TSUKUBA J. MATH.
Vol. 22 No. 2 (1998), 305–331

NORMAL FORMS FOR DERIVATIONS IN ARAI'S AI_{ξ}^{-}

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Abstract. In this paper, we shall consider normal forms for derivations in AI_{ξ}^{-}, where AI_{ξ}^{-} is a system introduced by Arai (cf. [4]) and its consistency implies the consistency of Feferman's ID_{ξ} (cf. [6]). We shall give two normal form theorems for derivations in AI_{ξ}^{-}. One (Theorem 1) implies the consistency of AI_{ξ}^{-}. The other (Theorem 2) implies the ω-consistency of AI_{ξ}^{-}.

0. Introduction

In this paper, we shall consider normal forms for derivations in AI_{ξ}^{-}, where AI_{ξ}^{-} is a system introduced by Arai (cf. [4]) and its consistency implies the consistency of Feferman's ID_{ξ} (cf. [6]).

Normal forms for derivations in LK have been studied by several authors (for example, Gentzen [7], Mints [10], Arai and Mints [5]). Gentzen's cut elimination theorem (cf. [7], [11]) is one of the most famous normal form theorems for derivations in LK. In [10], Mints gave an extended form of Gentzen's theorem. Moreover, extended forms of Mints' theorem were given by Arai and Mints (cf. [5]).

And also, normal forms for derivations in arithmetic formalized in the sequent style have been studied by several authors (for instance, Hinata [8], the author [9]). Hinata's theorem (cf. [8]) is considered as an analogue of Gentzen's theorem and implies the consistency of arithmetic. In [9], the author gave an extended form of Hinata's theorem, which is also considered as an analogue of Mints' theorem and implies the ω-consistency of arithmetic.

In this paper, we shall give some normal form theorems for derivations in AI_{ξ}^{-}. To prove these theorems, Takeuti's system of ordinal diagrams $O(\xi+1,2)$

Received October 31, 1996
Revised March 13, 1997

(cf. [11]) will be used. $O(\xi+1,2)$ is the structure consisting of the set of objects called *ordinal diagrams* and the well-orderings $<_i$ $(i \in I)$ over the ordinal diagrams, where I is the well-ordering set $(\xi+1) \cup \{\infty\}$, whose ordering is that of $\xi+1$ with the largest element ∞.

In [1] and [4], Arai showed that the consistency of AI_ξ^- can be proved by transfinite induction along $<_0$ up to the ordinal diagram $(\xi,1,0)$ but can not be proved by transfinite induction along $<_0$ up to any α, where $\alpha <_0 (\xi,1,0)$.

So, we want to give a normal form theorem for derivations in AI_ξ^-, which implies the fact that the consistency of AI_ξ^- can be proved by transfinite induction along $<_0$ up to the ordinal diagram $(\xi,1,0)$. Theorem 1 given in Section 2 below is just such a theorem. Moreover, it is considered as an analogue of Hinata's theorem (cf. [8]). Furthermore, we shall give another normal form theorem (Theorem 2) for derivations in AI_ξ^- in Section 2 below. It implies the ω-consistency of AI_ξ^- and is proved by transfinite induction along $<_0$ up to the ordinal diagram $(\xi,1,0\#0)$. Moreover, it is considered as an analogue of author's theorem (cf. [9]).

1. The system AI_ξ^-

The system considered here is obtained from Arai's original AI_ξ^- (cf. [3], [4]) by some modifications. In this section, we explain the system AI_ξ^- in detail.

DEFINITION 1.1. The language $\mathscr{L}$ is the first order language whose non-logical symbols consist of the following symbols:

1. Individual constant: 0;
2. Function constant: $'$ (successor) and $\bar{f}$ for each primitive recursive function f;
3. Predicate constant: $=$.

The language $\mathscr{L}+\{Y_0,Y_1,c_0,c_1\}$ is the language obtained from $\mathscr{L}$ by adding a unary predicate variable Y_0 and a binary predicate variable Y_1 and individual constants c_0 and c_1.

Let ξ be a fixed ordinal and let $\prec$ be a primitive recursive well-ordering on a primitive recursive subset of the set of natural numbers and $\lambda x \cdot x \oplus 1$ a primitive recursive successor function with respect to $\prec$. We assume that the order type of $\prec$ is $\xi+1$ and the least element of $\prec$ is the natural number 0. Moreover, we assume the same properties with respect to $\prec$ and $\oplus$ as ones assumed in [4]. We denote the largest element of $\prec$ by ξ. Furthermore, "ξ" is also used to denote the numeral corresponding to the largest element with respect to $\prec$. Let $f_\prec$ be the characteristic function of $\prec$. Then, to denote the formula "$\bar{f}_{\prec}(s,t)=0$", we use the expression "$s \prec t$".

Let t be a closed term in $\mathscr{L}$. Then $v(t)$ is used to denote the value of t under the standard interpretation.

DEFINITION 1.2. A formula $\mathfrak{B}(Y_0, Y_1, c_0, c_1)$ in $\mathscr{L} + \{Y_0, Y_1, c_0, c_1\}$ is said to be an *arithmetical form* if it includes no free individual variables.

DEFINITION 1.3. The language $\mathscr{L}'$ is the language obtained from $\mathscr{L}$ by adding unary predicate variables $X_i (i \in \omega)$ and adding binary predicate constants $Q^{\mathfrak{B}}$ and ternary predicate constants $Q^{\mathfrak{B}}_{\prec}$ for each arithmetical form $\mathfrak{B}$ in $\mathscr{L} + \{Y_0, Y_1, c_0, c_1\}$. We write $Q^{\mathfrak{B}}_{\prec u}ts$ for $Q^{\mathfrak{B}}_{\prec}uts$.

DEFINITION 1.4. AI^{-}_{ξ} is a system formalized in the language $\mathscr{L}'$ and consists of the following initial sequents and inference rules:

1. Initial sequents

(a) Logical initial sequents:

$$D \to D, \text{ where } D \text{ is an arbitrary atomic formula.}$$

(b) Mathematical initial sequents:

The sequents which consist of atomic formulas in $\mathscr{L}$ and are true under the standard interpretation.

2. Inference rules

(a) Inference rules of LK without inference rules for $\supset$.

(b) Cut:

$$\frac{\Gamma \to \Delta, D \quad D, \Lambda \to \Pi}{\Gamma, \Lambda \to \Delta, \Pi}$$

D is called the *cut formula* of this inference. This inference is said to be *inessential* if its cut formulas are of the form $Q^{\mathfrak{B}}ts$ and include at least one free individual variable.

(c) Inference rules for $\supset$:

$\supset$: left $\qquad\qquad$ $\supset$: right

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta} \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, A \supset B} \quad \text{and} \quad \frac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}$$

(d) Term-replacement:

$$\frac{\Gamma(s) \to \Delta(s)}{\Gamma(t) \to \Delta(t)}$$

s and t are closed terms such that $v(s) = v(t)$

This inference is considered as a structural inference.

(e) Equality rule:

$$\frac{\Gamma \to \Delta, t = s \quad \Gamma \to \Delta, F(t) \quad F(s), \Gamma \to \Delta}{\Gamma \to \Delta}$$

t and s are arbitrary terms

$t = s$, $F(t)$ and $F(s)$ are called the *auxiliary formulas* and also $F(t)$ and $F(s)$ are called the *equality formulas*. This inference is said to be *inessential* if $t = s$ includes at least one free individual variable and $F(t)$ is not identical with $F(s)$.

(f) Induction rule:

$$\frac{\Gamma \to \Delta, A(0) \quad A(a), \Gamma \to \Delta, A(a') \quad A(t), \Gamma \to \Delta}{\Gamma \to \Delta}$$

a does not occur in the lower sequent and t is an arbitrary term

$A(0)$, $A(a)$, $A(a')$ and $A(t)$ are called the *auxiliary formulas* and also $A(a)$ is called the *induction formula*. a and t are said to be the *eigenvariable* and the *induction term*, respectively. This inference is said to be *constant normal* if its induction formula contains at least one occurrence of its eigenvariable and its induction term contains at least one free individual variable.

(g) Inference rules for $Q^{\mathfrak{B}}$:

$Q^{\mathfrak{B}}$: left

$$\frac{\Gamma \to \Delta, t \prec \xi \quad \mathfrak{B}(V, Q^{\mathfrak{B}}_{\prec t}, t, s), \Gamma \to \Delta}{Q^{\mathfrak{B}}ts, \Gamma \to \Delta}$$

V is an arbitrary unary abstract and t, s are arbitrary terms

$Q^{\mathfrak{B}}$: right

$$\frac{\Gamma \to \Delta, t \prec \xi \quad \Gamma \to \Delta, \mathfrak{B}(X, Q^{\mathfrak{B}}_{\prec t}, t, s)}{\Gamma \to \Delta, Q^{\mathfrak{B}}ts}$$

X does not occur in the lower sequent and t, s are arbitrary terms

In $Q^{\mathfrak{B}}$: left, $t \prec \xi$ and $\mathfrak{B}(V, Q^{\mathfrak{B}}_{\prec t}, t, s)$ are called the *auxiliary formulas* and $Q^{\mathfrak{B}}ts$ is called the *principal formula*. In $Q^{\mathfrak{B}}$: right, $t \prec \xi$ and $\mathfrak{B}(X, Q^{\mathfrak{B}}_{\prec t}, t, s)$ are called the *auxiliary formulas*, $Q^{\mathfrak{B}}ts$ is called the *principal formula* and X is called the *eigenvariable* of this inference.

(h) Infernece rules for $Q^{\mathfrak{B}}_{\prec}$:

$Q^{\mathfrak{B}}_{\prec}$: left

$$\frac{t \prec u, \Gamma \to \Delta}{Q^{\mathfrak{B}}_{\prec u}ts, \Gamma \to \Delta} \quad \text{and} \quad \frac{Q^{\mathfrak{B}}ts, \Gamma \to \Delta}{Q^{\mathfrak{B}}_{\prec u}ts, \Gamma \to \Delta}$$

s, t and u are arbitrary terms

$Q^{\mathfrak{B}}_{\prec}$: right

$$\frac{\Gamma \to \Delta, t \prec u \quad \Gamma \to \Delta, Q^{\mathfrak{B}}ts}{\Gamma \to \Delta, Q^{\mathfrak{B}}_{\prec u}ts}$$

s, t and u are arbitrary terms

$t \prec u$ and $Q^{\mathfrak{B}}ts$ are called the *auxiliary formulas* and $Q^{\mathfrak{B}}_{\prec u}ts$ is called the *principal formula*.

2. Normal form theorems and their applications

In this section, we explain our normal form theorems and their applications. First of all, we give definitions necessary to state our theorems.

Definition 2.1. Let Γ be a sequence $A_1, \dots, A_n$ of formulas. Let $\langle i_1, i_2, \dots, i_k \rangle$ be a sequence of natural numbers such that $1 \le i_1 < i_2 < \cdots < i_k \le n$. Then, the sequence $A_{i_1}, \dots, A_{i_k}$ is called a *part* of Γ. Γ^* is used to denote a part of Γ. Let $\Lambda \to \Pi$ be a sequent. Then $\Lambda^* \to \Pi^*$ is called a *part* of $\Lambda \to \Pi$.

Definition 2.2. Let π be a derivation with the end sequent S in AI^{-}_{ξ}. And let S^* be a part of S and C a formula in π. Then C is said to be (S^*)-*implicit* if a descendant (cf. [11]) of C satisfies one of the following conditions:

1. It is a cut formula.
2. It is an auxiliary formula of an equality or an induction.
3. It is in S^*.
4. It is an atomic formula.

Otherwise C is said to be (S^*)-*explicit*. And also C is said to be *implicit* if a descendant of C satisfies one of the above conditions 1,2. Otherwise C is said to be *explicit*.

Let I be an inference in π. Then I is called (S^*)-*implicit* or (S^*)-*explicit* according as its principal formula is (S^*)-implicit or (S^*)-explicit. And also I is called *implicit* or *explicit* according as its principal formula is implicit or explicit.

DEFINITION 2.3. Let π be a derivation and let v be a free individual variable or a unary predicate variable in π. Then v is said to be *redundant* in π if it occurs in an upper sequent of an inference I and does not occur in the lower sequent of I and is not used as the eigenvariable of I.

DEFINITION 2.4. Let T be a subtheory of AI_ξ^- and let π be a derivation in AI_ξ^-. Then a logical inference I in π is said to be *reducible with respect to* T if one of the auxiliary formulas of I is derivable (refutable) in T provided that it belongs to the antecedent (succedent) of the sequent in which it occurs.

DEFINITION 2.5. Let π be a derivation with the end sequent S in AI_ξ^-. Then π is said to be *normal* if it satisfies the following conditions:

1. It includes no cuts except inessential ones.
2. It includes no redundant variables.
3. It includes no inductions except constant normal ones.
4. It includes no equalities except inessential ones.

Let S^* be a part of S. Then π is said to be (S^*)-*strongly normal* if it is normal and satisfies the following condition:

5. It includes no (S^*)-explicit inferences which are reducible with respect to AI_ξ^-.

Especially, we say that π is *strongly normal* if it is $(\rightarrow)$-strongly normal.

REMARK. Let π be a derivation with the end sequent S in AI_ξ^-. Then, π is (S)-strongly normal if it is normal.

Then we have the following theorems.

THEOREM 1. *We can transform any derivation in AI_ξ^- into a normal one with the same end sequent.*

THEOREM 2. *We can transform any derivation in AI_ξ^- into a strongly normal one with the same end sequent.*

In Section 4, Theorem 1 will be proved by transfinite induction along $<_0$ up to $(\xi, 1, 0)$ and Theorem 2 will be proved by transfinite induction along $<_0$ up to $(\xi, 1, 0\#0)$, where $(\xi, 1, 0)$ and $(\xi, 1, 0\#0)$ are ordinal diagrams and $<_0$ is a well-ordering over the ordinal diagrams in Takeuti's system of ordinal diagrams $O(\xi+1, 2)$ (cf. [11]).

Theorem 1 implies the following corollary. Thus, by transfinite induction along $<_0$ up to $(\xi, 1, 0)$ we can show that AI_ξ^- is consistent.

COROLLARY 1. *AI_ξ^- is consistent.*

PROOF. Similar to corollary 2 below. ■

Theorem 2 implies the following corollary. Thus, by transfinite induction along $<_0$ up to $(\xi, 1, 0\#0)$ we can show that AI_ξ^- is ω-consistent.

COROLLARY 2. *AI_ξ^- is ω-consistent.*

PROOF. Let $A(a)$ be an arbitrary formula which includes no free individual variable other then a and $\to A(\bar{n})$ is derivable in AI_ξ^- for all numeral $\bar{n}$. Then it suffices to show that $\forall x A(x) \to$ is not derivable in AI_ξ^-. Now, we suppose that $\forall x A(x) \to$ is derivable in AI_ξ^-. Then there exists a strongly normal derivation π of $\forall x A(x) \to$. Assume that π includes at least one non-structural inference. Note that the end-place of π includes no free individual variables and hence it includes no cuts. If an inference is an induction or an equality or an inference for $Q^{\mathfrak{B}}$ or an inference for $Q_{\prec}^{\mathfrak{B}}$, then it does not belong to the boundary of π. Thus every boundary inference is a $\forall$: left whose auxiliary formula is of the form $A(t)$ where t is a closed term. But it is impossible, because π is strongly normal and $\to A(t)$ is derivable in AI_ξ^- by our assumption. Thus π does not include non-structural inferences. But it is clear that there does not exist such a derivation. So AI_ξ^- is ω-consistent. ■

3. Preliminaries

In order to prove our theorems, we shall consider the system $\underline{\mathrm{AI}}_\xi^-$ obtained from AI_ξ^- by adding the following inference rule, called *substitution rule*,

$$\frac{\Gamma(X) \to \Delta(X)}{\Gamma(V) \to \Delta(V),}$$

where X does not occur in the lower sequent and $\Gamma(V) \to \Delta(V)$ is the sequent obtained from $\Gamma(X) \to \Delta(X)$ by substituting a unary abstract V for X. Then X is called the *eigenvariable* of this inference and V is called the *substituted abstract* of this inference. This inference is considered as a structural inference.

DEFINITION 3.1. The *grade* of a formula A, denoted by $g(A)$, is defined as follows:

1. $g(A) = 0$, if A is an atomic formula which is not of the form $Q^{\mathfrak{B}}_{\prec u}ts$.
2. $g(Q^{\mathfrak{B}}_{\prec u}ts) = 1$, where s, t and u are arbitrary terms.
3. $g(B \wedge C) = g(B \vee C) = g(B \supset C) = \max\{g(B), g(C)\} + 1$.
4. $g(\neg B) = g(\forall xB) = g(\exists xB) = g(B) + 1$.

DEFINITION 3.2. The *grade* of an inference I, denoted by $g(I)$, is defined as follows:

$$g(I) = \begin{cases} \max\{g(A) \mid A \text{ is an auxiliary formula of } I\} & \text{if } I \text{ is non-structural,} \\ \text{the grade of a cut formula of } I & \text{if } I \text{ is a cut,} \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 3.3. Let π be a derivation in $\underline{\mathrm{AI}}^{-}_{\xi}$ and S a sequent in π. For any natural number ρ, the *height* based on ρ of S in π, denoted by $h_\rho(S;\pi)$ or simply $h_\rho(S)$, is defined as follows:

1. $h_\rho(S) = \rho$, if S is the end sequent of π.
2. Let S be one of the upper sequents of an inference I in π and S' the lower sequent of I. Assume that $h_\rho(S')$ is defined. Then

$$h_\rho(S) = \begin{cases} 0 & \text{if } I \text{ is a substitution,} \\ \max\{h_\rho(S'), g(I)\} & \text{otherwise.} \end{cases}$$

DEFINITION 3.4. The *degree* of a formula A, denoted by $dg(A)$, is defined as follows:

1. $dg(t = s) = dg(Xt) = 0$, where s and t are arbitrary terms and X is an arbitrary unary predicate variable.
2. $dg(Q^{\mathfrak{B}}ts) = \begin{cases} v(t) \oplus 1 & \text{if } Q^{\mathfrak{B}}ts \text{ is closed and } v(t) \prec \xi, \\ \xi & \text{otherwise.} \end{cases}$
3. $dg(Q^{\mathfrak{B}}_{\prec u}ts) = \begin{cases} v(u) & \text{if } Q^{\mathfrak{B}}_{\prec u}ts \text{ is closed and } v(u) \prec \xi, \\ \xi & \text{otherwise.} \end{cases}$
4. $dg(\neg B) = dg(B)$.
5. $dg(B \wedge C) = dg(B \vee C) = dg(B \supset C) = \max_{\prec}\{dg(B), dg(C)\}$, where $\max_{\prec}$ is used to denote the maximum with respect to $\prec$.
6. $dg(\forall xB) = dg(\exists xB) = dg(B)$.

Let π be a derivation in $\underline{\mathrm{AI}}_{\xi}^{-}$. Then the *degree* of a formula F in π, denoted by $d(F;\pi)$ or simply $d(F)$, is defined as follows:

$$d(F) = \begin{cases} dg(F) & \text{if } F \text{ is implicit in } \pi, \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 3.5. Let π be a derivation in $\underline{\mathrm{AI}}_{\xi}^{-}$. We say that a sequent S in π *belongs to the end-place* of π if no non-structural inferences occur below S in π. And we say that an inference I in π *belongs to the boundary* of π or is a *boundary inference* of π if the lower sequent of I belongs to the end-place of π and the upper sequents of I do not belong to the end-place of π.

DEFINITION 3.6. Let π be a derivation with the end sequent S in $\underline{\mathrm{AI}}_{\xi}^{-}$ and let S^* be a part of S. Let d be a mapping from the set of substitutions in π to the set of ordinals less than ξ. For each substitution J in π, $d(J)$ is used to denote the value of the mapping d at J and is read "*degree* of J." Then the triple $\langle \pi; d; S^* \rangle$ is called a *derivation with degree* if it satisfies the following conditions for each substitution J in π and each formula B in the upper sequent of J:

1. The upper sequent of J belongs to the end-place of π.
2. If B is (S^*)-explicit, then it includes no eigenvariables of J.
3. If B is (S^*)-implicit, then so is its successor.
4. $d(B) \preceq d(J)$ holds.

DEFINITION 3.7. Let $\langle \pi; d; S^* \rangle$ be a derivation with degree. Then $\langle \pi; d; S^* \rangle$ is said to be *normal* if it satisfies the conditions $1 \sim 4$ in Definition 2.5. And also $\langle \pi; d; S^* \rangle$ is said to be (S^*)-*strongly normal* if it satisfies the conditions $1 \sim 5$ in Definition 2.5.

Since we shall use Takeuti's system of ordinal diagrams $O(\xi + 1, 2)$ to prove our theorems, we shall give some related definitions and propositions.

DEFINITION 3.8. Let i be an ordinal less than ξ. Then we shall define the order $\ll_i$ on ordinal diagrams. Let α and β be ordinal diagrams. Then

$$\alpha \ll_i \beta \Leftrightarrow \alpha <_j \beta \text{ for all } i \preceq j \preceq \xi.$$

$\alpha \underline{\ll}_i \beta$ is used to denote the statement "$\alpha \ll_i \beta$ or $\alpha = \beta$."

NOTATION. Let α be an ordinal diagram and let ζ be an ordinal less than or equal to ξ and n a natural number. Then an ordinal diagram $\zeta(n,0,\alpha)$ is defined as follows:

$$\zeta(0,0,\alpha) := \alpha, \qquad \zeta(n+1,0,\alpha) := (\zeta,0,\zeta(n,0,\alpha)).$$

PROPOSITION 1. *Let α, β and γ be ordinal diagrams and let $i \prec \zeta \preceq \xi$ and $n \in \omega$. Then,*

1. $\alpha \ll_0 \alpha\#\beta$.
2. $\alpha <_j (\zeta,0,\alpha)$ *for* $j \preceq \zeta$.
3. $(i,0,\alpha) \ll_{i+1} (\zeta,0,\beta)$.
4. $\alpha,\beta \ll_i (\zeta,0,\gamma) \Rightarrow \alpha\#\beta \ll_i (\zeta,0,\gamma)$.
5. *If* $\alpha \ll_i \beta$, *then* $(\zeta,0,\alpha) \ll_i (\zeta,0,\beta)$.
6. $(\zeta,0,\alpha)\#(\zeta,0,\beta) \ll_0 (\zeta,0,\alpha\#\beta)$.
7. *If* $\alpha \ll_i (\zeta,1,0)$, *then* $\zeta(n,0,\alpha) \ll_i (\zeta,1,0)$

PROPOSITION 2. *Let $j \preceq \xi$ and let γ and δ be ordinal diagrams for which there exists two finite sequences of ordinal diagrams $\delta = \delta_0,\ldots,\delta_m$ and $\gamma = \gamma_0,\ldots,\gamma_m$ which satisfies the following conditions:*

1. *Each γ_i is of the form $(k,a,\gamma_{i+1}\#\eta)$ for some $j \preceq k \preceq \xi$, $0 \le a \le 1$ and η.*
2. *Each δ_i is of the form $(k,a,\delta_{i+1}\#\eta)$ for some $\eta' \underline{\ll}_j \eta$ if γ_i is $(k,a,\gamma_{i+1}\#\eta)$.*
3. $\delta_m \ll_j \gamma_m$.

Then $\delta \ll_j \gamma$.

DEFINITION 3.9. Let π be a derivation with the end sequent $\check{S}$ in $\underline{\mathrm{AI}}_{\xi}^{-}$. Let $\check{S}^*$ be a part of $\check{S}$ and let d be a mapping from the set of substitutions in π to the set of ordinals less than ξ. Let ρ be a natural number. To each sequent S in π and each inference I in π, we assign ordinal diagrams $O_\rho(S;\pi;d;\check{S}^*)$ and $O_\rho(I;\pi;d;\check{S}^*)$, or simply $O_\rho(S)$ and $O_\rho(I)$, respectively, as follows:

1. If S is an initial sequent, then

$$O_\rho(S) = 0.$$

2. Let S_i $(1 \le i \le n)$ be the upper sequents of I. Assume that $O_\rho(S_i)$ are defined for each $1 \le i \le n$.

(2.1) If I is a weak inference or a term-replacement, then

$$O_\rho(I) = O_\rho(S).$$

(2.2) If I is a cut, then

$$O_\rho(I) = O_\rho(S_1)\#O_\rho(S_2).$$

(2.3) If I is an $(\check{S}^*)$-explicit logical inference, then

$$O_p(I) = \begin{cases} O_p(S_1)\#(\xi, 1, 0) & I \text{ has one upper sequent,} \\ O_p(S_1)\#O_p(S_2)\#(\xi, 1, 0) & I \text{ has two upper sequents.} \end{cases}$$

(2.4) If I is an $(\check{S}^*)$-implicit logical inference or a $Q^{\mathfrak{B}}$: right or an inference for $Q^{\mathfrak{B}}_{\prec}$, then

$$O_p(I) = \begin{cases} O_p(S_1)\#0 & I \text{ has one upper sequent,} \\ O_p(S_1)\#O_p(S_2) & I \text{ has two upper sequents.} \end{cases}$$

(2.5) If I is a $Q^{\mathfrak{B}}$: left, then

$$O_p(I) = O_p(S_1)\#O_p(S_2)\#(\xi, 0, 0).$$

(2.6) If I is an equality, then

$$O_p(I) = O_p(S_1)\#O_p(S_2)\#O_p(S_3).$$

(2.7) If I is an induction, then

$$O_p(I) = O_p(S_1)\#(\xi, 0, O_p(S_2))\#O_p(S_3).$$

(2.8) If I is a substitution, then

$$O_p(I) = (\xi, 0, O_p(S_1)).$$

3. Let S be the lower sequent of I.

(3.1) If I is a substitution, then

$$O_p(S) = (d(I), 0, O_p(I)).$$

(3.2) If I is not a substitution, then

$$O_p(S) = \xi(h_p(S_1) - h_p(S), 0, O_p(I)).$$

Finally, we define the ordinal diagram $O_p(\pi; d; \check{S}^*)$ by $(\xi, 0, O_p(\check{S}))$.

Then we have a proposition similar to one given by Arai (cf. [2]).

PROPOSITION 3. *Let $\langle \pi; d; S^* \rangle$ be a derivation with degree and S' a sequent in π. And let p and σ be natural numbers. If $\sigma \leq p$, then*

$$O_\sigma(S') \ll_0 \xi(h_p(S') - h_\sigma(S'), 0, O_p(S')).$$

4. Proofs of our theorems

Let α be an ordinal diagram such that $\alpha \leq_0 (\xi, 1, 0\#0)$. Then we shall show the following lemma by transfinite induction along $<_0$ up to α.

LEMMA 1. *For any derivation with degree* $\langle \pi; d; \check{S}^* \rangle$ *such that* $O_0(\pi; d; \check{S}^*) < \alpha$, *we can transform* $\langle \pi; d; \check{S}^* \rangle$ *into an* $(\check{S}^*)$*-strongly normal derivation in* AI_ξ^- *with the same end sequent.*

This lemma implies Theorem 1 and 2 as follows.

PROOF OF THEOREM 1. Let π be a derivation with the end sequent S in AI_ξ^-. Note that π includes no substitutions. So, $\langle \pi; \phi; S \rangle$ is a derivation with degree. Note that $O_0(\pi; \phi; S) <_0 (\xi, 1, 0)$. So, set $\alpha = (\xi, 1, 0)$. Then, by Lemma 1 and its proof, we can transform $\langle \pi; \phi; S \rangle$ to a normal derivation by transfinite induction along $<_0$ up to $(\xi, 1, 0)$. ∎

PROOF OF THEOREM 2. Let π be a derivation in AI_ξ^-. Note that π includes no substitutions. So, $\langle \pi; \phi; \rightarrow \rangle$ is a derivation with degree. Note that $O_0(\pi; \phi; \rightarrow) <_0 (\xi, 1, 0\#0)$. So, set $\alpha = (\xi, 1, 0\#0)$. Then, by Lemma 1 and its proof, we can transform $\langle \pi; \phi; \rightarrow \rangle$ to a strongly normal derivation by transfinite induction along $<_0$ up to $(\xi, 1, 0\#0)$. ∎

To prove Lemma 1, we need the following lemma.

LEMMA 2. *Let* $\langle \pi; d; S^* \rangle$ *be an* (S^*)*-strongly normal derivation with degree. Then we can transform* $\langle \pi; d; S^* \rangle$ *into an* (S^*)*-strongly normal derivation in* AI_ξ^- *with the same end sequent.*

PROOF. By induction on the number of substitutions in π. ∎

The rest of this section is devoted to proving Lemma 1.

PROOF OF LEMMA 1. We shall prove this lemma by transfinite induction along $<_0$ up to α.

Suppose that $\langle \pi; d; \check{S}^* \rangle$ be a derivation with degree such that $O_0(\pi; d; \check{S}^*) <_0 \alpha$. If $\langle \pi; d; \check{S}^* \rangle$ is $(\check{S}^*)$-strongly normal, we can transform $\langle \pi; d; \check{S}^* \rangle$ into an $(\check{S}^*)$-strongly normal derivation in AI_ξ^- with the same end sequent by Lemma 2. So, we assume that $\langle \pi; d; \check{S}^* \rangle$ is not $(\check{S}^*)$-strongly normal.

We suppose that $\check{S}$ is of the form $\Gamma \rightarrow \Delta$ and $\check{S}^*$ is of the form $\Gamma^* \rightarrow \Delta^*$. We can suppose that π includes no redundant variables, because $dg(F(t)) \preceq dg(F(a))$ for any formula F and any term t. And also we can suppose that if there exists a weakening I in the end-place of π then every inference below I is a weakening or an exchange, because if π does not satisfy the above condition then we can transform $\langle\pi;d;\check{S}^*\rangle$ to a derivation with degree $\langle\pi';d';\check{S}^*\rangle$ such that π' satisfies the above condition and every substitution in π' has same degree as the corresponding one in π and $O_0(\pi';d';\check{S}^*)\ll_0 O_0(\pi;d;\check{S}^*)$ by the usual method.

We shall divide our proof into some cases. When we shall consider a case, we assume that the proceeding case(s) do not hold.

In this proof, the letter "S" in the expression "$\Lambda \xrightarrow{S} \Pi$" is used to denote the sequent "$\Lambda \to \Pi$" itself. And also we shall omit the superscript $\mathfrak{B}$ in $Q^{\mathfrak{B}}$ or $Q^{\mathfrak{B}}_{\prec}$ if there is no danger of confusion.

(1) The case where π includes at least one logical initial sequent $\hat{S}$ in the end-place.

(1.1) The case where a descendant of a formula in $\hat{S}$ is a cut formula.

Assume that π is of the form:

$$\dfrac{\begin{array}{c} \\ \pi_1 \vdots \\ \Lambda \xrightarrow{S_1} \Pi, D' \end{array} \quad \begin{array}{c} D \xrightarrow{\hat{S}} D \\ \vdots \\ D' \xrightarrow{S_2} D'' \end{array}}{\Lambda \xrightarrow{S} \Pi, D''} \atop \vdots \qquad ,$$

where D' (D'') in S_2 is a descendant of D in the antecedent (succedent) of $\hat{S}$.

Note that D'' is $(\check{S}^*)$-implicit. Because, if D'' is atomic, it is clear that D'' is $(\check{S}^*)$-implicit. So, we assume that D'' contains at least one logical symbol. Since D is atomic, D'' is obtained from D by at least one substitution. Since $\langle\pi;d;\check{S}^*\rangle$ is a derivation with degree, D'' in π is $(\check{S}^*)$-implicit.

Let $h_0(S_1;\pi)=\rho$ and $h_0(S;\pi)=\sigma$ and let $\Lambda^* \to \Pi^*, D'$ be the sequent obtained from S_1 by deleting the $(\check{S}^*)$-explicit formulas. Then we reduce π to the derivation π':

$$\begin{array}{c} \pi_1 \vdots \\ \Lambda \xrightarrow{S_1} \Pi, D' \\ \hline \text{term-replacements} \\ \hline \Lambda \xrightarrow{S} \Pi, D'' \\ \vdots \end{array} \quad .$$

Here, note that D'' is also $(\check{S}^*)$-implicit in π'. Let d' be the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J'

in π', $d'(J') = d(J)$, where J is the corresponding one in π. The letter "d'" is also used to denote the restriction of d' to the set of substitutions in π_1. Then $\langle \pi'; d'; \check{S}^* \rangle$ is a derivation with degree. Next we shall prove $O_0(S; \pi'; d'; \check{S}^*) \ll_0 O_0(S; \pi; d; \check{S}^*)$. Note that $h_0(S_1; \pi') = \sigma$. Since

$$\begin{aligned} O_0(S_1; \pi'; d'; \check{S}^*) &= O_\sigma(S_1; \pi_1; d'; \Lambda^* \to \Pi^*, D') \\ &\lesssim_0 \xi(\rho - \sigma, 0, O_\rho(S_1; \pi_1; d'; \Lambda^* \to \Pi^*, D')) \\ &= \xi(\rho - \sigma, 0, O_0(S_1; \pi; d; \check{S}^*)), \end{aligned}$$

we have

$$\begin{aligned} O_0(S; \pi'; d'; \check{S}^*) &= O_0(S_1; \pi'; d'; \check{S}^*) \\ &\lesssim_0 \xi(\rho - \sigma, 0, O_0(S_1; \pi; d; \check{S}^*)) \\ &\ll_0 \xi(\rho - \sigma, 0, O_0(S_1; \pi; d; \check{S}^*) \sharp O_0(S_2; \pi; d; \check{S}^*)) \\ &= O_0(S; \pi; d; \check{S}^*). \end{aligned}$$

Thus, $O_0(\pi'; d'; \check{S}^*) \ll_0 (O_0(\pi; d; \check{S}^*)$ by proposition 2. Hence we can transform π' to an (S^*)-strongly normal derivation with the same end sequent, by induction hypothesis.

(1.2) The other case.

Since the proceeding case does not hold, there exists a formula A (B) which is a descendant of the antecedent (succedent) formula of $\hat{S}$ and occurs in $\check{S}$.

If A is atomic, then B is also atomic and hence it is clear that we can obtain a desired derivation.

So, we assume that A contains at least one logical symbol. Then both A and B are in $\check{S}^*$, because both A and B are obtained from the formulas in $\hat{S}$ by at least one substitution. Thus it is clear that we can obtain a desired derivation.

(2) The case where π includes no boundary inferences.

Then π includes no logical initial sequents. Thus we can obtain a desired derivation, since the mathematical initial sequents are closed under cuts.

(3) The case where π includes at least one $(\check{S}^*)$-explicit inference which is reducible with respect to AI_ξ^-.

Let I be such an inference. Since the other cases are treated similarly, we shall consider the case where I is a $\wedge$: left.

Assume that π is of the form:

$$\dfrac{\begin{array}{c}\pi_1\vdots\\ A, \Lambda \xrightarrow{S_1} \Pi\end{array}}{A \wedge B, \Lambda \xrightarrow{S} \Pi}\\ \vdots \quad .$$

Let $h_0(S_1;\pi) = \rho$ and $h_0(S;\pi) = \sigma$ and let $\Lambda^* \to \Pi^*$ be the sequent obtained from S by deleting the $(\check{S}^*)$-explicit formulas. By our assumption, $\to A$ is derivable in AI_{ξ}^{-}. So, let $\hat{\pi}$ be a derivation of $\to A$. Note that $\hat{\pi}$ contains no substitutions. Then we reduce π to the derivation π':

$$\dfrac{\dfrac{\begin{array}{c}\hat{\pi}\vdots\\ \xrightarrow{\hat{S}} A\end{array} \quad \begin{array}{c}\pi_1\vdots\\ A, \Lambda \xrightarrow{S_1} \Pi\end{array}}{\Lambda \to \Pi}}{A \wedge B, \Lambda \xrightarrow{S} \Pi}\\ \vdots \quad .$$

Let d' by the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J' of in π', $d'(J') = d(J)$, where J is the corresponding one in π. The letter "d'" is also used to denote the restriction of d' to the set of substitutions in π_1. Since π_1 and $\hat{\pi}$ include no substitutions, $\langle \pi'; d'; \check{S}^* \rangle$ is a derivation with degree. Then we shall prove $O_0(S;\pi';d';\check{S}^*) \ll_0 O_0(S;\pi;d;\check{S}^*)$. At first, we have

$$\begin{aligned} O_0(S_1;\pi';d';\check{S}^*) &= O_\rho(S_1;\pi_1;d';A,\Lambda^* \to \Pi^*)\\ &\underline{\ll}_0\, O_\rho(S_1;\pi_1;d';\Lambda^* \to \Pi^*)\\ &= O_0(S_1;\pi;d;S^*). \end{aligned}$$

Next we shall note that every logical inference in $\hat{\pi}$ is $(\check{S}^*)$-implicit in π'. Thus, $O_0(\hat{S};\pi';d';\check{S}^*) \ll_0 (\xi,1,0)$. So

$$\begin{aligned} O_0(S;\pi';d';\check{S}^*) &= \xi(\rho-\sigma,0,O_0(\hat{S};\pi';d';\check{S}^*)\# O_0(S_1;\pi';d';\check{S}^*))\\ &\ll_0 \xi(\rho-\sigma,0,(\xi,1,0)\# O_0(S_1;\pi;d;\check{S}^*))\\ &= O_0(S;\pi;d;\check{S}^*). \end{aligned}$$

So, $O_0(\pi';d';\check{S}^*) \ll_0 O_0(\pi;d;\check{S}^*)$ by proposition 2. Hence we can transform π' to an (S^*)-strongly normal derivation with the same end sequent, by induction hypothesis.

(4) The case where π includes at least one equality which belongs to the boundary of π.

Assume that π is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ \Lambda \xrightarrow{S_1} \Pi, t=s\end{array} \quad \begin{array}{c}\vdots\\ \Lambda \xrightarrow{S_2} \Pi, F(t)\end{array} \quad \begin{array}{c}\vdots\\ F(s), \Lambda \xrightarrow{S_3} \Pi\end{array}}{\begin{array}{c}\Lambda \xrightarrow{S} \Pi\\ \vdots\\ \Gamma \to \Delta\end{array}} \quad .$$

Let $h_0(S_1;\pi) = \rho$ and $h_0(S;\pi) = \sigma$ and let $\Lambda^* \to \Pi^*$ be the sequent obtained from S by deleting the (S^*)-explicit formulas in π.

(4.1) The case where $t = s$ has no free individual variables.

(4.1.1) The case where $t = s$ is true under the standard interpretation.

We reduce π to the following derivation π':

$$\dfrac{\begin{array}{c}\vdots\\ \Lambda \xrightarrow{S_2} \Pi, F(t)\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ F(s), \Lambda \xrightarrow{S_3} \Pi\end{array}}{F(t), \Lambda \to \Pi}}{\begin{array}{c}\Lambda \xrightarrow{S} \Pi\\ \vdots\end{array}} \quad .$$

Let d' be the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J' in π', $d'(J') = d(J)$, where J is the corresponding one in π. Then $\langle \pi'; d'; \check{S}^* \rangle$ is a derivation with degree. Next we shall show that $O_0(S;\pi';d';\check{S}^*) \ll_0 O_0(S;\pi;d;\check{S}^*)$.

$$\begin{aligned} O_0(S;\pi';d';\check{S}^*) &= \xi(\rho - \sigma, 0, O_0(S_2;\pi';d';\check{S}^*) \# O_0(S_3;\pi';d';\check{S}^*)) \\ &\ll_0 \xi(\rho - \sigma, 0, O_0(S_1;\pi;d;\check{S}^*) \# O_0(S_2;\pi;d;\check{S}^*) \# O_0(S_3;\pi;d;\check{S}^*)) \\ &= O_0(S;\pi;d;\check{S}^*). \end{aligned}$$

Thus, $O_0(\pi';d';\check{S}^*) \ll_0 O_0(\pi;d;\check{S}^*)$ by proposition 2. Hence we can transform π' to an (S^*)-strongly normal derivation with the same end sequent, by induction hypothesis.

(4.1.2) The case where $t = s$ is false under the standard interpretation.

Then the sequent $t = s \to$ is a mathematical initial sequent. So, we reduce π to the following derivation π':

$$\begin{array}{c} \vdots \\ \dfrac{\Lambda \to \Pi, t = s \quad t = s \to}{\Lambda \xrightarrow{S} \Pi} \\ \vdots \end{array} \quad .$$

Let d' be the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J' in π', $d'(J') = d(J)$, where J is the corresponding one in π. Then $\langle \pi'; d'; \check{S}^* \rangle$ is a derivation with degree. We can show that $O_0(S; \pi'; d'; \check{S}^*) \ll_0 O_0(S; \pi; d; \check{S}^*)$. Thus, $O_0(\pi'; d'; \check{S}^*) \ll_0 O_0(\pi; d; \check{S}^*)$ by proposition 2. Hence we can transform π' to an (S^*)-strongly normal derivation with the same end sequent, by induction hypothesis.

(4.2) The case where $F(t)$ is identical with $F(s)$.
Similar to the case (4.1.1).

(4.3) The case where I is inessential.
Then we construct the following derivations π_1, π_2 and π_3 from π.

$$\begin{array}{ccc}
\pi_1 & \pi_2 & \pi_3 \\
\vdots & \vdots & \vdots \\
\Lambda \to \Pi, t = s & \Lambda \to \Pi, F(t) & F(s), \Lambda \to \Pi \\
\hline\hline
\Lambda \to t = s, \Pi & \Lambda \to F(t), \Pi & \Lambda, F(s) \to \Pi \\
\vdots & \vdots & \vdots \\
\Gamma \to t = s, \Delta & \Gamma \to F'(t), \Delta & \Gamma, F'(s) \to \Delta,
\end{array}$$

where $F'(t)$ and $F'(s)$ are formulas obtained from $F(t)$ and $F(s)$ by some substitutions, respectively. Let d_i be the mapping from the set of substitutions in π_i to the ordinals less than ξ such that, for each substitution J' in π_i, $d_i(J') = d(J)$, where J is the corresponding one in π. Then $\langle \pi_1; d_1; \Gamma^* \to t = s, \Delta^* \rangle$, $\langle \pi_2; d_2; \Gamma^* \to F'(t), \Delta^* \rangle$ and $\langle \pi_3; d_3; \Gamma^*, F'(s) \to \Delta^* \rangle$ are derivations with degree. Because $t = s$, $F'(t)$ and $F'(s)$ are explicit in π_1, π_2 and π_3, respectively. We can prove the following facts:

$$O_0(\pi_1; d_1; \Gamma^* \to t = s, \Delta^*) \ll_0 O_0(\pi; d; \check{S}^*).$$

$$O_0(\pi_2; d_2; \Gamma^* \to F'(t), \Delta^*) \ll_0 O_0(\pi; d; \check{S}^*).$$

$$O_0(\pi_3; d_3; \Gamma^*, F'(s) \to \Delta^*) \ll_0 O_0(\pi; d; \check{S}^*).$$

By induction hypothesis, we can transform π_1 to a derivation π_1' whose end sequent is $\Gamma \to t = s$, Δ and which is $(\Gamma^* \to t = s, \Delta^*)$-strongly normal, and π_2 to a derivation π_2' whose end sequent is $\Gamma \to F'(t)$, Δ and which is $(\Gamma^* \to F'(t), \Delta^*)$-strongly normal, and π_3 to a derivation π_3' whose end sequent is Γ, $F'(s) \to \Delta$ and which is $(\Gamma^*, F'(s) \to \Delta^*)$-strongly normal. We define the derivation π' as follows:

$$\dfrac{\dfrac{\begin{matrix}\pi_1' \vdots \\ \Gamma \to t = s, \Delta\end{matrix}}{\Gamma \to \Delta, t = s} \quad \dfrac{\begin{matrix}\pi_2' \vdots \\ \Gamma \to F'(t), \Delta\end{matrix}}{\Gamma \to \Delta, F'(t)} \quad \dfrac{\begin{matrix}\pi_3' \vdots \\ \Gamma, F'(s) \to \Delta\end{matrix}}{F'(s), \Gamma \to \Delta}}{\Gamma \to \Delta} \; .$$

Then π' is $(\check{S}^*)$-strongly normal, because the free individual variables in t or s occur in Γ or Δ.

(5) The case where π includes at least one induction which belongs to the boundary of π.

Similar to the case (4) (cf. [9]).

(6) The case where π includes at least one explicit logical inference which belongs to the boundary of π.

Let I be such an inference. Since the other cases are treated similarly, we shall consider the case where I is a $\forall$:left.

Assume that π is of the form:

$$\begin{matrix} \vdots \\ \dfrac{A(t), \Lambda \to \Delta}{\forall x A(x), \Lambda \to \Delta} I \\ \vdots \\ \Gamma \to \Delta \end{matrix} \quad .$$

(6.1) The case where I is $(\check{S}^*)$-explicit.

We shall note that Γ includes the formula which is a descendant of $\forall x A(x)$ and is of the form $\forall x A'(x)$, where $A'(x)$ is a formula obtained from $A(x)$ by some term-replacements. We reduce π to the following derivation π':

$$\begin{matrix} \vdots \\ \dfrac{A(t), \Lambda \to \Delta}{\forall x A(x), \Lambda, A(t) \to \Delta} \\ \vdots \\ \Gamma, A'(t) \to \Delta \end{matrix} \quad ,$$

where $A'(t)$ is the formula obtained from $A'(x)$ by substituting t for x. Note that $A(t)$ and its descendants in π' contain no eigenvariables of substitutions in π', since $\forall x A(x)$ is $(\check{S}^*)$-explicit in π. Let d' be the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J' in π', $d'(J') = d(J)$, where J is the corresponding one in π. Then, $\langle \pi'; d'; \Gamma^* \to \Delta^* \rangle$ is a derivation with degree. We can show that $O_0(\pi'; d'; \Gamma^* \to \Delta^*) \ll_0 O_0(\pi; d; \check{S}^*)$. Thus, we can transform π' to a derivation $\hat{\pi}$ whose end sequent is Γ, $A'(t) \to \Delta$ and which is $(\Gamma^* \to \Delta^*)$-strongly normal, by induction hypothesis. Then we shall define the derivation $\tilde{\pi}$ as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\hat{\pi}\vdots}{\Gamma, A'(t) \to \Delta}}{A'(t), \Gamma \to \Delta}}{\forall x A'(x), \Gamma \to \Delta}}{\Gamma \to \Delta} \quad .$$

Then $\tilde{\pi}$ is $(\check{S}^*)$-strongly normal, because the free individual variables in t occur in Γ or Δ and $\to A'(t)$ is not derivable in AI_ξ^- by our assumption.

(6.2) The case where I is $(\check{S}^*)$-implicit.

At first, note that Γ includes the formula which is a descendant of $\forall x A(x)$ and of the form $\forall x A'(x)$, where $A'(x)$ is a formula obtained from $A(x)$ by some substitutions and some term-replacements. We reduce π to a derivation π'' similar to π' in the case (6.1). Let d' be the mapping from the set of substitutions in π'' to the ordinals less than ξ such that, for each substitution J' in π'', $d'(J') = d(J)$, where J is the corresponding one in π. Then $\langle \pi''; d'; \Gamma^*, A'(t) \to \Delta^* \rangle$ is a derivation with degree. We can show that $O_0(\pi''; d'; \Gamma^*, A'(t) \to \Delta^*) \ll_0 O_0(\pi; d; \check{S}^*)$. So, we can transform π'' to a derivation $\hat{\pi}$ whose end sequent is Γ, $A'(t) \to \Delta$ and which is $(\Gamma^*, A'(t) \to \Delta^*)$-strongly normal, by induction hypothesis. From $\hat{\pi}$, we shall construct a derivation $\tilde{\pi}'$ similar to $\tilde{\pi}$ in the case (6.1). Then $\tilde{\pi}'$ is $(\check{S}^*)$-strongly normal.

(7) The case where π includes at least one explicit inference for $Q^{\mathfrak{B}}$ or $Q^{\mathfrak{B}}_{\prec}$, which belongs to the boundary of π.

Let I be such an inference. Since the other cases are treated similarly, we shall consider the case where I is a $Q^{\mathfrak{B}}$:left.

Assume that π is of the form:

$$\cfrac{\begin{matrix}\vdots & & \vdots \\ \Lambda \to \Pi, t \prec \xi & & \mathfrak{B}(V, Q_{\prec t}, t, s), \Lambda \to \Pi\end{matrix}}{Qts, \Lambda \to \Pi}\,I$$
$$\begin{matrix}\vdots \\ \Gamma \to \Delta\end{matrix} \quad .$$

We shall note that Γ includes the formula which is a descendant of Qts and is of the form $Qt's'$, where $Qt's'$ are a formula obtained from Qts by some term-replacements. We reduce π to the following derivations π_1 and π_2:

$$
\begin{array}{cc}
\pi_1 & \pi_2 \\
\vdots & \vdots \\
\dfrac{\Lambda \to \Pi, t \prec \xi}{Qts, \Lambda \to t \prec \xi, \Pi} & \dfrac{\mathfrak{B}(V, Q_{\prec t}, t, s), \Lambda \to \Pi}{Qts, \Lambda, \mathfrak{B}(V, Q_{\prec t}, t, s) \to \Pi} \\
\vdots & \vdots \\
\Gamma \to t' \prec \xi, \Delta & \Gamma, \mathfrak{B}(V', Q_{\prec t'}, t', s') \to \Delta
\end{array} ,
$$

where $\mathfrak{B}(V', Q_{\prec t'}, t', s')$ is a formula obtained from $\mathfrak{B}(V, Q_{\prec t}, t, s)$ by some substitutions and some term-replacements. Let d_i be the mapping from the set of substitutions in π_i to the ordinals less than ξ such that, for each substitution J' in π_i, $d_i(J') = d(J)$, where J is the corresponding one in π. Note that, in π_2, $\mathfrak{B}(V, Q_{\prec t}, t, s)$ and its descendants are $(\Gamma^*, \mathfrak{B}(V', Q_{\prec t'}, t', s') \to \Delta^*)$-implicit and explicit. Thus $\langle \pi_1; d_1; \Gamma^* \to t' \prec \xi, \Delta^* \rangle$ and $\langle \pi_2; d_2; \Gamma^*, \mathfrak{B}(V', Q_{\prec t'}, t', s') \to \Delta^* \rangle$ are derivations with degree. We can prove the following facts:

$$O_0(\pi_1; d_1; \Gamma^* \to t' \prec \xi, \Delta^*) \ll_0 O_0(\pi; d; \check{S}^*)$$

$$O_0(\pi_2; d_2; \Gamma^*, \mathfrak{B}(V', Q_{\prec t'}, t', s') \to \Delta^*) \ll_0 O_0(\pi; d; \check{S}^*).$$

By induction hypothesis, we can transform π_1 to a derivation π_1' whose end sequent is $\Gamma \to t' \prec \xi,\ \Delta$ and which is $(\Gamma^* \to t' \prec \xi, \Delta^*)$-strongly normal. And also we can transform π_2 to a derivation π_2' whose end sequent is Γ, $\mathfrak{B}(V', Q_{\prec t'}, t', s') \to \Delta$ and which is $(\Gamma^*, \mathfrak{B}(V', Q_{\prec t'}, t', s') \to \Delta^*)$-strongly normal. Then we shall define the derivation π' as follows:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1' \vdots \\ \Gamma \to t' \prec \xi, \Delta\end{array}}{\Gamma \to \Delta, t' \prec \xi} \qquad \dfrac{\begin{array}{c}\pi_2' \vdots \\ \Gamma, \mathfrak{B}(V', Q_{\prec t'}, t', s') \to \Delta\end{array}}{\mathfrak{B}(V', Q_{\prec t'}, t', s'), \Gamma \to \Delta}}{Qt's', \Gamma \to \Delta}}{\Gamma \to \Delta} .
$$

Then π' is $(\check{S}^*)$-strongly normal, because the free individual variables in V', t' or s' occur in Γ or Δ.

(8) The case where all the inferences which belong to the boundary of π are implicit inferences.

Then there is at least one suitable cut. Let I be a suitable cut. We shall consider the cases where the cut formula of I is of the form Qts or $Q_{\prec u}ts$.

(8.1) The case where the cut formula of I is of the form Qts.

Assume that π is of the form:

$$
\dfrac{\begin{array}{c}\vdots\\ \Lambda_1 \to \Pi_1, t_1 \prec \xi\end{array} \quad \begin{array}{c}\vdots\\ \Lambda_1 \to \Pi_1, \mathfrak{B}(X, Q_{\prec t_1}, t_1, s_1)\end{array}}{\Lambda_1 \xrightarrow{S_1} \Pi_1, Qt_1, s_1}
\qquad
\dfrac{\begin{array}{c}\pi_{2l}\,\vdots\\ \Lambda_2 \xrightarrow{S_{2l}} \Pi_2, t_2 \prec \xi\end{array} \quad \begin{array}{c}\vdots\\ \mathfrak{B}(V, Q_{\prec t_2}, t_2, s_2), \Lambda_2 \xrightarrow{S_{2r}} \Pi_2\end{array}}{Qt_2s_2, \Lambda_2 \xrightarrow{S_2} \Pi_2}
$$

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Lambda_3 \xrightarrow{S_3} \Pi_3, Qts\end{array} \qquad\qquad \begin{array}{c}\vdots\\ Qts, \Lambda_4 \xrightarrow{S_4} \Pi_4\end{array}}{\Lambda_3, \Lambda_4 \xrightarrow{S_5} \Pi_3, \Pi_4}\ I \\
\vdots \\
\dfrac{\Lambda \xrightarrow{S} \Pi}{}\ I_0 \\
\vdots \\
\Gamma \longrightarrow \Delta
\end{array}
$$

Let $j = d(\mathfrak{B}(X, Q_{\prec t}, t, s))$ and let S be the *j-resolvent* of S_5, i.e. the upper sequent of the uppermost substitution I_0 under S_5 whose degree is not greater than j, if such exists; otherwise, the end sequent of π. Assume that $h_0(S_{2l}; \pi) = \rho_{2l}$ and $h_0(S_2; \pi) = \rho_2$. And also assume that the sequent $\Lambda_2^* \to \Pi_2^*$, $t_2 \prec \xi$ is the sequent obtained from S_{2l} by deleting the $(\check{S}^*)$-explicit formulas in π.

(8.1.1) The case where Qts is not closed.

We reduce π to the following derivations π_1 and π_2:

$$
\begin{array}{c}
\pi_1 \\
\vdots \\
\dfrac{\Lambda_3 \xrightarrow{S_3} \Pi_3, Qts}{\Lambda_3, \Lambda_4 \xrightarrow{S_5} Qts, \Pi_3, \Pi_4} \\
\vdots \\
\Gamma \to Qts, \Delta
\end{array}
\qquad\qquad
\begin{array}{c}
\pi_2 \\
\vdots \\
\dfrac{Qts, \Lambda_4 \xrightarrow{S_4} \Pi_4}{\Lambda_3, \Lambda_4, Qts \xrightarrow{S_5} \Pi_3, \Pi_4} \\
\vdots \\
\Gamma, Qts \to \Delta
\end{array}
$$

Let d_i be the mapping from the set of substitutions in π_i to the ordinals less than ξ such that, for each substitution J' in π_i, $d_i(J') = d(J)$, where J is the corresponding one in π. Then $\langle \pi_1; d_1; \Gamma^* \to Qts, \Delta^* \rangle$ and $\langle \pi_2; d_2; \Gamma^*, Qts \to \Delta^* \rangle$ are derivations with degree. We shall prove $O_0(S_5; \pi_1; d_1; \Gamma^* \to Qts, \Delta^*) \ll_0 O_0(S_5; \pi; d; \check{S}^*)$.

$$\begin{aligned} O_0(S_5; \pi_1; d_1; \Gamma^* \to Qts, \Delta^*) &= O_0(S_3; \pi_1; d_1; \Gamma^* \to Qts, \Delta^*) \\ &= O_0(S_3; \pi; d; \check{S}^*) \\ &\ll_0 O_0(S_3; \pi; d; \check{S}^*) \# O_0(S_4; \pi; d; \check{S}^*) \\ &= O_0(S_5; \pi; d; \check{S}^*) \end{aligned}$$

So, we can transform π_1 into a derivation π_1' whose end sequent is $\Gamma \to Qts$, Δ and which is ($\Gamma^* \to Qts$, Δ^*)-strongly normal by induction hypothesis. Similarly, we have $O_0(S_5; \pi_2; d_2; \Gamma^*, Qts \to \Delta^*) \ll_0 O_0(S_5; \pi; d; \check{S}^*)$. Hence, we can transform π_2 into a derivation π_2' whose end sequent is Γ, $Qts \to \Delta$ and which is $(\Gamma^*, Qts \to \Delta^*)$-strongly normal. We shall define π' as follows:

$$\dfrac{\dfrac{\dfrac{\pi_1' \vdots}{\Gamma \to Qts, \Delta}}{\Gamma \to \Delta, Qts} \qquad \dfrac{\dfrac{\pi_2' \vdots}{\Gamma, Qts \to \Delta}}{Qts, \Gamma \to \Delta}}{\dfrac{\Gamma, \Gamma \to \Delta, \Delta}{\Gamma \to \Delta}} \;.$$

Then π' is $(\check{S}^*)$-strongly normal, because the free individual variables in t or s occur in Γ or Δ.

(8.1.2) The case where Qts is closed.

(8.1.2.1) The case where $t \prec \xi$ is true under the standard interpretation.
We reduce π to the derivation π':

$$
\begin{array}{c}
\vdots \\
\dfrac{\Lambda_1 \to \Pi_1, \mathfrak{B}(X, Q_{\prec t_1}, t_1, s_1)}{\Lambda_1 \to \mathfrak{B}(X, Q_{\prec t_1}, t_1, s_1), \Pi_1, Qt_1s_1} \\
\vdots \qquad\qquad \vdots \\
\dfrac{\Lambda_3 \to \mathfrak{B}(X, Q_{\prec t}, t, s), \Pi_3, Qts \quad Qts, \Lambda_4 \to \Pi_4}{\Lambda_3, \Lambda_4 \to \mathfrak{B}(X, Q_{\prec t}, t, s), \Pi_3, \Pi_4} \\
\vdots
\end{array}
$$

$$
\dfrac{
\dfrac{\dfrac{\Lambda \to \mathfrak{B}(X, Q_{\prec t}, t, s), \Pi}{\Lambda \to \Pi, \mathfrak{B}(X, Q_{\prec t}, t, s)}}{\Lambda \to \Pi, \mathfrak{B}(V, Q_{\prec t}, t, s)}\, J_0
\qquad
\dfrac{\begin{array}{c}\vdots \\ \mathfrak{B}(V, Q_{\prec t_2}, t_2, s_2), \Lambda_2 \xrightarrow{S_{2r}} \Pi_2\end{array}}{\mathfrak{B}(V, Q_{\prec t}, t, s), \Lambda_2 \to \Pi_2}
}{
\dfrac{\Lambda, \Lambda_2 \to \Pi, \Pi_2}{Qt_2s_2, \Lambda_2, \Lambda \xrightarrow{S_2'} \Pi, \Pi_2}
}
$$

$$
\begin{array}{c}
\vdots \qquad\qquad\qquad \vdots \\
\dfrac{\Lambda_3 \to \Pi_3, Qts \qquad Qts, \Lambda_4, \Lambda \to \Pi, \Pi_4}{\Lambda_3, \Lambda_4, \Lambda \to \Pi, \Pi_3, \Pi_4} \\
\vdots \\
\dfrac{\dfrac{\Lambda, \Lambda \to \Pi, \Pi}{\Lambda \xrightarrow{S} \Pi}}{} \, I_0 \\
\vdots \\
\Gamma \to \Delta
\end{array}
$$

Let d' be the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J' in π' except J_0, $d'(J') = d(J)$, where J is the corresponding one in π and $d(J_0) = j$. We shall note the following facts:

1. $d(\mathfrak{B}(X, Q_{\prec t}, t, s)) = j \prec j \oplus 1 = d(Qts) = d(Qt_1s_2) = d(Qt_2s_2)$.
2. For each formula A in Λ or Π, $d(A) \preceq j$ by the definition of I_0.

By the above facts, we can show that $\langle \pi'; d'; \check{S}^* \rangle$ is a derivation with degree. Next we shall prove $O_0(I_0; \pi'; d'; \check{S}^*) \ll_0 O_0(I_0; \pi; d; \check{S}^*)$. Since

$$O_0(S_2; \pi; d; \check{S}^*) = \xi(\rho_{2l} - \rho_2, 0, O_0(S_{2l}; \pi; d; \check{S}^*))\# O_0(S_{2r}; \pi; d; \check{S}^*)\#(\xi, 0, 0))$$

and

$$O_0(S_2'; \pi'; d'; \check{S}^*) = \xi(\rho_{2l} - \rho_2, 0, (j, 0, O_0(J_0; \pi'; d'; \check{S}^*))\# O_0(S_{2r}; \pi'; d'; \check{S}^*)),$$

$O_0(S_2'; \pi'; d'; \check{S}^*) \ll_{j+1} O_0(S_2; \pi; d; \check{S}^*)$. Hence $O_0(I_0; \pi'; d'; \check{S}^*) \ll_{j+1} O_0(I_0; \pi; d; \check{S}^*)$. We shall note that $O_0(J_0; \pi'; d'; \check{S}^*)$ is the only one j-section (cf. [11]) which occurs in $O_0(I_0; \pi'; d'; \check{S}^*)$ and does not occur in $O_0(I_0; \pi; d; \check{S}^*)$ and every k-section $(k < j)$ in $O_0(I_0; \pi'; d'; \check{S}^*)$ occurs in $O_0(I_0; \pi; d; \check{S}^*)$. So, in order to show that $O_0(I_0; \pi'; d'; \check{S}^*) \ll_0 O_0(I_0; \pi; d; \check{S}^*)$, it suffices to show that $O_0(J_0; \pi'; d'; \check{S}^*) <_j O_0(I_0; \pi; d; \check{S}^*)$. But it is clear, because $O_0(J_0; \pi'; d'; \check{S}^*) \ll_0 O_0(I_0; \pi; d; \check{S}^*)$. Hence we have $O_0(I_0; \pi'; d'; \check{S}^*) \ll_0 O_0(I_0; \pi; d; \check{S}^*)$. Thus, we have $O_0(\pi'; d'; \check{S}^*) \ll_0 O_0(\pi; d; \check{S}^*)$ by proposition 2. Hence we can transform π' to an (S^*)-strongly normal derivation with the same end sequent, by induction hypothesis.

(8.1.2.2) The case where $t \prec \xi$ is false under the standard interpretation.

We reduce π to the derivation π':

$$\dfrac{\dfrac{\begin{matrix}\pi_{2l}\vdots \\ \Lambda_2 \xrightarrow{S_{2l}} \Pi_2, t_2 \prec \xi \end{matrix} \quad t_2 \prec \xi \xrightarrow{\hat{S}}}{\Lambda_2 \rightarrow \Pi_2}}{\begin{matrix} Qts, \Lambda_2 \xrightarrow{S_2} \Pi_2 \\ \vdots \end{matrix}} \quad .$$

Let d' be the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J' in π', $d'(J') = d(J)$, where J is the corresponding one in π. Then $\langle \pi'; d'; \check{S}^* \rangle$ is a derivation with degree. The letter "d'" is also used to denote the restriction of d' to the set of substitutions in π_{2l}. We shall show that $O_0(S_2; \pi'; d'; \check{S}^*) \ll_0 O_0(S_2; \pi; d; \check{S}^*)$. Then, note that $h_0(S_{2l}; \pi') = \rho_2$.

$$\begin{aligned} O_0(S_{2l}; \pi'; d'; \check{S}^*) &= O_{\rho_2}(S_{2l}; \pi_{2l}; d'; \Lambda_2^* \rightarrow \Pi_2^*, t_2 \prec \xi) \\ &\underline{\ll}_0\, \xi(\rho_{2l} - \rho_2, 0, O_{\rho_{2l}}(S_{2l}; \pi_{2l}; d'; \Lambda_2^* \rightarrow \Pi_2^*, t_2 \prec \xi)) \\ &= \xi(\rho_{2l} - \rho_2, 0, O_0(S_{2l}; \pi; d; \check{S}^*)). \end{aligned}$$

Thus,

$$\begin{aligned} O_0(S_2; \pi'; d'; \check{S}^*) &= O_0(S_{2l}; \pi'; d'; \check{S}^*) \# 0 \\ &\underline{\ll}_0\, \xi(\rho_{2l} - \rho_2, 0, O_0(S_{2l}; \pi; d; \check{S}^*)) \# 0 \\ &\ll_0 \xi(\rho_{2l} - \rho_2, 0, O_0(S_{2l}; \pi; d; \check{S}^*) \# O_0(S_{2r}; \pi; d; \check{S}^*) \# (\xi, 0, 0)) \\ &= O_0(S_2; \pi; d; \check{S}^*). \end{aligned}$$

So, $O_0(\pi'; d'; \check{S}^*) \ll_0 O_0(\pi; d; \check{S}^*)$ by proposition 2. Hence we can transform π' to an (S^*)-strongly normal derivation with the same end sequent, by induction hypothesis.

(8.2) The case where the cut formulas of I are of the form $Q_{\prec u}ts$.

Assume that π is of the form:

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\dfrac{\begin{array}{cc}\vdots & \vdots\\ \Lambda_1 \xrightarrow{S_{1l}} \Pi_1, t_1 \prec u_1 & \Lambda_1 \xrightarrow{S_{1r}} \Pi_1, Qt_1s_1\end{array}}{\Lambda_1 \xrightarrow{S_1} \Pi_1, Q_{\prec u_1}t_1s_1}\\ \vdots \\ \Lambda_3 \xrightarrow{S_3} \Pi_3, Q_{\prec u}ts\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ Qt_2s_2, \Lambda_2 \rightarrow \Pi_2\end{array}}{Q_{\prec u_2}t_2s_2, \Lambda_2 \xrightarrow{S_2} \Pi_2}\\ \vdots \\ Q_{\prec u}ts, \Lambda_4 \xrightarrow{S_4} \Pi_4\end{array}}{\Lambda_3, \Lambda_4 \xrightarrow{S_5} \Pi_3, \Pi_4}\, I\\
\vdots\\
\dfrac{}{\Lambda \xrightarrow{S} \Pi}\, I_0\\
\vdots\\
\Gamma \rightarrow \Delta
\end{array},
$$

where S denotes the uppermost sequent below I whose height based on 0 is less than that of the upper sequents of I. Assume that $h_0(S_3; \pi) = \rho$ and $h_0(S; \pi) = \sigma$. Then note that $\sigma < \rho$ by our choice of I_0.

(8.2.1) The case where $Q_{\prec u}ts$ is not closed.

We reduce π to the derivation π':

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{cc}\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Lambda_1 \xrightarrow{S_{1r}} \Pi_1, Qt_1s_1\end{array}}{\overline{\Lambda_1 \xrightarrow{S_1} Qt_1s_1, \Pi_1, Q_{\prec u_1}t_1s_1}}\\ \vdots \\ \Lambda_3 \rightarrow Qts, \Pi_3, Q_{\prec u}ts\end{array} & \begin{array}{c}\vdots\\ \\ \\ Q_{\prec u}ts, \Lambda_4 \rightarrow \Pi_4\end{array}\end{array}}{\Lambda_3, \Lambda_4 \xrightarrow{S_5'} Qts, \Pi_3, \Pi_4}\\ \vdots\end{array}}{\Lambda \xrightarrow{S'} Qts, \Pi}\, I'}{\overline{\Lambda \rightarrow \Pi, Qts}} \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{cc}\begin{array}{c}\vdots\\ \\ \\ \Lambda_3 \rightarrow \Pi_3, Q_{\prec u}ts\end{array} & \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ Qt_2s_2, \Lambda_2 \rightarrow \Pi_2\end{array}}{\overline{Q_{\prec u_2}t_2s_2, \Lambda_2, Qt_2s_2 \rightarrow \Pi_2}}\\ \vdots \\ Q_{\prec u}ts, \Lambda_4, Qts \rightarrow \Pi_4\end{array}\end{array}}{\Lambda_3, \Lambda_4, Qts \xrightarrow{S_5''} \Pi_3, \Pi_4}\\ \vdots\end{array}}{\Lambda, Qts \xrightarrow{S''} \Pi}\, I''}{\overline{Qts, \Lambda \rightarrow \Pi}}}{\dfrac{\Lambda, \Lambda \rightarrow \Pi, \Pi}{\overline{\Lambda \xrightarrow{S} \Pi}}}\\
\vdots
\end{array}
$$

Let d' be the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J' in π', $d'(J') = d(J)$, where J is the corresponding one in π. We shall note the following facts:

1. $d(Qts) \preceq \xi = d(Q_{\prec u}ts)$.
2. There exist no substitutions between S_5' and S'.
3. There exist no substitutions between S_5'' and S''.

By the above facts, it is clear that $\langle \pi'; d'; \check{S}^* \rangle$ is a derivation with degree. We shall prove $O_0(S;\pi';d';\check{S}^*) \ll_0 O_0(S;\pi;d;\check{S}^*)$. Since we have $O_0(S_1;\pi';d';\check{S}^*) \ll_0 O_0(S_1;\pi;d;\check{S}^*)$, we have $O_0(I';\pi';d';\check{S}^*) \ll_0 O_0(I;\pi;d;\check{S}^*)$. Similarly, we have $O_0(I'';\pi';d';\check{S}^*) \ll_0 O_0(I;\pi;d;\check{S}^*)$. Note that $h_0(S';\pi') = h_0(S'';\pi') = \sigma$. Thus,

$$
\begin{aligned}
O_0(S;\pi';d';\check{S}^*) &= \xi(\rho-\sigma, 0, O_0(I';\pi';d';\check{S}^*)) \# \xi(\rho-\sigma, 0, O_0(I';\pi';d';\check{S}^*)) \\
&\ll_0 \xi(\rho-\sigma, 0, O_0(I;\pi;d;\check{S}^*)) \quad (\text{becauses } \sigma < \rho) \\
&= O_0(S;\pi;d;\check{S}^*).
\end{aligned}
$$

So, $O_0(\pi';d';\check{S}^*) \ll_0 O_0(\pi;d;\check{S}^*)$ by proposition 2. Hence we can transform π' to an (S^*)-strongly normal derivation with the same end sequent, by induction hypothesis.

(8.2.2) The case where $Q_{\prec u}ts$ is closed.

(8.2.2.1) The case where $t \prec u$ is true under the standard interpretation. Similar to the case (8.2.1).

(8.2.2.2) The case where $t \prec u$ is false under the standard interpretation. We reduce π to the derivation π':

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Lambda_1 \to \Pi_1, t_1 \prec u_1\end{array}}{\Lambda_1 \xrightarrow{S_1} t_1 \prec u_1, \Pi_1, Q_{\prec u_1} t_1 s_1}}{\begin{array}{c}\vdots\\ \Lambda_3 \to t \prec u, \Pi_3, Q_{\prec u}ts\end{array}} \quad \begin{array}{c}\vdots\\ Q_{\prec u}ts, \Lambda_4 \to \Pi_4\end{array}}{\Lambda_3, \Lambda_4 \to t \prec u, \Pi_3, \Pi_4} \\ \vdots \end{array}}{\Lambda \to t \prec u, \Pi}\, I'}{\Lambda \to \Pi, t \prec u} \qquad t \prec u \to}{\begin{array}{c}\Lambda \xrightarrow{S} \Pi \\ \vdots \end{array}}
$$

.

Let d' be the mapping from the set of substitutions in π' to the ordinals less than ξ such that, for each substitution J' in π', $d'(J') = d(J)$, where J is the corresponding one in π. Note that $d(t \prec u) = 0$. Then it is clear that $\langle \pi'; d'; \check{S}^* \rangle$ is a derivation with degree. Next, we shall prove $O_0(S; \pi'; d'; \check{S}^*) \ll_0 O_0(S; \pi; d; \check{S}^*)$. Since we have $O_0(S_1; \pi'; d'; \check{S}^*) \ll_0 O_0(S_1; \pi; d; \check{S}^*)$, we have $O_0(I'; \pi'; d'; \check{S}^*) \ll_0 O_0(I; \pi; d; \check{S}^*)$. Thus,

$$\begin{aligned} O_0(S; \pi'; d'; \check{S}^*) &= \xi(\rho - \sigma, 0, O_0(I', \pi'; d'; \check{S}^*))\#0 \\ &\ll_0 \xi(\rho - \sigma, 0, O_0(I; \pi; d; \check{S}^*)) \qquad (\text{because } \sigma < \rho) \\ &= O_0(S; \pi; d; \check{S}^*). \end{aligned}$$

Thus, $O_0(\pi'; d'; \check{S}^*) \ll_0 O_0(\pi; d; \check{S}^*)$ by proposition 2. Hence we can transform π' to an (S^*)-strongly normal derivation with the same end sequent, by induction hypothesis.

This completes a proof of Lemma. ∎

Acknowledgement. The author would like to thank the referee for the valuable suggestions.

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