$.

PROOF. Let $\Omega \subset [-1, 1] \times \mathbf{R}$ be a neighbourhood of the point $(0, 0)$ and let the distribution $u \in \mathcal{D}'(\Omega)$ be a solution of $L_{\varphi_0}u = f$, where f belongs to $C^\infty(\Omega)$. We have to prove that there exists a neighbourhood $\Omega_1 \subset \Omega$ such that $(0, 0) \in \Omega_1, u \in C^\infty(\Omega_1)$.

The ellipticity of L_{φ_0} outside of $t = 0$ implies $u \in C^\infty(t \neq 0)$.

Further, let $\psi \in C^\infty(\mathbf{R})$ be such that $\psi(z) = 1$ when $|z| \leq \varepsilon$ while $\psi(z) = 0$ for $|z| \geq 2\varepsilon$. Then the distribution $v = \psi(t)\psi(x)u$ solves $L_{\varphi_0}(v) = f_1$, where f_1 is the distribution $f_1 = \psi(t)\psi(x)f(t, x) + [L_{\varphi_0}, \psi(t)\psi(x)]u$ and $[L_{\varphi_0}, \psi(t)\psi(x)]$ denotes the commutator. The last term of f_1 has support in $[-2\varepsilon, 2\varepsilon] \times [-2\varepsilon, 2\varepsilon]$ and vanishes outside of $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$, while the first one is a smooth function.

It is known [12] that the structure of the characteristic set of L_{φ_0} implies $WF(u) \subset \{(t, x; \tau, \xi) | t = 0, \tau = 0\}$. Hence,

$$WF(v), WF(f_1) \subset \{(t, x; \tau, \xi) | t = 0, \tau = 0\}.$$

Besides $v, f_1 \in \mathcal{E}'(\Omega)$. Therefore, there is a cone $\Gamma_\gamma = \{(\tau, \xi) | |\tau| \geq \gamma|\xi|\}$ and for every N there exists a constant C_N such that

$$(4.10) \quad |\hat{v}(\tau, \xi)| \leq C_N(1 + |\tau| + |\xi|)^{-N},$$

$$(4.11) \quad |\hat{f}_1(\tau, \xi)| \leq C_N(1 + |\tau| + |\xi|)^{-N}$$

for all $(\tau, \xi) \in \Gamma_\gamma$. Now let us denote by Γ^c the cone $\{(\tau, \xi) | |\tau| \leq 2\gamma|\xi|\}$. The property of v to be a distribution with compact support in Ω implies that there exist constants M_v, C_v such that

$$(4.12) \quad |\hat{v}(\tau, \xi)| \leq C_v(1 + |\tau| + |\xi|)^{M_v} \quad \text{for all } (\tau, \xi) \in \mathbf{R}^2.$$

According to (4.10) one has for $(\tau, \xi) \in \Gamma_\gamma$

$$|\hat{v}(\tau, \xi)| \leq C_k(1 + |\tau|)^{-k},$$

while according to (4.12) one has for $(\tau, \xi) \in \Gamma^c$ with an universal constant C_v

$$\begin{aligned} |\hat{v}(\tau, \xi)| &\leq C_v(1 + |\tau| + |\xi|)^{M_v} \leq C_v \frac{1}{(1 + |\tau|)^k} (1 + |\tau| + |\xi|)^{M_v+k} \\ &\leq C_v(1 + |\tau|)^{-k} (1 + |\xi|)^{M_v+k}. \end{aligned}$$

Thus, we obtain

$$(4.13) \quad |\hat{v}(\tau, \xi)| \leq C_v(1 + |\tau|)^{-k} (1 + |\xi|)^{M_v+k} \quad \text{for all } (\tau, \xi) \in \mathbf{R}^2.$$

In the similar way one can prove for f_1 the estimate

$$(4.14) \quad |\hat{f}_1(\tau, \xi)| \leq C_{f_1} (1 + |\tau|)^{-k} (1 + |\xi|)^{M_{f_1} + k} \quad \text{for all } (\tau, \xi) \in \mathbf{R}^2.$$

The estimates (4.13) and (4.14) imply that one can regard these distributions as elements v and f_1 of $C^k([-1, 1]; H_{(s_v - k)})$ and $C^k([-1, 1]; H_{(s_{f_1} - k)})$, respectively, defined as follows:

$$(4.15) \quad \langle v(t), \varphi(x) \rangle = \int \int e^{it\tau} \hat{v}(\tau, \xi) \hat{\varphi}(-\xi) d\tau d\xi, \quad \varphi \in C_0^\infty(\mathbf{R}^1),$$

$$(4.16) \quad \langle f_1(t), \varphi(x) \rangle = \int \int e^{it\tau} \hat{f}_1(\tau, \xi) \hat{\varphi}(-\xi) d\tau d\xi, \quad \varphi \in C_0^\infty(\mathbf{R}^1),$$

Now let us set (with a cut-off function $\chi = \chi(\xi)$ vanishing when (φ_0, ξ) and $(\varphi_0, -\xi) \notin Z_e(M, N, \delta)$ and equal to 1 outside of the ball $|\xi| \geq M(\varphi_0)$)

$$(4.17) \quad (\text{Op}G)w(t, x) = \frac{1}{2\pi} \int \int \int_{-1}^1 e^{i\xi(x-y)} \chi(\xi) G(t, s, \varphi_0, \xi) w(s, y) dy d\xi ds,$$

$$w \in C_0^\infty([-1, 1] \times \mathbf{R}),$$

with the Green's function $G(t, s, \varphi_0, \xi)$ was constructed before, (4.2). This operator can be extended to a bounded operator from $C([-1, 1]; H_{(s)})$ into $C^2([-1, 1]; H_{(s - m_{\varphi_0})})$, where m_{φ_0} is the constant from the estimate (4.8).

It is easy to see, that

$$(4.18) \quad (\text{Op}G)L_{\varphi_0}w = w - Kw,$$

where

$$Kw(t, x) = \frac{1}{2\pi} \int e^{ix\xi} W(u, v)(\varphi_0, \xi) (1 - \chi(\xi)) \hat{w}(t, \xi) d\xi, \quad w \in C_0^\infty([-1, 1] \times \mathbf{R}),$$

is a smoothing operator. Therefore

$$Kv(t, x) = \frac{1}{2\pi} \int e^{ix\xi} W(u, v)(\varphi_0, \xi) (1 - \chi(\xi)) \langle v(t), e^{-ix\xi} \rangle d\xi$$

belongs to $C_0^\infty([-1, 1] \times \mathbf{R})$. If we set in (4.18) $w = v$, then $v = (\text{Op}G)f_1 + Kv$. It remains only to consider $(\text{Op}G)f_2$, where $f_2 = [L_{\varphi_0}, \psi(t)\psi(x)]u$. Using $f_2 \in \mathcal{E}'(\Omega)$ we have the existence of a continuous function f_3 with support in $[-3\varepsilon, 3\varepsilon]^2$ such that $f_2 = \partial_x^{\alpha_1} \partial_t^{\alpha_2} f_3$. Moreover, f_3 vanishes on $B_{\varepsilon/3} = \{(x, t) \mid |x|^2 + |t|^2 \leq \varepsilon^2/9\}$.

Hence, we have for every k and for all $(t, x) \in B_{\varepsilon/4}$

$$\begin{aligned}
 (\text{Op}G)f_2 &= \frac{(-1)^{\alpha+k}}{2\pi} \int \int \int \frac{i^{\alpha_1}}{(x-y)^k} e^{i\xi(x-y)} f_3(s, y) \left(\frac{\partial}{\partial \xi}\right)^k \left(\frac{\partial}{\partial s}\right)^{\alpha_2} \\
 &\quad \times (\xi^{\alpha_1} \chi(\xi) G(t, s, \varphi_0, \xi)) dy ds d\xi.
 \end{aligned}$$

Taking into consideration (4.9) v is smooth in some neighbourhood of $(0, 0)$. Finally, this gives the smoothness of u in the same neighbourhood. The proof is finished.

COROLLARY 4.2. *Let $Z_e(M, N, \delta)$ be such a zone that the Stokes coefficient T_{11} satisfies for a given $\varphi_0 \in (-\pi/2, \pi/2)$ the estimate $|T_{11}(\pm \bar{b}, -\varphi_0, \xi)| \geq C_{\varphi_0} \langle \xi \rangle^{m_0}$ for all $(-\varphi_0, \xi) \in Z_e(M, N, \delta)$. Then L_{φ_0} is locally solvable at $(0, 0)$.*

PROOF. By Theorem 4.2 the adjoint operator $L_{\varphi_0}^*$ is hypoelliptic at $(0, 0)$. But this means that L_{φ_0} is locally solvable at $(0, 0)$.

4.2. Uniqueness property for a Stokes coefficient

In this section we want to discuss the following problem:

“The Stokes coefficient $T_{11}(b, \varphi, \xi)$ is defined in $Z_e(M, N, \delta)$. If it vanishes on some subset of $Z_e(M, N, \delta)$, can we conclude that it vanishes everywhere on $Z_e(M, N, \delta)$.”

THEOREM 4.3. *If for a given $\varphi_0 \in (-\pi/2, \pi/2)$ the Stokes coefficient $T_{11}(b, \varphi_0, \xi)$ vanishes for all ξ with $(\varphi_0, \xi) \in Z_e(M, N, \delta)$, then $T_{11}(b, \varphi, \xi)$ vanishes everywhere on $Z_e(M, N, \delta)$.*

PROOF. By definition $T_{11} = (u_1^- \partial_t u_2^+ - u_2^+ \partial_t u_1^-) / W(u_1^+, u_2^+)$. From (3.1) we know that $W(u_1^+, u_2^+) \neq 0$ on $Z_e(M, N, \delta)$. Due to Corollary 2.2 the functions $u_{2,g}^+$ and $u_{1,g}^-$ depend holomorphically on $\zeta = \xi e^{i\varphi}$ in $Z_e(M, N, \delta)$. In the interior zone we define for $t \in [0, t_\xi]$ a new solution of (4.1) in the form

$$w(t, \varphi, \xi) = C_1(\varphi, \xi) u_1^+(t, \varphi, \xi) + C_2(\varphi, \xi) u_2^+(t, \varphi, \xi).$$

We want to assume that $w(+0, \varphi, \xi) = 1$ or 0 , $\partial_t w(+0, \varphi, \xi) = 0$ or 1 , respectively. For the Wronskian $W(w, u_{2,g}^+)$ we obtain

$$\begin{aligned}
 W(w, u_{2,g}^+) &= (C_1(\varphi, \xi) u_1^+(t, \varphi, \xi) + C_2(\varphi, \xi) u_{2,g}^+(t, \varphi, \xi)) \partial_t u_{2,g}^+(t, \varphi, \xi) \\
 &\quad - u_{2,g}^+(t, \varphi, \xi) (C_1(\varphi, \xi) \partial_t u_1^+(t, \varphi, \xi) + C_2(\varphi, \xi) \partial_t u_{2,g}^+(t, \varphi, \xi)) \\
 &= C_1(\varphi, \xi) W(u_1^+, u_{2,g}^+),
 \end{aligned}$$

where

$$C_1(\varphi, \xi) = \partial_t u_{2,g}^+(+0, \varphi, \xi) / W(u_1^+, u_2^+) \quad \text{if } w(+0, \varphi, \xi) = 1, \quad \text{or}$$

$$C_1(\varphi, \xi) = -u_{2,g}^+(+0, \varphi, \xi) / W(u_1^+, u_2^+) \quad \text{if } w(+0, \varphi, \xi) = 0.$$

From (1.13) and (1.14) we conclude that the new Stokes coefficient $T_{w,11}(b, \varphi, \xi) = T_{11}(b, \varphi, \xi) / C_1(\varphi, \xi)$. Using estimate (2.16) for $u_{2,g}^+(t_\xi, \varphi, \xi)$ we have

$$\mathcal{E}(u_{2,g}^+)(t = t_\xi) \geq C \langle \xi \rangle^{m_0}.$$

By the energy estimate from Corollary 2.3

$$\mathcal{E}(u_{2,g}^+)(t = 0) = |u_{2,g}^+(+0, \varphi, \xi)|^2 + |\partial_t u_{2,g}^+(+0, \varphi, \xi)|^2 \geq C \langle \xi \rangle^{m_1} \mathcal{E}(u_{2,g}^+)(t = t_\xi).$$

Let us fix an inner point $\zeta_0 = \xi_0 e^{i\varphi_0} \in Z_e(M, N, \delta)$. Using the last energy estimate then there exists a neighbourhood $U(\zeta_0)$ such that $|u_{2,g}^+(+0, \varphi, \xi)| \geq C > 0$ or $|\partial_t u_{2,g}^+(+0, \varphi, \xi)| \geq C > 0$ for all $(\varphi, \xi) \in U(\zeta_0)$. In the first case we choose $w(+0, \varphi, \xi) = 0$, in the second $w(+0, \varphi, \xi) = 1$. In $U(\zeta_0)$ we have $W(w, u_2^+) \neq 0$ and

$$T_{w,11}(b, \varphi, \xi) = (u_1^- \partial_t u_2^+ - u_2^+ \partial_t u_1^-) / W(w, u_2^+).$$

But $W(w, u_2^+)$ depends holomorphically on $\zeta \in U(\zeta_0)$. By Corollary 2.2 the Stokes coefficient $T_{w,11}(b, \varphi, \xi)$ depends holomorphically on $\zeta \in U(\zeta_0)$, too. Moreover, $T_{w,11}(b, \varphi_0, \xi) = 0$. This gives $T_{w,11}(b, \varphi, \xi) = 0$ in $U(\zeta_0)$. Consequently, $T_{11}(b, \varphi, \xi) = 0$ in $U(\zeta_0)$. Especially $T_{11}(b, \varphi, \xi_0) = 0$ for all $\varphi \in U_\varepsilon(\varphi_0)$, ε sufficiently small. From Corollary 2.1 we conclude that $T_{11}(b, \varphi, \xi)$ depends real analytically w.r.t. φ for $(\varphi, \xi_0) \in Z_e(M, N, \delta)$. Hence, $T_{11}(b, \varphi, \xi_0) = 0$ for the same set of φ . By the same reasoning $T_{11}(b, \varphi, \xi) = 0$ in $Z_e(M, N, \delta)$. This completes the proof.

4.3. Sufficient condition for local non-solvability

THEOREM 4.4. *Let L_{φ_0} , $\varphi_0 \in (-\pi/2, \pi/2)$, be the given differential operator. Let $Z_e(M, N, \delta)$ be such a zone that one of the Stokes coefficients $T_{11}(\pm \bar{b}, -\varphi_0, \xi) = 0$ for all ξ satisfying $(-\varphi_0, \xi) \in Z_e(M, N, \delta)$. Then L_{φ_0} is locally non-solvable at $(0, 0)$.*

PROOF. We restrict ourselves to the proof for $T_{11}(-\bar{b}, -\varphi_0, \xi) = 0$. If L_{φ_0} is locally solvable in $(0, 0)$, then $L_{\varphi_0} u = f$ has a solution $u \in \mathcal{D}'(\Omega)$ for every

$f \in C_0^\infty(\Omega)$. Here Ω is a subset of $[-1, 1] \times \mathbb{R}$ containing the origin. If ω is an open set with compact closure contained in ω , then

$$(4.19) \quad \left| \int_{\mathbb{R}^1} \int_{\mathbb{R}^1} f v \, dx \, dt \right| \leq C \sup_{\alpha+\beta \leq m} |D_x^\alpha D_t^\beta f| \sup_{\alpha+\beta \leq m} |D_x^\alpha D_t^\beta L_{\varphi_0}^* v|$$

for all $f, v \in C_0^\infty(\omega)$, see [11]. Functions which violate this inequality will be constructed.

For the construction we need the following auxiliary functions ($i = 1, 2$):

1) $t_i = t_i(\tau)$ are defined by the following equations:

$N_1 \ln \tau^2 = \Lambda(t_1)\tau^2$, $(N_1 + \tau^{-2}) \ln \tau^2 = \Lambda(t_2)\tau^2$, where $N < N_1$ and N is used for the definition of $Z_{e,ext}(M, N, \delta)$. It is clear, that $0 < t_1(\tau) < t_2(\tau)$, and $\lim_{\tau \rightarrow \infty} t_2(\tau) = 0$.

2) $F_{2,\tau} = F_{2,\tau}(t)$ are cut-off functions with $\text{supp } F_{2,\tau} \subset [t_1(\tau), t_2(\tau)]$ and $\int_{-\infty}^\infty F_{2,\tau}(t) \, dt = 1$.

3) $g_\tau = g(\tau\rho)$ are cut-off functions with $\text{supp } g_\tau \subset \tau[1, 2]$ and $\int_{-\infty}^\infty g(\tau\rho) \, d\rho = \tau$.

Let us choose the function

$$v_\tau(x, t) = \chi(x, t) \int_{\mathbb{R}^1} g(\tau\rho) e^{ix\rho\tau^2} \frac{\hat{u}(t, -\varphi_0, \rho\tau^2)}{T_{12}(-\bar{b}, -\varphi_0, \rho\tau^2)} \, d\rho,$$

where

$$\hat{u}(t, -\varphi_0, \rho\tau^2) = \begin{cases} \chi_+(-\varphi_0, \rho\tau^2) u_1^{-c}(t, -\varphi_0, \rho\tau^2) = \\ \chi_+(-\varphi_0, \rho\tau^2) T_{12}(-\bar{b}, -\varphi_0, \rho\tau^2) e^{-\Lambda(t)\rho\tau^2 e^{-i\varphi_0}} a_{2,\tau}^+(t, -\varphi_0, \rho\tau^2), & t \geq 0, \\ \chi_+(-\varphi_0, \rho\tau^2) u_1^-(t, -\varphi_0, \rho\tau^2) = \\ \chi_+(-\varphi_0, \rho\tau^2) e^{\Lambda(t)\rho\tau^2 e^{-i\varphi_0}} a_{1,\tau}^-(t, -\varphi_0, \rho\tau^2), & t \leq 0. \end{cases}$$

The cut-off function $\chi \in C_0^\infty(\omega)$ satisfies $\chi = 1$ in a neighbourhood of $(0, 0)$. Using $T_{11}(-\bar{b}, -\varphi_0, \xi) = 0$, (1.13) and (1.14) we have $\partial_l^1 u_1^{-c} = T_{12} \partial_l^1 u_2^+$, $l = 0, 1$. By Lemma 2.6 and (2.30) we conclude that

$$|T_{12}(-\bar{b}, -\varphi_0, \rho\tau^2)|^2 = \frac{E(u_1^{-c})(+0, -\varphi_0, \rho\tau^2)}{E(u_2^+)(+0, -\varphi_0, \rho\tau^2)} \geq C(\rho\tau^2)^{m_0}$$

with some real constant m_0 . The assumptions concerning g and the fact that $T_{12}(-\bar{b}, -\varphi_0, \xi)$ has polynomial growth in $Z_c(M, N, \delta)$ imply that $v_\tau \in C_0^\infty(\omega)$, too. Moreover let us choose

$$f_\tau(x, t) = F_{1,\tau}(\tau^2 x) F_{2,\tau}(t),$$

where $F_{1,\tau} = F_{1,\tau}(\tau^2 x)$ are cut-off functions which support we shall fix later and $\int_{-\infty}^{\infty} F_{1,\tau}(\tau^2 x) dx = \tau^2$. Then

$$\int_{R^1} \int_{R^1} f_{\tau} v_{\tau} dx dt = \int_{R^1} \int_{R^1} \int_{R^1} g(\tau \rho) F_{1,\tau}(\tau^2 x) F_{2,\tau}(t) \chi(x, t) e^{i(x + \Lambda(t) \sin \varphi_0) \rho \tau^2} \times e^{-\Lambda(t) \rho \tau^2 \cos \varphi_0} a_{2,\tau}^+(t, -\varphi_0, \rho \tau^2) \chi_+(-\varphi_0, \rho \tau^2) d\rho dx dt.$$

From construction in section 2.1 we have

$$a_{2,\tau,0}^+(t, -\varphi_0, \rho \tau^2) = e^{\int_t^T \frac{\lambda'(s) + \operatorname{Re} \overline{b(s)}}{2\lambda(s)} ds + i \int_t^T \frac{\operatorname{Im} \overline{b(s)}}{2\lambda(s)} ds}.$$

Using that

$$\left| \sum_{k=1}^{\infty} a_{2,\tau,k}^+(t, -\varphi_0, \rho \tau^2) \right| \leq \frac{C}{\Lambda(t) \rho \tau^2} \leq \frac{C}{N_1 \ln \tau^2}$$

in the zone $N_1 \ln \tau^2 \leq \Lambda(t) \tau^2 \leq (N_1 + \tau^{-2}) \ln \tau^2$ it is enough to consider $a_{2,\tau,0}^+(t, -\varphi_0, \rho \tau^2)$. This term of formal asymptotic solution majorizes the others. Consequently,

$$\left| \int_{R^1} \int_{R^1} f_{\tau} v_{\tau} dx dt \right| \geq \frac{1}{2} \left| e^{i \int_{t_1(\tau)}^T \frac{\operatorname{Im} \overline{b(s)}}{2\lambda(s)} ds} \int_{R^1} \int_{R^1} \int_{R^1} g(\tau \rho) F_{1,\tau}(\tau^2 x) F_{2,\tau}(t) \chi(x, t) \times e^{i(x + \Lambda(t) \sin \varphi_0) \rho \tau^2} e^{i \int_t^{t_1(\tau)} \frac{\operatorname{Im} \overline{b(s)}}{2\lambda(s)} ds} \times e^{-\Lambda(t) \rho \tau^2 \cos \varphi_0} e^{\int_t^T \frac{\lambda'(s) + \operatorname{Re} \overline{b(s)}}{2\lambda(s)} ds} \chi_+(-\varphi_0, \rho \tau^2) d\rho dx dt \right|.$$

The new variable z is defined by $z = \Lambda(t) \sin \varphi_0$ runs through an interval $[z_1(\tau), z_2(\tau)]$ having length $o(\tau^{-3})$. This follows from

$$|(\Lambda(s_1) - \Lambda(s_2)) \sin \varphi_0| \leq \tau^{-4} \ln \tau^2$$

for all $s_1, s_2 \in [t_1(\tau), t_2(\tau)]$. Now let us choose $x \in [x_2(\tau), x_1(\tau)]$ with $x_i(\tau) = -z_i(\tau)$, $i = 1, 2$, and suppose that the support of $F_{1,\tau}(\tau^2 x)$ is contained in $\tau^2 [x_2(\tau), x_1(\tau)]$. Using $\lim_{\tau \rightarrow \infty} z_i(\tau) = 0$ for $i = 1, 2$ the same holds for $\lim_{\tau \rightarrow \infty} x_i(\tau) = 0$, too. Consequently, $[t_1(\tau), t_2(\tau)] \times [x_2(\tau), x_1(\tau)]$ belongs to ω if τ is large. But this guarantees that $\chi(x, t) = 1$ on $[x_2(\tau), x_1(\tau)] \times [t_1(\tau), t_2(\tau)]$ and $\chi_+(-\varphi_0, \rho \tau^2) = 1$ if τ is large. The Levi condition (1.5) implies

$$\left| \int_t^{t_1(\tau)} \frac{\operatorname{Im} \overline{b(s)}}{2\lambda(s)} ds \right| \leq C \ln \frac{\Lambda(t_2(\tau))}{\Lambda(t_1(\tau))} = C \ln (1 + (\tau^2 N_1)^{-1})$$

for all $t \in [t_1(\tau), t_2(\tau)]$. Hence,

$$\lim_{\tau \rightarrow \infty} e^{i \int_t^{t_1(\tau)} \frac{\text{Im } \overline{b(s)}}{2\lambda(s)} ds} = 1$$

for all $t \in [t_1(\tau), t_2(\tau)]$. Using the substitution $y = x + \Lambda(t) \sin \varphi_0$ the new variable y runs through an interval $[y_1(\tau), y_2(\tau)]$ around the origin with length $o(\tau^{-3})$. Hence,

$$\lim_{\tau \rightarrow \infty} e^{i(x + \Lambda(t) \sin \varphi_0) \rho \tau^2} = \lim_{\tau \rightarrow \infty} e^{iy \rho \tau^2} = 1$$

for all $y \in [y_1(\tau), y_2(\tau)]$ and $\rho \in [1, 2]$. It remains to estimate

$$e^{-\Lambda(t) \rho \tau^2 \cos \varphi_0} e^{\int_t^T \frac{\lambda'(s) + \text{Re } \overline{b(s)}}{2\lambda(s)} ds}$$

From definition of $t_i(\tau)$ and Levi condition (1.5) for $b = b(t)$ we conclude

$$\begin{aligned} e^{-\Lambda(t) \rho \tau^2 \cos \varphi_0} e^{\int_t^T \frac{\lambda'(s) + \text{Re } \overline{b(s)}}{2\lambda(s)} ds} &\geq C_1 e^{-2(N_1 + \tau^{-2}) \ln \tau^2 \cos \varphi_0} e^{c \ln \Lambda(t_1(\tau))} \\ &\geq C_1 \tau^{-5N_1 \cos \varphi_0} \left(\frac{N_1 \ln \tau^2}{\tau^2} \right)^c \end{aligned}$$

for all $t \in [t_1(\tau), t_2(\tau)]$ and $\rho \in [1, 2]$.

Using $\int_{-\infty}^{\infty} F_{2,\tau}(t) dt = 1$, $\int_{-\infty}^{\infty} g(\tau \rho) d\rho = \tau$ and $\int_{-\infty}^{\infty} F_{1,\tau}(x \tau^2) dx = \int_{-\infty}^{\infty} F_{1,\tau}(\tau^2(y - \Lambda(t) \sin \varphi_0)) dy = \tau^2$ then we obtain

$$(4.20) \quad \left| \int_{R^1} \int_{R^1} f_\tau v_\tau dx dt \right| \geq C \tau^{m_0}$$

with a suitable real constant m_0 . Then it is easy to see that the function

$$w_\tau(x, t) = \int_{R^1} g(\tau \rho) e^{ix \rho \tau^2} \hat{u}(t, -\varphi_0, \rho \tau^2) d\rho$$

solves the equation $L_{\varphi_0}^* w_\tau(x, t) = 0$ in a neighbourhood of the origin. Therefore,

$$L_{\varphi_0}^* \chi w_\tau = [L_{\varphi_0}^*, \chi] w_\tau + \chi L_{\varphi_0}^* w_\tau = \zeta(t, x) Q(t, x, D_t, D_x) w_\tau(x, t),$$

where $\zeta(t, x) = 0$ inside of some neighbourhood of $(0, 0)$ of the form $[-\varepsilon, \varepsilon]^2$. The inequality

$$(4.21) \quad \sup_{\alpha + \beta \leq m} \sum |D_x^\alpha D_t^\beta L_{\varphi_0}^* v_\tau| \leq C_N \tau^{-N}$$

follows from

$$|D_x^\alpha D_t^\beta w_\tau(x, t)| \leq C_N \tau^{-N}$$

for all (x, t) belonging to $\text{supp } \zeta$. If $t \geq \varepsilon$, then for all x

$$D_x^\alpha D_t^\beta w_\tau(x, t) = (-i)^\alpha \int_1^2 g(\tau\rho)(\tau^2\rho)^\alpha e^{ix\tau^2\rho} D_t^\beta (e^{-\Lambda(t)\rho\tau^2 e^{-i\varphi_0}} a_{2,\tau}^+(t, -\varphi_0, \rho\tau^2)) d\rho.$$

It follows for $\alpha + \beta \leq m$

$$|D_x^\alpha D_t^\beta w_\tau(x, t)| \leq C_m \int_1^2 g(\tau\rho)(1 + 2\tau^2)^{\alpha+\beta+m_2^+} e^{-\Lambda(\varepsilon)2\tau^2 \cos \varphi_0} d\rho \leq C_{N,m} \tau^{-N}$$

for every N . Furthermore if $|x| \geq \varepsilon$ and $0 \leq t \leq \varepsilon$ then for every k

$$\begin{aligned} & |\tau^2 x|^k |D_x^\alpha D_t^\beta w_\tau(x, t)| \\ & \leq \left| \int_1^2 g(\tau\rho)(\tau^2\rho)^\alpha \left(\left(\frac{\partial}{\partial \rho} \right)^k e^{ix\tau^2\rho} \right) D_t^\beta (e^{-\Lambda(t)\rho\tau^2 e^{-i\varphi_0}} a_{2,\tau}^+(t, -\varphi_0, \rho\tau^2)) d\rho \right| \\ & = \tau^{2\alpha} \left| \int_1^2 e^{ix\tau^2\rho} \left(\frac{\partial}{\partial \rho} \right)^k (\rho^\alpha g(\tau\rho)) D_t^\beta (e^{-\Lambda(t)\rho\tau^2 e^{-i\varphi_0}} a_{2,\tau}^+(t, -\varphi_0, \rho\tau^2)) d\rho \right| \\ & \leq C_{k,m} \tau^{2\alpha+2\beta+2m_2^++k}, \end{aligned}$$

respectively,

$$|D_x^\alpha D_t^\beta w_\tau(x, t)| \leq C_{k,m} \tau^{2\alpha+2\beta+2m_2^+-k} \leq C_{k,m} \tau^{2m+2m_2^+-k}$$

for all $\alpha, \beta \leq m$ and all k . It follows the desired inequality (4.21). Comparing (4.20) with (4.21) the inequality (4.19) does not hold for f_τ and v_τ . In the same way we prove the statement for $T_{11}(\bar{b}, -\varphi_0, \xi) = 0$.

THEOREM 4.5. *If for a given $\varphi_0 \in (-\pi/2, \pi/2)$ one of the Stokes coefficients $T_{11}(\pm \bar{b}, -\varphi_0, \xi)$ vanishes for all ξ with $(-\varphi_0, \xi) \in Z_e(M, N, \delta)$, then all operators $L_\varphi, \varphi \in (-\pi/2, \pi/2)$, are locally non-solvable at $(0, 0)$.*

PROOF. It follows immediately by Theorems 4.3 and 4.4.

4.4. Sufficient condition for non-hypoellipticity

In the previous section we have discussed local non-solvability of L_{φ_0} if one of the Stokes coefficients $T_{11}(\pm \bar{b}, -\varphi_0, \xi) = 0$ for all $(-\varphi_0, \xi) \in Z_e(M, N, \delta)$. In this section we shall prove a result of non-hypoellipticity under weaker conditions concerning $T_{11}(\pm b, \varphi_0, \xi)$.

THEOREM 4.6. *Let $Z_e(M, N, \delta)$ be such a zone that the Stokes coefficient T_{11} satisfies*

$$(4.22) \quad |T_{11}(b, \varphi_0, \xi)| \leq h(\varphi_0, \xi)e^{-2\Lambda(T)\xi} \quad \text{or} \quad |T_{11}(-b, \varphi_0, \xi)| \leq h(\varphi_0, \xi)e^{-2\Lambda(T)\xi},$$

for all ξ with $(\varphi_0, \xi) \in Z_e(M, N, \delta)$. Here $h(\varphi_0, \xi)$ is a C^∞ -function having polynomial growth together with all derivatives. Then L_{φ_0} is non-hypoelliptic at $(0, 0)$.

PROOF. We restrict ourselves to the proof for $T_{11}(b, \varphi_0, \xi)$. Let us choose the solution

$$(4.23) \quad \hat{u}(t, \varphi_0, \xi) = \begin{cases} \chi_+(\varphi_0, \xi)u_1^{-c}(t, \varphi_0, \xi) = \chi_+(\varphi_0, \xi)(T_{11}(b, \varphi_0, \xi)e^{\xi e^{\varphi_0}\Lambda(t)}a_1^+ + T_{12}(b, \varphi_0, \xi)e^{-\xi e^{\varphi_0}\Lambda(t)}a_2^+), & t \geq 0, \\ \chi_+(\varphi_0, \xi)u_1^-(t, \varphi_0, \xi) = \chi_+(\varphi_0, \xi)e^{\xi e^{\varphi_0}\Lambda(t)}a_1^-, & t \leq 0. \end{cases}$$

Here $\chi_+ = \chi_+(\varphi_0, \xi)$ is a C^∞ -function such that $\chi_+(\varphi_0, \xi) = 0$ if $(\varphi_0, \xi) \notin Z_e(M, N, \delta)$ and $\chi_+(\varphi_0, \xi) = 1$ for $\xi \geq M(\varphi_0)$.

The condition concerning T_{11} and the fact that T_{12} has polynomial growth in $Z_e(M, N, \delta)$ imply

$$(4.24) \quad \hat{u}(t, \varphi_0, \xi) = \begin{cases} \chi_+(\varphi_0, \xi)e^{-\xi e^{\varphi_0}\Lambda(t)}h_1^+, & t \geq 0, \\ \chi_+(\varphi_0, \xi)e^{\xi e^{\varphi_0}\Lambda(t)}a_1^-, & t \leq 0, \end{cases}$$

where h_1^+ has polynomial growth together with all derivatives. We see that for all $t \neq 0$ $\hat{u}(t, \varphi_0, \xi)$ and $\partial_t \hat{u}(t, \varphi_0, \xi)$ belong to S . Therefore its partial Fourier transform u is a solution of (0.3). Our goal is to show that under the conditions of the theorem the point $(0, 0) \in \text{sing supp } u$.

For this reason let us choose a function $\psi(x) \in C_0^\infty(R^1)$ with $\psi = 1$ in a neighbourhood $(-\varepsilon, \varepsilon)$ of $x = 0$. Then $(\widehat{\psi u})(t = 0) = (\widehat{\psi} * \hat{u})(t = 0) = \int_{R^1} \widehat{\psi}(\xi)\hat{u}(-0, \varphi_0, \eta - \xi) d\xi = \int_{R^1} \widehat{\psi}(\xi)\chi_+(\varphi_0, \xi)a_1^-(-0, \varphi_0, \eta - \xi) d\xi$. Analogously, $\partial_t(\widehat{\psi u})(t = 0) = \int_{R^1} \widehat{\psi}(\xi)\chi_+(\varphi_0, \xi)a_{1,t}^-(-0, \varphi_0, \eta - \xi) d\xi$. The integrands belong to C^∞ w.r.t. ξ . Consequently,

$$\begin{aligned} (\widehat{\psi u})(t = 0) &= \int_{R^1} \widehat{\psi}(\xi) \sum_{k=0}^{N-1} \frac{1}{k!} (\chi_+ a_1^-)^{(k)}(-0, \varphi_0, \eta) (-\xi)^k d\xi \\ &+ \frac{1}{N!} \int_{R^1} \widehat{\psi}(\xi) \int_0^1 (\chi_+ a_1^-)^{(N+1)}(-0, \varphi_0, \eta - \rho\xi) d\rho (-\xi)^{N+1} d\xi. \end{aligned}$$

Now we have $\psi(x) = \int_{R^1} e^{ix\xi} \hat{\psi}(\xi) d\xi = 1$ and $\psi^{(k)}(x) = \int_{R^1} e^{ix\xi} (i\xi)^k \hat{\psi}(\xi) d\xi = 0$ for all $x \in (-\varepsilon, \varepsilon)$. Consequently, $\int_{R^1} \hat{\psi}(\xi) d\xi = 1$ and $\int_{R^1} \xi^k \hat{\psi}(\xi) d\xi = 0$ for all $k \in N_0$. With this property we conclude

$$\begin{aligned} (\widehat{\psi u})(t = 0) &= (\chi_+ a_1^-)(-0, \varphi_0, \eta) \\ &\quad + \frac{1}{N!} \int_{R^1} \hat{\psi}(\xi) (-\xi)^{N+1} \int_0^1 (\chi_+ a_1^-)^{(N+1)}(-0, \varphi_0, \eta - \rho\xi) d\rho d\xi. \end{aligned}$$

Now let us estimate the inner integral of the second term. We have from (2.3)

$$|(\chi_+ a_1^-)^{(N+1)}(-0, \varphi_0, \eta - \rho\xi)| \leq C_{N+1} (1 + |\eta - \rho\xi|^2)^{1/2(m_- - N - 1)}$$

for all $\eta \geq M(\varphi_0)$, $\rho \in (0, 1)$ and $\xi \in R^1$. Using Petree's inequality [13], that is

$$(1 + |\eta - \rho\xi|^2)^{1/2(m_- - N - 1)} \leq (1 + |\eta|^2)^{1/2(m_- - N - 1)} (1 + |\rho\xi|^2)^{1/2(m_- - N - 1)},$$

we conclude that if m_{φ_0} satisfies $m_- - N - 1 \leq m_{\varphi_0} - 1$, then

$$\begin{aligned} &\frac{1}{N!} \int_{R^1} \hat{\psi}(\xi) (-\xi)^{N+1} \int_0^1 (\chi_+ a_1^-)^{(N+1)}(-0, \varphi_0, \eta - \rho\xi) d\rho d\xi \\ &\leq C \langle \eta \rangle^{\frac{1}{2}(m_{\varphi_0} - 1)} \int_{-\infty}^{\infty} |\hat{\psi}(\xi)| |\langle \xi \rangle^{N + |m_{\varphi_0} - 1|} d\xi \\ &\leq C \langle \eta \rangle^{\frac{1}{2}(m_{\varphi_0} - 1)} \end{aligned}$$

for all $\eta \geq M(\varphi_0)$ because of $\hat{\psi}(\xi) \in S$. But this gives

$$|(\widehat{\psi u})(t = 0)| \geq |a_1^-(-0, \varphi_0, \eta)| - C \langle \eta \rangle^{\frac{1}{2}(m_{\varphi_0} - 1)}.$$

In the same way we derive

$$|\partial_t (\widehat{\psi u})(t = 0)| \geq |\partial_t a_1^-(-0, \varphi_0, \eta)| - C \langle \eta \rangle^{\frac{1}{2}(m_{\varphi_0} - 1)}$$

for all $\eta \geq M(\varphi_0)$. Hence,

$$|(\widehat{\psi u})(t = 0)| + |\partial_t (\widehat{\psi u})(t = 0)| \geq |a_1^-(-0, \varphi_0, \eta)| + |\partial_t a_1^-(-0, \varphi_0, \eta)| - C \langle \eta \rangle^{\frac{1}{2}(m_{\varphi_0} - 1)}$$

for all $\eta \geq M(\varphi_0)$. Using, finally, the energy estimate (2.30) from Corollary 2.3 we have

$$\mathcal{E}(u_1^-)(t = 0) = |a_1^-(-0, \varphi_0, \eta)|^2 + |\partial_t a_1^-(-0, \varphi_0, \eta)|^2 \geq C \langle \eta \rangle^{m_{\varphi_0}}$$

for all $\eta \geq M(\varphi_0)$. Hence,

$$|(\widehat{\psi u})(t=0)| + |\partial_t(\widehat{\psi u})(t=0)| \geq C\langle \eta \rangle^{\frac{1}{2}m_{\varphi_0}}.$$

This is a contradiction to $u, \partial_t u \in C^\infty$ in $(0,0)$. The theorem can be proved for $T_{11}(-b, \varphi_0, \xi)$ in a similar way.

THEOREM 4.7. *If for a given $\varphi_0 \in (-\pi/2, \pi/2)$ one of the Stokes coefficients $T_{11}(\pm b, \varphi_0, \xi)$ vanishes for all ξ with $(\varphi_0, \xi) \in Z_e(M, N, \delta)$, then all operators $L_\varphi, \varphi \in (-\pi/2, \pi/2)$, are non-hypoelliptic at $(0,0)$.*

PROOF. It follows immediately by Theorems 4.3 and 4.6.

4.5. On the Stokes matrix for our starting example

We have formulated by the aid of Stokes coefficients sufficient conditions for hypoellipticity, local solvability, local non-solvability and non-hypoellipticity. These conditions are basing on the representations (2.2) with suitable properties of Theorem 2.1. These representations we have obtained by construction in section 2.

Of course, all above results hold if one can propose another construction principle which leads to representations (2.2) with the same properties. For example, for (0.1) a construction is given by theory of special functions in [17]. Indeed, the calculation of Stokes coefficients for (0.1) basing on the representations

$$\begin{aligned} u_1^+(t, \varphi, \xi) &= e^{\xi e^{i\varphi} e^{-1/|t|}} t\Psi(\alpha_+, 1; -2\xi e^{i\varphi} e^{-1/|t|}), \\ u_2^+(t, \varphi, \xi) &= e^{-\xi e^{i\varphi} e^{-1/|t|}} t\Psi(1 - \alpha_+, 1; 2\xi e^{i\varphi} e^{-1/|t|}), \\ u_1^-(t, \varphi, \xi) &= e^{-\xi e^{i\varphi} e^{-1/|t|}} t\Psi(1 - \alpha_-, 1; 2\xi e^{i\varphi} e^{-1/|t|}), \\ u_2^-(t, \varphi, \xi) &= e^{\xi e^{i\varphi} e^{-1/|t|}} t\Psi(\alpha_-, 1; -2\xi e^{i\varphi} e^{-1/|t|}), \end{aligned}$$

where $\alpha_+ = (1 + b_+)/2$, $\alpha_- = (1 + b_-)/2$ and

$$\Psi(\alpha, 1; z) = \frac{1}{2i\pi} e^{-i\pi\alpha} \Gamma(1 - \alpha) \int_{\infty e^{i\varphi}}^{(0+)} e^{-zt} t^{\alpha-1} (1+t)^{-\alpha} dt$$

$(-\pi/2 < \varphi + \arg z < \pi/2, \arg t = \varphi$ at the starting point, $\Gamma(\alpha)$ is Euler's function [3], Ψ is a solution of the confluent hypergeometric equation), leads to the following results:

a) under the assumptions of (A1)

$$|T_{11}(b, \varphi, \xi)| \geq C_\varphi \langle \xi \rangle^{-1}, \quad |T_{11}(-b, \varphi, \xi)| \geq C_\varphi \langle \xi \rangle^{-1},$$

$$|T_{11}(\bar{b}, -\varphi, \xi)| \geq C_\varphi \langle \xi \rangle^{-1}, \quad |T_{11}(-\bar{b}, -\varphi, \xi)| \geq C_\varphi \langle \xi \rangle^{-1},$$

b) if $b_+ = 2n + 1$, $b_- = 2l + 1$, where n and l are non-negative integer, then $T_{11}(b, \varphi, \xi) = 0$ for all $(\varphi, \xi) \in (-\pi/2, \pi/2) \times (0, \infty)$,

c) if $b_+ = -2n - 1$, $b_- = -2l - 1$, where n and l are non-negative integer, then $T_{11}(-b, \varphi, \xi) = 0$ for all $(\varphi, \xi) \in (-\pi/2, \pi/2) \times (0, \infty)$.

Consequently, the application of Theorems 4.2, 4.4, 4.6 and Corollary 4.2 gives us:

- 1) under the assumptions of (A1) every operator P_φ is hypoelliptic as well as locally solvable at $(0, 0)$,
- 2) under the assumptions of (A2) every operator P_φ is locally non-solvable as well as non-hypoelliptic at $(0, 0)$,
- 3) the statement of Theorem 4.3 holds in the exceptional cases b) and c) because of $T_{11}(b, \varphi, \xi) = 0$ or $T_{11}(-b, \varphi, \xi) = 0$ for all $(\varphi, \xi) \in (-\pi/2, \pi/2) \times (0, \infty)$.

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