TSUKUBA J. MATH.
Vol. 24 No. 2 (2000), 387–417

ON WARING'S PROBLEM: TWO CUBES AND SEVEN BIQUADRATES

By

Jörg Brüdern and Trevor D. Wooley[1]

1. Introduction

Additive problems involving sums of small powers of natural numbers form a convenient area for the development and investigation of new techniques within the Hardy-Littlewood method. In accordance with recent progress in Waring's problem for cubes, and for biquadrates, respectable advances have been made in our understanding of the representation of large integers as mixed sums of cubes and biquadrates. For each natural number r, let $B(r)$ denote the least integer s with the property that all sufficiently large integers are the sum of r cubes and s biquadrates. Then in the current state of knowledge, one has the upper bound $B(r) \leq F(r)$, where the values $F(r)$ are described in the following table.

r	1	2	3	4	5	6	7
$F(r)$	9	8	6	5	3	2	0

Here, the values of $F(r)$ for $1 \leq r \leq 3$ are due to Kawada and Wooley [11], and for $r = 5, 6$ they are due to Brüdern [3]. For $r = 7$, of course, the conclusion implicit in the table is the celebrated result of Linnik [12] that all sufficiently large integers are the sum of seven positive integral cubes. When $r = 4$, meanwhile, the desired conclusion is not immediately available from the literature, but follows directly from a result of Brüdern [3] to the effect that almost all positive integers are the sum of three positive integral cubes and a biquadrate, together with a result of Kawada and Wooley [11] to the effect that almost all positive integers are the sum of four biquadrates and a positive integral cube (indeed, such a conclusion was noted in [11]). The purpose of this paper is to exploit recent

---

[1] Packard Fellow, and supported in part by NSF grant DMS-9970440. The second author is grateful to the Mathematisches Institut A at Stuttgart for its generous hospitality during the period in which this work was undertaken.
Received October 28, 1999
Revised February 7, 2000

developments in the circle method so as to improve the upper bound for $B(2)$ implicit in the above table.

THEOREM 1. *Let $v(n)$ denote the number of representations of the natural number n as the sum of two positive integral cubes and seven biquadrates. Then* $v(n) \gg n^{43/36}(\log n)^{-1}$.

The conclusion of Theorem 1 shows, in particular, that the entry $F(2) = 8$ in the above table may now be replaced by $F(2) = 7$. We note that our methods are of sufficient power to establish a lower bound for $v(n)$ in which the factor $(\log n)^{-1}$ is replaced by a very small positive power of n. However, we fall short of the lower bound $v(n) \gg n^{17/12}$ suggested by a formal application of the Hardy-Littlewood method.

The ideas underlying our proof of Theorem 1 are simple to describe, though putting them into effect entails some technical difficulties. Our starting point in §2 is an estimate for the seventh moment of a biquadratic smooth Weyl sum significantly sharper than any available hitherto, and this we establish by means of the "breaking classical convexity" device of Wooley [19]. Following some preliminary work in §3, we move on in §4 to incorporate the latter estimate into an upper bound for a mean value, involving two cubic exponential sums and seven biquadratic exponential sums, by means of an efficient differencing process restricted to minor arcs. The bound here is obtained via a modification of a method due to Vaughan [13], the novel use of an odd number of generating functions having recently arisen in work of the authors [7] concerning sums of three cubes and a sixth power. The estimates established thus far narrowly provide a satisfactory bound for the contribution of a rather thin set of minor arcs in our application of the Hardy-Littlewood method, and there remains the problem of handling the corresponding set of major arcs. The latter difficulty is surmounted in §§4–6 by negotiating a sequence of pruning manoeuvres, both on the differenced and undifferenced sides of the efficient differencing procedure. Pruning techniques of Brüdern [2, 4] and Brüdern, Kawada and Wooley [5] play a prominent role in this somewhat lengthy process.

The estimation of the seventh moment of a biquadratic smooth Weyl sum entails bounding a sequence of moments of such smooth Weyl sums. Since these estimates may be of some independent interest in future work associated with sums of biquadrates, we record these estimates in a table in §2 for ease of reference. We highlight the most interesting of these estimates in Theorem 2 below, and to facilitate the statement of the latter we require some notation.

When P is a positive number, and $2 \le R \le P$, write $\mathscr{A}(P,R)$ for the set of R-smooth numbers up to P, that is

$$\mathscr{A}(P,R) = \{n \in [1,P] \cap \boldsymbol{Z} : p|n,\ p \text{ prime} \Rightarrow p \le R\}.$$

Also, define

$$f(\alpha; Q, R) = \sum_{x \in \mathscr{A}(Q,R)} e(\alpha x^4) \quad \text{and} \quad F(\alpha; Q) = \sum_{1 \le y \le Q} e(\alpha y^4), \tag{1.1}$$

where, as usual, we write $e(z)$ for $e^{2\pi i z}$.

THEOREM 2. *There is a positive number η with the property that whenever $R \le P^\eta$, one has*

$$\int_0^1 |f(\alpha; P, R)|^7 \, d\alpha \ll P^{\mu_7},$$

where $\mu_7 = 3.849408$. *Also, under the same hypotheses one has*

$$\int_0^1 |F(\alpha; P)^2 f(\alpha; P, R)^{s-2}| \, d\alpha \ll P^{\mu_s} \quad (s = 6, 8, 10),$$

where

$$\mu_6 = 3.183428, \quad \mu_8 = 4.594193, \quad \mu_{10} = 6.213431.$$

For comparison, the sharpest bounds available in the literature hitherto arise by combining methods of Vaughan [14] and [15], and yield analogues of the above conclusions in which

$$\mu_6 = 3.1861407, \quad \mu_8 = 4.5951377, \quad \mu_{10} = 6.2142036,$$

whence, by convexity,

$$\mu_7 = 3.8906392.$$

Here we note that both Ford [8] and Israilov and Allakov [10] have recorded exponents which implicitly contain stronger estimates for μ_8 and μ_{10} than those recorded above. Unfortunately, however, both these sets of estimates are based on the assumption that the parameter θ_4 arising in Vaughan's iterative method is independent of the exponent λ_3, and a cursory examination of § 4 of Vaughan [14] reveals this assumption to be unfounded. The exponents arising in [8] and [10] are consequently invalid (see the erratum [9]).

Throughout, ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, with implicit constants depending at most on ε. In an effort to simplify our analysis, we adopt the following convention concerning the number ε. Whenever ε appears in a statement, either implicitly or explicitly, we assert that for each $\varepsilon > 0$, the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. Finally, when y is a real number we write $[y]$ for the greatest integer not exceeding y.

The authors are grateful to the referee for useful comments.

2. Mean Value Estimates for Smooth Weyl Sums

Before advancing to the main theme of our argument, we first discuss the non-trivial estimates for fractional moments of smooth Weyl sums mentioned in connection with Theorem 2. We first require some further notation. Let P and R be positive real numbers, and recall the definition (1.1). When s is a real number, we define the mean value $U_s(P,R)$ by

$$U_s(P,R) = \int_0^1 |f(\alpha; P, R)|^s \, d\alpha.$$

We say that an exponent μ_s is *permissible* whenever the exponent has the property that, for each $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon, s)$ such that whenever $R \leq P^\eta$, one has

$$U_s(P,R) \ll_{\varepsilon,s} P^{\mu_s+\varepsilon}.$$

It is a fact (see [19]) that for every positive number s, one has $\mu_s \geq \max\{s/2, s-4\}$. Moreover, in view of the trivial estimate for $U_s(P,R)$, one may always take $\mu_s \leq s$ for each s. It is convenient to describe an exponent δ_s as *associated* if the exponent $\mu_s = s/2 + \delta_s$ is permissible, and to describe an exponent Δ_s as *admissible* if the exponent $\mu_s = s - 4 + \Delta_s$ is permissible.

A discussion of the broad ideas underlying the calculation of permissible exponents μ_s may be found in Wooley [19], and the particular case of cubic smooth Weyl sums is discussed in detail in §2 of Baker, Brüdern and Wooley [1], and Brüdern and Wooley [6]. In order to treat biquadratic smooth Weyl sums efficiently, we are forced to work somewhat harder with the methods of [19].

LEMMA 2.1. *Let u be a real number with $u > 2$, let v be a real number with $4u/3 \leq v \leq 2u$, and suppose that δ_u and δ_v are associated exponents. Then the exponent δ_{u+2} is associated, where*

$$\delta_{u+2} = \delta_u(1-\theta) + \tfrac{1}{2}u\theta$$

and

$$\theta = \frac{2u - v + 2(u\delta_v - v\delta_u)}{6u + 2v + 2(u\delta_v - v\delta_u)}.$$

PROOF. Suppose that u and v satisfy the hypotheses of the statement of the lemma, and write $s = u + 2$. Take ϕ to be a real number with $0 \le \phi \le 1/4$ to be chosen later, and write

$$M = P^{\phi}, \quad H = PM^{-4} \quad \text{and} \quad Q = PM^{-1}.$$

We apply the argument of §4 of Wooley [19] with $t = 1$. Since we may suppose that $\mu_u = u/2 + \delta_u$ and $\mu_v = v/2 + \delta_v$ are permissible exponents, on following the argument of [19, §4] surrounding equations (4.2) and (4.3) of that paper, we find that our choice for ϕ is determined from the equation

$$PMQ^{\mu_u} = P^{1/2}H^{u/v}M^{u/v}Q^{u\mu_v/v}.$$

Thus we take $\phi = \min\{\theta, 1/4\}$, where θ is defined as in the statement of the lemma. We may now mimic the argument of the proof of Theorem 1.1 in [19, §4] to deduce that

$$\mu_s^* = \mu_u(1 - \theta) + 1 + u\theta$$

is permissible, and the conclusion of the lemma follows immediately.

We augment this lemma with a special method for estimating μ_{10}.

LEMMA 2.2. *Suppose that δ_8 is an associated exponent. Then the exponent*

$$\delta_{10} = (9\delta_8 + 8)/11$$

is associated, provided only that $\delta_{10} \ge 68/57$.

PROOF. This is immediate from the proof of Theorem 4.3 of Vaughan [14] in the case $s = 5$, given our definition of an associated exponent.

We note that the condition on δ_{10} could be weakened with little additional effort to $\delta_{10} \ge 1$, but in the present circumstances such is surplus to our requirements. Next we consider the twelfth moment of the biquadratic smooth Weyl sum.

LEMMA 2.3. *The exponent $\delta_{12} = 2$ is associated.*

PROOF. By considering the underlying diophantine equations, one obtains

$$U_{12}(P,R) \le \int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^{10}|\,d\alpha,$$

and by Lemma 5.2 of Vaughan [14], the latter integral is $O(P^8)$. The desired conclusion is therefore immediate from our definition of an associated exponent.

Finally, we recall a consequence of classical convexity bounds.

LEMMA 2.4. *Suppose that $s > 2$ and that $t < s$. Whenever δ_{s-t} and δ_{s+t} are associated exponents, then the exponent $\delta_s = (\delta_{s+t} + \delta_{s-t})/2$ is also associated.*

PROOF. This is merely the analogue for biquadratic smooth Weyl sums of Lemma 4.3 of Brüdern and Wooley [6].

We now indicate how to calculate relatively strong associated exponents using a computer in combination with Lemmata 2.1–2.4. One sets up an array of known associated exponents δ_{jh} $(0 \le j \le J)$, for some step size $h > 0$ and upper limit J (with $J \ge 24/h$), by using known bounds for δ_s. Thus we have the associated exponent $\delta_s = 0$ for $0 \le s \le 4$ which follows from Hua's Lemma (see Vaughan [16, Lemma 2.5]), on considering the underlying diophantine equations, and the associated exponents

$$\delta_6 = 0.250, \quad \delta_8 = 0.619, \quad \delta_{10} = 1.233,$$

which are immediate from Table 4.1 of Vaughan [14]. Of course, sharper values are available through the methods of Vaughan [15], but these do not assist us in any significant manner. Furthermore, it is immediate from Lemma 2.3 that $\delta_s = s/2 - 4$ is an associated exponent for $s \ge 12$. By making use of the convexity argument of Lemma 2.4, one readily obtains associated exponents for intermediate values of s. Next, for the interesting values of j with $4/h < j < 12/h$, one may now calculate new associated exponents δ_{jh} by means of Lemmata 2.1, 2.2 and 2.4. Observe that Lemma 2.1 may be applied with a range of possible values for the parameter v, and so we examine all such permissible values of the shape ih, for suitable integers i, in order to locate the optimal choice for this parameter. By iterating this process for $4/h < j < 12/h$, one derives new collections of associated exponents, and eventually this collection converges to some set of limiting values. We note also that Lemma 2.4 may be employed in the form

$$\delta_{jh} \le \tfrac{1}{2}(\delta_{(j-k)h} + \delta_{(j+k)h}) \quad (1 \le k \le K),$$

for a suitable parameter K, in order to economise on the number of operations required to obtain convergence of the iterative process.

In the table below we record associated exponents δ_s for $4 \le s \le 12$, rounded up in the final displayed decimal place. These values were calculated by using a step size of $h = 0.005$, although we record values only at intervals of 0.1. Since it is convenient in many circumstances to make use of admissible exponents Δ_s, we record also in the table the values of Δ_s corresponding to each δ_s, these being related by means of the formula

$$\delta_s = \tfrac{1}{2}s - 4 + \Delta_s.$$

We record also for each entry s the corresponding value of v employed in the application of Lemma 2.1, when indeed Lemma 2.1 gives the optimal associated exponent. When Lemma 2.4 is instead optimal, we record an asterisk in this column, and in any other circumstance we leave this entry blank. We note that for $8 < s < 10$, and for $10 < s < 12$, the associated exponents obtained by our calculations were simply linear interpolations between δ_8 and δ_{10}, and between δ_{10} and δ_{12} respectively. We omit intermediate values, therefore, in the interests of saving space, leaving it to the readers to perform such an interpolation for themselves.

s	δ_s	Δ_s	v	s	δ_s	Δ_s	v
4.0	0.000000	2.000000		6.0	0.183428	1.183428	6.455
4.1	0.000382	1.950382	4.200	6.1	0.198000	1.148000	6.400
4.2	0.001816	1.901816	4.400	6.2	0.212998	1.112998	6.355
4.3	0.004376	1.854376	4.600	6.3	0.228430	1.078430	6.325
4.4	0.008265	1.808265	4.800	6.4	0.244340	1.044340	6.300
4.5	0.013131	1.763131	5.000	6.5	0.260608	1.010608	6.275
4.6	0.019098	1.719098	5.200	6.6	0.277254	0.977254	6.255
4.7	0.026416	1.676416	5.400	6.7	0.294345	0.944345	*
4.8	0.034677	1.634677	5.600	6.8	0.312077	0.912077	6.400
4.9	0.043573	1.593573	5.800	6.9	0.330429	0.880429	*
5.0	0.053081	1.553081	6.000	7.0	0.349408	0.849408	*
5.1	0.063332	1.513332	6.200	7.1	0.369175	0.819175	6.800
5.2	0.074546	1.474546	6.400	7.2	0.389840	0.789840	*
5.3	0.086687	1.436687	6.600	7.3	0.411416	0.761416	*
5.4	0.099788	1.399788	6.710	7.4	0.434081	0.734081	7.200
5.5	0.113290	1.363290	6.680	7.5	0.457775	0.707775	*
5.6	0.126989	1.326989	6.645	7.6	0.482580	0.682580	*
5.7	0.140865	1.290865	6.600	7.7	0.508530	0.658530	7.600
5.8	0.154904	1.254904	6.555	7.8	0.535705	0.635705	*
5.9	0.169096	1.219096	6.510	7.9	0.564198	0.614198	*
				8.0	0.594193	0.594193	8.000
				10.0	1.213431	0.213431	
				12.0	2.000000	0.000000	

We conclude by noting that Theorem 2 follows immediately from the above discussion and table of exponents. A remark is in order, however, concerning the conclusion of Theorem 2 in regards to the cases $s = 6, 8, 10$. Here one should perform the efficient differencing process not by means of the methods of Wooley [19], but instead by using the approach of Wooley [18] (see especially the remark concluding § 3 therein). This facilitates the introduction of the classical Weyl sums $F(\alpha; P)$ into the picture. We then find ourselves in the situation discussed in [19, § 4], and so the exponents claimed in the table do indeed imply the estimates recorded in Theorem 2 for $s = 6, 8, 10$.

3. Preliminaries to the Proof of Theorem 1

Let n be a large natural number, and write

$$P = [(\tfrac{1}{4}n)^{1/3}], \quad Y = n^{1/27}, \quad H = 4PY^{-4} \quad \text{and} \quad Q = n^{1/4}Y^{-1}. \tag{3.1}$$

Further, let $\delta = 10^{-5}$, and write

$$M = n^{\delta}, \quad L = M(2Y)^4 \quad \text{and} \quad V = (\log n)^{\delta}. \tag{3.2}$$

We take η to be a positive number sufficiently small in the context of Theorem 2, and put $R = P^{\eta}$. It is convenient to modify the notation for exponential sums introduced in §§ 1 and 2 by writing

$$F_p(\alpha) = \sum_{\substack{P < x \le 2P \\ (x,p)=1}} e(\alpha x^3), \quad g(\alpha) = \sum_{1 \le y \le Q} e(\alpha y^4), \tag{3.3}$$

$$f(\alpha) = \sum_{z \in \mathscr{A}(Q,R)} e(\alpha z^4). \tag{3.4}$$

We define also

$$S(\alpha) = |g(\alpha) f(\alpha)^6|, \tag{3.5}$$

and

$$\mathscr{F}(\alpha) = \sum_{\substack{Y < p \le 2Y \\ p \equiv 2 \pmod 3}} F_p(\alpha)^2 g(\alpha p^4) f(\alpha p^4)^6, \tag{3.6}$$

where here, and throughout, the summation in p is over prime numbers. Finally, define

$$r(n) = \int_0^1 \mathscr{F}(\alpha) e(-n\alpha)\, d\alpha. \tag{3.7}$$

Then by orthogonality one finds that $r(n)$ is equal to the number of integral solutions of the equation

$$n = x_1^3 + x_2^3 + (py)^4 + (pz_1)^4 + \cdots + (pz_6)^4,$$

with

$$P < x_i \le 2P \quad \text{and} \quad (x_i, p) = 1 \quad (i = 1, 2),$$

$$1 \le y \le Q, \quad z_j \in \mathscr{A}(Q, R) \quad (1 \le j \le 6),$$

$$Y < p \le 2Y \quad \text{and} \quad p \equiv 2 \pmod 3.$$

Since each integer w with $1 \le w \le 2n^{1/4}$ has at most 8 representations in the shape $w = pv$ with $Y < p \le 2Y$ and p prime, it follows from the definition of $v(n)$ in the introduction that $v(n) \gg r(n)$, and so the proof of Theorem 1 will be completed on showing that $r(n) \gg n^{43/36}(\log n)^{-1}$.

We estimate the integral (3.7) by means of the circle method. Our primary Hardy-Littlewood dissection is defined as follows. Denote by $\mathfrak{M}$ the union of the major arcs

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \le Ln^{-1}\}$$

with $0 \le a \le q \le L$ and $(a, q) = 1$, and let $\mathfrak{m} = [0, 1) \backslash \mathfrak{M}$. In the first phase of the analysis, we apply an efficient differencing process restricted to minor arcs in order to estimate the contribution of the arcs $\mathfrak{m}$ to the integral (3.7). This process transforms the latter quantity into a mean value over arcs $\mathfrak{n}$, defined to be the set of real numbers $\alpha \in [0, 1)$ such that whenever $a \in \boldsymbol{Z}$ and $q \in \boldsymbol{N}$ satisfy $(a, q) = 1$ and $|q\alpha - a| \le MQ^{-4}$, then one has $q > M$. The analysis of this new mean value over $\mathfrak{n}$ entails several pruning procedures, and we introduce new sets of major and minor arcs to facilitate the latter as the discussion unfolds. The second phase of the analysis is devoted to obtaining an asymptotic formula for the contribution of the arcs $\mathfrak{M}$ to the integral (3.7). This again involves certain pruning operations, the object being to thin the arcs $\mathfrak{M}$ down to a controllable set $\mathfrak{P}$, which we define to be the union of the arcs

$$\mathfrak{P}(q, a) = \{\alpha \in [0, 1) : |\alpha - a/q| \le Vn^{-1}\}$$

with $0 \le a \le q \le V$ and $(a, q) = 1$. The arcs comprising $\mathfrak{P}$ are sufficiently few and narrow that technology by now familiar to experts is adequate to analyse the behaviour of our generating functions on $\mathfrak{P}$.

We finish this section by extracting from §2 the mean value estimate for smooth Weyl sums which underlies our treatment in the first phase devoted to the analysis of $\mathfrak{m}$.

LEMMA 3.1. *One has*

$$\int_0^1 S(\alpha)\,d\alpha \ll Q^{3.86911}.$$

PROOF. We apply the conclusion of Theorem 2 in combination with Hölder's inequality to obtain from (3.5) the upper bound

$$\int_0^1 S(\alpha)\,d\alpha \ll \left(\int_0^1 |f(\alpha)|^7\,d\alpha\right)^{1/2}\left(\int_0^1 |g(\alpha)^2 f(\alpha)^4|\,d\alpha\right)^{1/4}\left(\int_0^1 |g(\alpha)^2 f(\alpha)^6|\,d\alpha\right)^{1/4}$$
$$\ll Q^{\mu},$$

where $\mu = \mu_7/2 + (\mu_6 + \mu_8)/4 < 3.86911$. The conclusion of the lemma is therefore immediate.

4. Efficient Differencing Restricted to Minor Arcs

In this section we establish an upper bound for the integral

$$r_1(n) = \int_{\mathfrak{m}} \mathscr{F}(\alpha)e(-n\alpha)\,d\alpha \tag{4.1}$$

by engineering an efficient differencing process restricted to minor arcs only. Such procedures originate in work of Vaughan [13] on sums of four cubes, although in the present situation one has differing degrees and odd moments on the scene. Nonetheless, we are able to accomplish a relatively concise treatment by heavy use of Vaughan's work.

Before proceeding further, we define the differenced exponential sum

$$\Phi_p(\alpha) = \sum_{\substack{P<y\leq 2P\\ (y,p)=1}} 1 + 2\Re\left(\sum_{1\leq h\leq H}\sum_{\substack{2P+hp^4<y\leq 4P-hp^4\\ (y,p)=1\\ y\equiv h \pmod 2}} e\left(\frac{3}{4}\alpha hy^2 + \frac{1}{4}\alpha h^3p^8\right)\right). \tag{4.2}$$

LEMMA 4.1. *One has*

$$|r_1(n)| \leq \sum_{\substack{Y<p\leq 2Y\\ p\equiv 2 \pmod 3}} \int_{\mathfrak{n}} \Phi_p(\alpha)S(\alpha)\,d\alpha.$$

PROOF. Observe first that on applying the triangle inequality to (4.1), it follows from (3.5) and (3.6) that

$$r_1(n) \leq \sum_{\substack{Y<p\leq 2Y \\ p\equiv 2 \pmod 3}} \int_{\mathfrak{m}} |F_p(\alpha)|^2 S(\alpha p^4)\, d\alpha. \tag{4.3}$$

We now follow the argument of the proof of [13, Lemma 10], noting the modest adjustments in notation between our setup here and that therein. Writing here $d = p^4$ for a given prime number p occurring in the summation of (4.3), the argument leading to equation (8.5) of [13] reveals that

$$\int_{\mathfrak{m}} |F_p(\alpha)|^2 S(\alpha p^4)\, d\alpha \leq \int_{\mathfrak{n}} d^{-1} \sum_{k=0}^{d-1} \left| F_p\left(\frac{\alpha+k}{d}\right)\right|^2 S(\alpha)\, d\alpha, \tag{4.4}$$

and moreover the argument completing the proof of [13, Lemma 10] again applies, thus yielding the equation

$$d^{-1}\sum_{k=0}^{d-1} \left| F_p\left(\frac{\alpha+k}{d}\right)\right|^2 = \Phi_p(\alpha). \tag{4.5}$$

The conclusion of the lemma is completed on collecting together (4.3), (4.4), (4.5).

In order to transform the conclusion of Lemma 4.1 into a satisfactory bound for $r_1(n)$, we must rework Lemmata 6, 8 and 10 of Vaughan [13] in the new setting resulting from our definition (4.2). Our first step along this path is to remove the coprimality condition $(y,p) = 1$ from the definition of $\Phi_p(\alpha)$, and for this purpose we employ the argument of [13, §5]. Write

$$F(\beta,\gamma;h) = \sum_{\substack{2P<y\leq 4P \\ y\equiv h \pmod 2}} e\left(\frac{3}{4}\beta y^2 - \gamma y\right),$$

$$G_h(\rho,\sigma) = \sum_{\substack{Y<p\leq \min\{2Y,(P/h)^{1/4}\} \\ p\equiv 2 \pmod 3}} e\left(\frac{1}{4}\rho p^8 + \sigma p^4\right),$$

$$\Xi_p(\alpha) = 2\Re\left(\sum_{1\leq h\leq H} e\left(\frac{1}{4}\alpha h^3 p^8\right) \sum_{\substack{2Pp^{-1}+hp^3<y\leq 4Pp^{-1}-hp^3 \\ y\equiv h \pmod 2}} e\left(\frac{3}{4}\alpha h p^2 y^2\right)\right),$$

$$\Upsilon(\alpha;\gamma,\theta) = \sum_{1\leq h\leq H} |F(\alpha h,\gamma;h) G_h(\alpha h^3, \theta\gamma h)|,$$

and then define

$$T_2(p) = \int_{\mathfrak{n}} \Xi_p(\alpha) S(\alpha)\, d\alpha, \quad T_3 = \int_0^1 S(\alpha)\, d\alpha, \tag{4.6}$$

$$T_5(\gamma, \theta) = \int_{\mathfrak{n}} \Upsilon(\alpha; \gamma, \theta) S(\alpha)\, d\alpha. \tag{4.7}$$

LEMMA 4.2. *One has*

$$\sum_{\substack{Y<p\le 2Y \\ p\equiv 2 \pmod 3}} \int_{\mathfrak{n}} \Phi_p(\alpha) S(\alpha)\, d\alpha \ll PYT_3 + (\log P) \sup_{\substack{0\le\gamma\le 1 \\ \theta=\pm 1}} T_5(\gamma, \theta) + \left| \sum_{\substack{Y<p\le 2Y \\ p\equiv 2 \pmod 3}} T_2(p) \right|. \tag{4.8}$$

PROOF. The proof of [13, Lemma 6] (see, in particular, equations (5.13) and (5.26) of [13]) establishes the conclusion of the lemma, on accounting for the substitution here of p^4 in place of p^3. Two remarks are in order for the purpose of clarification. First, on combining (5.14) and (5.20) of [13], it is evident that the error term in (5.22) therein may be replaced here by a term of order

$$H \int_0^1 S(\alpha)\, d\alpha \le PT_3,$$

whence the contribution arising from this term in (4.8) is $O(PYT_3)$ (the reader may compare equation (5.26) of [13]). Second, on the bottom of page 142 of [13], the condition $p \le (P/(2h))^{1/3}$ should read $p \le (P/h)^{1/3}$, and we have corrected this minor error in the definition of $G_h(\rho, \sigma)$ above. This adjustment is harmless both in the argument of [13, §6], and in what follows herein.

The estimation of the first term on the right hand side of (4.8) is essentially trivial in view of the conclusion of Lemma 3.1.

LEMMA 4.3. *One has*

$$PYT_3 \ll n^{43/36-2\delta}.$$

PROOF. Recalling (3.1), (4.6) and the conclusion of Lemma 3.1, one finds that

$$PYT_3 = PY \int_0^1 S(\alpha)\, d\alpha \ll PYQ^{3.86911} \ll n^{43/36-9\delta},$$

and hence the conclusion of the lemma is immediate.

We next discuss the contribution of the second term on the right hand side of (4.8).

LEMMA 4.4. *One has*

$$(\log P) \sup_{\substack{0 \le \gamma \le 1 \\ \theta = \pm 1}} T_5(\gamma, \theta) \ll n^{43/36 - \delta/26}.$$

PROOF. We make use of §6 of Vaughan [13], adjusting the argument there to fit our present situation. We begin by observing that the argument of the proof of [13, Lemma 7] (see also the proof of Lemma 3.1 of Vaughan [14]) shows that whenever $\alpha \in \boldsymbol{R}$, and $a \in \boldsymbol{Z}$ and $q \in \boldsymbol{N}$ satisfy $|q\alpha - a| \le q^{-1}$ and $(a, q) = 1$, then one has

$$\sum_{1 \le h \le H} |F(\alpha h, \gamma; h)|^2 \ll P^{\varepsilon}(P^2 H(q + Q^4|q\alpha - a|)^{-1} + PH + q + Q^4|q\alpha - a|). \quad (4.9)$$

Next, since $H^{3/4}Y^2 \ll HY$, the argument used to establish [13, Lemma 8] shows that whenever $\alpha \in \boldsymbol{R}$, and $a \in \boldsymbol{Z}$ and $q \in \boldsymbol{N}$ satisfy $(a, q) = 1$, $q \le Q^4 H^{-3/4}$ and $|q\alpha - a| \le H^{3/4} Q^{-4}$, then one has

$$\sum_{1 \le h \le H} |G_h(\alpha h^3, \theta \gamma h)|^2 \ll P^{\varepsilon}(HY^2(q + Q^4|q\alpha - a|)^{-1/3} + HY). \quad (4.10)$$

Experts may care to note that the exponent $1/3$ in the last expression is a consequence of the summation in the definition of $G_h(\rho, \sigma)$ being over prime numbers, the corresponding exponent arising from composite numbers being $1/4$.

Equipped with the basic estimates (4.9) and (4.10), our attack on the proof of the lemma is achieved in three stages. Let $\mathfrak{K}$ denote the union of the intervals

$$\mathfrak{K}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \le PQ^{-4}\}$$

with $0 \le a \le q \le P$ and $(a, q) = 1$, and write $\mathfrak{k} = [0, 1) \backslash \mathfrak{K}$. In the first stage of our offensive, we estimate the contribution to the mean value (4.7) arising from the arcs $\mathfrak{k} \subseteq \mathfrak{n}$. Suppose that $\alpha \in \mathfrak{k}$. By Dirichlet's approximation theorem, there exist $a \in \boldsymbol{Z}$ and $q \in \boldsymbol{N}$ with $1 \le q \le P^{-1}Q^4$, $(a, q) = 1$ and $|q\alpha - a| \le PQ^{-4} \le q^{-1}$. By the definition of $\mathfrak{k}$, moreover, one necessarily has $q > P$, and hence by (4.9),

$$\sup_{\alpha \in \mathfrak{k}} \left(\sum_{1 \le h \le H} |F(\alpha h, \gamma; h)|^2 \right) \ll P^{1+\varepsilon} H + P^{-1} Q^4 \ll P^{1+\varepsilon} H.$$

A second application of Dirichlet's approximation theorem reveals that there exist

$a \in \boldsymbol{Z}$ and $q \in \boldsymbol{N}$ with $1 \le q \le Q^4 H^{-3/4}$, $(a,q) = 1$ and $|q\alpha - a| \le H^{3/4} Q^{-4}$. The definition of $\mathfrak{k}$ again requires that $q > P$, and consequently it follows from (4.10) that

$$\sup_{\alpha \in \mathfrak{k}} \left(\sum_{1 \le h \le H} |G_h(\alpha h^3, \theta \gamma h)|^2 \right) \ll P^{\varepsilon}(HY^2 P^{-1/3} + HY) \ll P^{\varepsilon} HY.$$

On recalling (4.6) and (3.1), therefore, an application of Cauchy's inequality reveals that

$$\int_{\mathfrak{k}} \Upsilon(\alpha; \gamma, \theta) S(\alpha)\, d\alpha \ll P^{\varepsilon}(PY)^{1/2} HT_3 \ll P^{1+\varepsilon} YT_3,$$

whence by Lemma 4.3,

$$\int_{\mathfrak{k}} \Upsilon(\alpha; \gamma, \theta) S(\alpha)\, d\alpha \ll n^{43/36-\delta}. \tag{4.11}$$

In the second stage of our offensive, we prune down to a set of arcs narrower and sparser than $\mathfrak{K}$. Let $\mathfrak{L}$ denote the union of the intervals

$$\mathfrak{L}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \le Y^3 Q^{-4}\}$$

with $0 \le a \le q \le Y^3$ and $(a,q) = 1$, and write $\mathfrak{l} = [0,1) \backslash \mathfrak{L}$. Suppose that $\alpha \in \mathfrak{l}$. By Dirichlet's approximation theorem, there exist $a \in \boldsymbol{Z}$ and $q \in \boldsymbol{N}$ with $1 \le q \le H^{-3/4} Q^4$, $(a,q) = 1$ and $|q\alpha - a| \le H^{3/4} Q^{-4}$. The definition of $\mathfrak{l}$ implies, moreover, that $q + Q^4 |q\alpha - a| > Y^3$, and hence by (4.10),

$$\sup_{\alpha \in \mathfrak{l}} \left(\sum_{1 \le h \le H} |G_h(\alpha h^3, \theta \gamma h)|^2 \right) \ll P^{\varepsilon} HY.$$

Thus we deduce from Cauchy's inequality and (4.9) that whenever $\alpha \in \mathfrak{K}(q,a) \cap \mathfrak{l} \subseteq \mathfrak{K} \cap \mathfrak{l}$, one has

$$\Upsilon(\alpha; \gamma, \theta) \ll P^{1+\varepsilon} HY^{1/2} (q + Q^4 |q\alpha - a|)^{-1/2}.$$

Define the function $\Delta(\alpha)$ for $\alpha \in [0,1)$ by taking

$$\Delta(\alpha) = (q + Q^4 |q\alpha - a|)^{-1},$$

when $\alpha \in \mathfrak{K}(q,a) \subseteq \mathfrak{K}$, and by taking $\Delta(\alpha)$ to be zero otherwise. Then on recalling (3.5) and (4.6), we obtain

$$\int_{\mathfrak{K} \cap \mathfrak{l}} \Upsilon(\alpha; \gamma, \theta) S(\alpha)\, d\alpha \ll I_1, \tag{4.12}$$

where

$$I_1 = P^{1+\varepsilon} H Y^{1/2} \int_{\mathfrak{R}} \Delta(\alpha)^{1/2} |g(\alpha) f(\alpha)^6| \, d\alpha. \tag{4.13}$$

Observe next that Theorem 4.1 and Lemma 4.6 of Vaughan [16] together imply that for $\alpha \in \mathfrak{R}(q, a) \subseteq \mathfrak{R}$, one has

$$g(\alpha) \ll Q\Delta(\alpha)^{1/4} + (q + Q^4|q\alpha - a|)^{1/2+\varepsilon} \ll Q\Delta(\alpha)^{1/4} + P^{1/2+\varepsilon}.$$

Then on substituting into (4.13), we obtain

$$I_1 \ll P^{1+\varepsilon} H Y^{1/2} Q I_2 + P^{3/2+\varepsilon} H Y^{1/2} I_3, \tag{4.14}$$

where

$$I_2 = \int_{\mathfrak{R}} \Delta(\alpha)^{3/4} |f(\alpha)|^6 \, d\alpha \quad \text{and} \quad I_3 = \int_{\mathfrak{R}} \Delta(\alpha)^{1/2} |f(\alpha)|^6 \, d\alpha. \tag{4.15}$$

Plainly, by (3.4) one has

$$|f(\alpha)|^4 = \sum_{l \in \mathbf{Z}} \psi(l) e(l\alpha),$$

where $\psi(l)$ denotes the number of solutions of the equation $z_1^4 + z_2^4 - z_3^4 - z_4^4 = l$ with $z_i \in \mathscr{A}(Q, R)$ $(1 \le i \le 4)$. Observe that by Hua's Lemma (see Lemma 2.5 of Vaughan [16]) and an elementary counting argument, one has

$$\psi(0) \ll Q^{2+\varepsilon} \quad \text{and} \quad \sum_{l \in \mathbf{Z}} \psi(l) = f(0)^4 \ll Q^4.$$

Then by Lemma 2 of Brüdern [2], we have

$$\int_{\mathfrak{R}} \Delta(\alpha) |f(\alpha)|^4 \, d\alpha \ll Q^{\varepsilon - 4}(PQ^{2+\varepsilon} + Q^4) \ll Q^{\varepsilon}. \tag{4.16}$$

Then on applying Hölder's inequality to (4.15) in combination with (4.16), and recalling Lemma 2.3 and Theorem 2, respectively, we deduce that

$$I_2 \le \left(\int_{\mathfrak{R}} \Delta(\alpha) |f(\alpha)|^4 \, d\alpha \right)^{3/4} \left(\int_0^1 |f(\alpha)|^{12} \, d\alpha \right)^{1/4} \ll Q^{2+\varepsilon}$$

and

$$I_3 \le \left(\int_{\mathfrak{R}} \Delta(\alpha) |f(\alpha)|^4 \, d\alpha \right)^{1/2} \left(\int_0^1 |f(\alpha)|^8 \, d\alpha \right)^{1/2} \ll Q^{\mu_8/2+\varepsilon},$$

where $\mu_8 = 4.594193$. Collecting together (3.1), (4.12) and (4.14) we conclude that

$$\int_{\mathfrak{K}\cap\mathfrak{l}} \Upsilon(\alpha;\gamma,\theta)S(\alpha)\,d\alpha \ll P^{1+\varepsilon}HY^{1/2}Q^3 + P^{3/2+\varepsilon}HY^{1/2}Q^{\mu_8/2}$$
$$\ll n^{43/36-\delta}. \tag{4.17}$$

We now come to the third and final phase of our attack. When $\alpha \in \mathfrak{L}(q,a) \subseteq \mathfrak{L}$, one has $q + Q^4|q\alpha - a| \leq Y^3 < P$, and moreover the estimates (4.9) and (4.10) hold. An application of Cauchy's inequality therefore yields

$$\int_{\mathfrak{L}\cap\mathfrak{n}} \Upsilon(\alpha;\gamma,\theta)S(\alpha)\,d\alpha \ll P^{1+\varepsilon}HYI_4, \tag{4.18}$$

where

$$I_4 = \int_{\mathfrak{L}\cap\mathfrak{n}} \Delta(\alpha)^{2/3}|g(\alpha)f(\alpha)^6|\,d\alpha. \tag{4.19}$$

But by Weyl's inequality (see, for example, Lemma 2.4 of Vaughan [16]), one has

$$\sup_{\alpha\in\mathfrak{n}} |g(\alpha)| \ll Q^{1+\varepsilon}M^{-1/8}.$$

Then on applying Hölder's inequality to (4.19), recalling (4.16), and noting the estimate reported in the proof of Lemma 2.3, we obtain

$$I_4 \leq \left(\sup_{\alpha\in\mathfrak{n}} |g(\alpha)|\right)^{1/3} \left(\int_{\mathfrak{K}} \Delta(\alpha)|f(\alpha)|^4\,d\alpha\right)^{2/3} \left(\int_0^1 |g(\alpha)^2 f(\alpha)^{10}|\,d\alpha\right)^{1/3}$$
$$\ll Q^3M^{-1/24}.$$

On recalling (3.1), we therefore find from (4.18) that

$$\int_{\mathfrak{L}\cap\mathfrak{n}} \Upsilon(\alpha;\gamma,\theta)S(\alpha)d\alpha \ll P^{1+\varepsilon}HYQ^3M^{-1/24} \ll n^{43/36-\delta/25}. \tag{4.20}$$

Since $\mathfrak{n}$ is the union of the sets $\mathfrak{k}$, $\mathfrak{K}\cap\mathfrak{l}$ and $\mathfrak{L}\cap\mathfrak{n}$, we conclude from (4.11), (4.17) and (4.20) that

$$\int_{\mathfrak{n}} \Upsilon(\alpha;\gamma,\theta)S(\alpha)\,d\alpha \ll n^{43/36-\delta/25},$$

and hence the conclusion of the lemma is immediate from (4.7).

Finally, we consider the contribution of the third term on the right hand side of (4.8).

Lemma 4.5. *One has*

$$\sum_{\substack{Y<p\leq 2Y\\ p\equiv 2 \pmod 3}} T_2(p) \ll n^{43/36-\delta}.$$

Proof. We apply the argument on pages 155 and 156 of Vaughan [13], but first pause to record some notation. Let $\mathfrak{T}$ denote the union of the intervals

$$\mathfrak{T}(q,a)=\{\alpha\in[0,1):|q\alpha-a|\leq H^2Y^{-2}Q^{-4}\}$$

with $0\leq a\leq q\leq H^2Y^{-2}$. Also, define the function $\Delta^*(\alpha)$ for $\alpha\in[0,1)$ by taking

$$\Delta^*(\alpha)=(q+Q^4|q\alpha-a|)^{-1},$$

when $\alpha\in\mathfrak{T}(q,a)\subseteq\mathfrak{T}$, and by taking $\Delta^*(\alpha)$ to be zero otherwise. Suppose that $\alpha\in\mathfrak{n}$. Then an inspection of the argument on pages 155 and 156 of [13] reveals that in the current setting,

$$\sum_{\substack{Y<p\leq 2Y\\ p\equiv 2 \pmod 3}} |\Xi_p(\alpha)| \ll P^{1+\varepsilon}Y+P^{3/2+\varepsilon}Y^{-7/2}, \tag{4.21}$$

except possibly when there is a natural number q with

$$\|q\alpha\|\leq (H^2Y^{-2})Q^{-4} \quad\text{and}\quad q\leq H^2Y^{-2}. \tag{4.22}$$

Note, in particular, that the parameter Q used in the argument of pages 155 and 156 of [13] is, in our notation, of order PY^{-1}. Since any real number $\alpha\in[0,1)$ satisfying (4.22) must lie in $\mathfrak{T}$, we deduce from the argument on page 156 of [13] that the contribution to

$$\sum_{\substack{Y<p\leq 2Y\\ p\equiv 2 \pmod 3}} T_2(p)$$

arising from these exceptional α is at most

$$P^{1+\varepsilon}H\int_{\mathfrak{T}\cap\mathfrak{n}} \Delta^*(\alpha)^{1/2}S(\alpha)\,d\alpha. \tag{4.23}$$

In view of (3.1), moreover, one has $H^2Y^{-2}<P$, and hence we have $\mathfrak{T}\cap\mathfrak{n}\subseteq\mathfrak{K}$, so that a comparison of (4.13) and (4.23) reveals that the expression (4.23) is $O(I_1)$. Consequently, on recalling (3.1), (4.6) and (4.21), we arrive at the upper bound

$$\sum_{\substack{Y<p\le 2Y\\ p\equiv 2 \pmod 3}} T_2(p) \ll P^{1+\varepsilon}YT_3 + I_1,$$

and thus the conclusion of the lemma follows from Lemma 4.3 and the argument leading from (4.12) to (4.17).

In order to complete the analysis of this section, it remains now only to collect together the conclusions of Lemmata 4.1–4.5 in order to deduce that

$$|r_1(n)| \ll n^{43/36-\delta/26}. \tag{4.24}$$

5. Major Arcs without a Difference, I

We now turn our attention to the estimation of the integral

$$r_2(n) = \int_{\mathfrak{M}} \mathscr{F}(\alpha)e(-n\alpha)\,d\alpha \tag{5.1}$$

that supplies the expected main term in our application of the Hardy-Littlewood method. The object of this section is to prune the fairly wide major arcs $\mathfrak{M}$ down to the thin set $\mathfrak{P}$ on which a conventional analysis of smooth Weyl sums is available. Before launching into the first pruning operation, we pause to discuss the approximants to generating functions on $\mathfrak{M}$.

We define

$$S(q,a) = \sum_{r=1}^{q} e(ar^3/q), \quad S(q,a,p) = S(q,a) - p^{-1}S(q,ap^3) \tag{5.2}$$

and

$$T(q,a) = \sum_{r=1}^{q} e(ar^4/q). \tag{5.3}$$

Notice that for $1 \le q \le Y$ and $p > Y$ one has $p \nmid q$, and hence by a change of variables one obtains

$$S(q,ap^3) = S(q,a) \quad \text{and} \quad T(q,ap^4) = T(q,a). \tag{5.4}$$

We define the multiplicative function $\kappa(q)$ on prime powers π^l $(l \in \mathbf{N})$ by means of the equations

$$\kappa(\pi^{3l}) = \pi^{-l}, \quad \kappa(\pi^{3l+1}) = 2\pi^{-l-1/2}, \quad \kappa(\pi^{3l+2}) = \pi^{-l-1}, \tag{5.5}$$

and similarly define the multiplicative function $\lambda(q)$ on prime powers π^l $(l \in \mathbf{N})$

via the equations

$$\lambda(\pi^{4l}) = \pi^{-l}, \quad \lambda(\pi^{4l+1}) = 3\pi^{-l-1/2}, \quad \lambda(\pi^{4l+2}) = \pi^{-l-1}, \quad \lambda(\pi^{4l+3}) = \pi^{-l-1}. \tag{5.6}$$

Then it follows from Lemmata 4.3–4.5 of Vaughan [16] that whenever $q \in \boldsymbol{N}$ and $a \in \boldsymbol{Z}$ satisfy $(a, q) = 1$, one has

$$q^{-1}|S(q,a)| \ll \kappa(q), \quad q^{-1}|T(q,a)| \ll \lambda(q), \tag{5.7}$$

and moreover that when $p \equiv 2 \pmod 3$ is a prime,

$$q^{-1}|S(q,a,p)| \ll \kappa(q). \tag{5.8}$$

We also define

$$v(\beta) = \int_P^{2P} e(\beta\gamma^3)\,d\gamma \quad \text{and} \quad w(\beta) = \int_0^Q e(\beta\gamma^4)\,d\gamma. \tag{5.9}$$

Next define the functions F_p^* and g_p^* for $\alpha \in [0,1)$ by taking

$$F_p^*(\alpha) = q^{-1}S(q,a,p)v(\alpha - a/q) \tag{5.10}$$

and

$$g_p^*(\alpha) = q^{-1}T(q,ap^4)w(p^4(\alpha - a/q)), \tag{5.11}$$

when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, and by taking each of these functions to be zero otherwise. Then it follows from Theorem 4.1 of Vaughan [16] that for any $\alpha \in \boldsymbol{R}$, $a \in \boldsymbol{Z}$, and $q \in N$, and for any prime $p \le 2Y$, one has

$$F_p(\alpha) - q^{-1}S(q,a,p)v(\alpha - a/q) \ll q^{1/2+\varepsilon}(1 + P^3|\alpha - a/q|)^{1/2}$$

and

$$g(\alpha p^4) - q^{-1}T(q,ap^4)w(p^4(\alpha - a/q)) \ll q^{1/2+\varepsilon}(1 + Q^4Y^4|\alpha - a/q|)^{1/2},$$

whence for $\alpha \in \mathfrak{M}$ one has

$$F_p(\alpha) - F_p^*(\alpha) \ll L^{1/2+\varepsilon} \quad \text{and} \quad g(\alpha p^4) - g_p^*(\alpha) \ll L^{1/2+\varepsilon}. \tag{5.12}$$

Before proceeding further, we define an auxiliary set of major arcs. When X is a real number with $1 \le X \le L$, define the set of major arcs $\mathfrak{W}(X)$ to be the union of the intervals

$$\mathfrak{W}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \le Xn^{-1}\},$$

with $0 \le a \le q \le X$ and $(a,q) = 1$. We now describe the main pruning lemma of this section, the proof of which should be compared with that of Lemma 3.3 of

Brüdern, Kawada and Wooley [5]. Write

$$\mathscr{G}(\alpha) = \sum_{\substack{Y<p\le 2Y \\ p\equiv 2 \pmod 3}} |F_p^*(\alpha)^2 f(\alpha p^4)^4|.$$

LEMMA 5.1. *Suppose that X is a real number with $1 \le X \le L$. Then*

$$\int_{\mathfrak{W}(X)} \mathscr{G}(\alpha)\, d\alpha \ll X^\varepsilon P^{-1} Y Q^4 (\log Y)^{-1}.$$

PROOF. For each prime $p \in (Y, 2Y]$ with $p \equiv 2 \pmod 3$, and for each $\alpha \in \mathfrak{W}(q,a) \subseteq \mathfrak{W}(X)$, it follows from the argument leading to equation (6.5) of Brüdern [4] that

$$F_p^*(\alpha)^2 \ll \kappa(q)^2 P^2 (1 + P^3|\alpha - a/q|)^{-2},$$

and moreover that when $p^2|q$, one has

$$F_p^*(\alpha)^2 = 0.$$

Thus we find that

$$\int_{\mathfrak{W}(X)} \mathscr{G}(\alpha)\, d\alpha \ll P^2 \sum_{1\le q\le X} \kappa(q)^2 \sum_{\substack{Y<p\le 2Y \\ p^2\nmid q}} \int_{-\infty}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{|f(p^4(a/q+\beta))|^4}{(1+P^3|\beta|)^2}\, d\beta. \tag{5.13}$$

Let $c_q(h)$ be Ramanujan's sum, which we define by

$$c_q(h) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e(ah/q).$$

Write also

$$\psi(\mathbf{x}) = x_1^4 + x_2^4 - x_3^4 - x_4^4.$$

Then it follows that

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} |f(p^4(a/q+\beta))|^4 = \sum_{x_1,\dots,x_4 \in \mathscr{A}(Q,R)} c_q(p^4\psi(\mathbf{x})) e(\beta p^4 \psi(\mathbf{x})).$$

Making use of the convention that $(q,0) = q$, the familiar estimate $|c_q(h)| \le (q,h)$ therefore leads from (5.13) to the upper bound

$$\int_{\mathfrak{W}(X)} \mathscr{G}(\alpha)\,d\alpha \ll P^{-1} \sum_{1 \le q \le X} \kappa(q)^2 \sum_{\substack{Y < p \le 2Y \\ p^2 \nmid q}} \sum_{x_1,\dots,x_4 \in \mathscr{A}(Q,R)} (q, p^4\psi(\mathbf{x})). \tag{5.14}$$

When $p^2 \nmid q$, one has

$$(q, p^4\psi(\mathbf{x})) \le (p,q)(q,\psi(\mathbf{x})).$$

Also, as in the argument leading to (3.7) of Brüdern, Kawada and Wooley [5], one readily establishes that

$$\sum_{Y < p \le 2Y} (p,q) \ll X^{\varepsilon} Y (\log Y)^{-1}.$$

Then it follows from (5.14) that

$$\int_{\mathfrak{W}(X)} \mathscr{G}(\alpha)\,d\alpha \ll X^{\varepsilon} P^{-1} Y (\log Y)^{-1} \sum_{1 \le q \le X} \kappa(q)^2 \sum_{1 \le x_1,\dots,x_4 \le Q} (q,\psi(\mathbf{x})). \tag{5.15}$$

Next write $\rho(d)$ for the number of solutions of the congruence $\psi(\mathbf{x}) \equiv 0 \pmod{d}$ with $1 \le x_i \le d$ $(1 \le i \le 4)$. Then by sorting the x_i into residue classes modulo d, it follows that whenever $q \le Q$, one has

$$\sum_{1 \le x_1,\dots,x_4 \le Q} (q,\psi(\mathbf{x})) \le \sum_{d|q} (Qd^{-1}+1)^4 d\rho(d) \ll Q^4 \sum_{d|q} d^{-3}\rho(d). \tag{5.16}$$

On the other hand, by orthogonality one has

$$\rho(d) = d^{-1} \sum_{a=1}^{d} |T(d,a)|^4 \le d^{-1} \sum_{a=1}^{d} (d,a)^4 \left| T\left(\frac{d}{(d,a)}, \frac{a}{(d,a)} \right) \right|^4.$$

In view of (5.7), therefore, one has $\rho(d) \ll \sigma(d)$, where

$$\sigma(d) = d^3 \sum_{a=1}^{d} \lambda(d/(d,a))^4 = d^3 \sum_{f|d} f\lambda(f)^4. \tag{5.17}$$

But $\lambda(f)$ is a multiplicative function of f, and thus $\sigma(d)$ is likewise a multiplicative function of d. Since from (5.17) we find that for each prime π and natural number h one has

$$\sigma(\pi^h) = \pi^{3h} \sum_{l=0}^{h} \pi^l \lambda(\pi^l)^4,$$

we deduce from (5.6) that

$$\sigma(\pi) \leq 81\pi^3(1+\pi^{-1}),$$

and for $h \geq 2$,

$$\sigma(\pi^h) \leq 81\pi^{3h}\left(1+\pi^{-1}+\sum_{l=2}^{4}\pi^{l-4}+\sum_{l=5}^{h}\pi^l(\pi^{-l/4})^4\right) \leq 81h\pi^{3h}.$$

Consequently, when $h = 1$,

$$\sum_{l=0}^{h}\pi^{-3l}\sigma(\pi^l) < 200$$

and when $h \geq 2$,

$$\sum_{l=0}^{h}\pi^{-3l}\sigma(\pi^l) \leq \sum_{l=0}^{h}81l < 81h^2.$$

In view of the multiplicative property of $\sigma(d)$, we therefore conclude from (5.16) that for $q \leq Q$, one has

$$\sum_{1\leq x_1,\ldots,x_4\leq Q}(q,\psi(\mathbf{x})) \ll Q^4\nu(q), \tag{5.18}$$

where $\nu(q)$ denotes the multiplicative function of q defined on prime powers π^l ($l \in \boldsymbol{N}$) by taking $\nu(\pi^h) = 200h^2$ ($h \geq 1$).

We next recall the definition (5.5) of $\kappa(q)$, and note that for each prime π, one has

$$\kappa(\pi)^2\nu(\pi) < 2000\pi^{-1}$$

and

$$\sum_{h=2}^{\infty}\kappa(\pi^h)^2\nu(\pi^h) < 2000\sum_{h=2}^{\infty}h^2\pi^{-2h/3} \ll \pi^{-4/3}.$$

Then since $\kappa(q)^2\nu(q)$ is a multiplicative function of q, we deduce that

$$\sum_{1\leq q\leq X}\kappa(q)^2\nu(q) \ll \prod_{\pi\leq X}(1+2000\pi^{-1}) \ll X^{\varepsilon}.$$

On substituting (5.18) into (5.15), we therefore conclude that whenever $X \leq L$,

$$\begin{aligned}\int_{\mathfrak{W}(X)}\mathscr{G}(\alpha)\,d\alpha &\ll X^{\varepsilon}P^{-1}Y(\log Y)^{-1}Q^4\sum_{1\leq q\leq X}\kappa(q)^2\nu(q)\\ &\ll X^{2\varepsilon}P^{-1}YQ^4(\log Y)^{-1}.\end{aligned}$$

This completes the proof of the lemma.

Our first step in the analysis of the integral in (5.1) is to replace the implicit occurrences of $F_p(\alpha)$ by their approximations $F_p^*(\alpha)$. In this context, define

$$\mathscr{F}_1(\alpha) = \sum_{\substack{Y<p\le 2Y \\ p\equiv 2 \pmod 3}} F_p^*(\alpha)^2 g(\alpha p^4) f(\alpha p^4)^6. \tag{5.19}$$

Lemma 5.2. *One has*

$$r_2(n) = \int_{\mathfrak{M}} \mathscr{F}_1(\alpha) e(-n\alpha)\, d\alpha + O(n^{43/36-\delta}).$$

Proof. Observe first that by (5.12),

$$F_p(\alpha)^2 - F_p^*(\alpha)^2 \ll L^{1/2+\varepsilon}|F_p^*(\alpha)| + L^{1+\varepsilon},$$

and so by (3.6), (5.1) and (5.19), one has

$$r_2(n) - \int_{\mathfrak{M}} \mathscr{F}_1(\alpha) e(-n\alpha)\, d\alpha \ll J_1 + J_2,$$

where

$$J_1 = L^{1+\varepsilon} \sum_{\substack{Y<p\le 2Y \\ p\equiv 2 \pmod 3}} \int_{\mathfrak{M}} |g(\alpha p^4) f(\alpha p^4)^6|\, d\alpha$$

and

$$J_2 = L^{1/2+\varepsilon} \sum_{\substack{Y<p\le 2Y \\ p\equiv 2 \pmod 3}} \int_{\mathfrak{M}} |F_p^*(\alpha) g(\alpha p^4) f(\alpha p^4)^6|\, d\alpha.$$

But by a change of variable, it follows from Lemma 3.1 that

$$\int_{\mathfrak{M}} |g(\alpha p^4) f(\alpha p^4)^6|\, d\alpha \le \int_0^1 |g(\alpha) f(\alpha)^6|\, d\alpha \ll Q^{3.86911},$$

and so from the definition (3.2) of L together with (3.1),

$$J_1 \ll n^{2\delta} Y^5 Q^{3.86911} \ll n^{43/36-3\delta}.$$

On the other hand, by the Cauchy-Schwarz inequalities, one has the upper bound

$$J_2 \ll L^{\varepsilon} J_1^{1/2} J_3^{1/2},$$

where

$$J_3 = \sum_{\substack{Y<p\le 2Y \\ p\equiv 2 \pmod 3}} \int_{\mathfrak{M}} |F_p^*(\alpha)^2 g(\alpha p^4) f(\alpha p^4)^6|\, d\alpha.$$

But by estimating $g(\alpha p^4)$ and $f(\alpha p^4)$ trivially, we obtain

$$J_3 \ll Q^3 \int_{\mathfrak{W}(L)} \mathscr{G}(\alpha)\, d\alpha,$$

whence by Lemma 5.1 we deduce from (3.1) and (3.2) that

$$J_3 \ll P^{\varepsilon-1} Y Q^7 \ll n^{43/36+\varepsilon}.$$

Thus we conclude that

$$r_2(n) - \int_{\mathfrak{M}} \mathscr{F}_1(\alpha) e(-n\alpha)\, d\alpha \ll n^{43/36-3\delta} + n^{\varepsilon}(n^{43/36-3\delta})^{1/2}(n^{43/36})^{1/2},$$

and the conclusion of the lemma is immediate.

As the next step in our analysis of $r_2(n)$, we replace the exponential sum $g(\alpha p^4)$ by its approximant $g_p^*(\alpha)$. In this context, define

$$\mathscr{F}_2(\alpha) = \sum_{\substack{Y<p\leq 2Y \\ p\equiv 2 \pmod 3}} F_p^*(\alpha)^2 g_p^*(\alpha) f(\alpha p^4)^6. \tag{5.20}$$

LEMMA 5.3. *One has*

$$r_2(n) = \int_{\mathfrak{M}} \mathscr{F}_2(\alpha) e(-n\alpha)\, d\alpha + O(n^{43/36-\delta}).$$

PROOF. First observe that by (5.19), (5.20) and Lemma 5.2, we have

$$r_2(n) - \int_{\mathfrak{M}} \mathscr{F}_2(\alpha) e(-n\alpha)\, d\alpha \ll K + n^{43/36-\delta}, \tag{5.21}$$

where

$$K = \sum_{\substack{Y<p\leq 2Y \\ p\equiv 2 \pmod 3}} \int_{\mathfrak{M}} |F_p^*(\alpha)^2 f(\alpha p^4)^6 (g(\alpha p^4) - g_p^*(\alpha))|\, d\alpha.$$

But on recalling (5.12), and applying a trivial estimate for $f(\alpha p^4)$, one obtains

$$K \ll L^{1/2+\varepsilon} Q^2 \sum_{\substack{Y<p\leq 2Y \\ p\equiv 2 \pmod 3}} \int_{\mathfrak{M}} |F_p^*(\alpha)^2 f(\alpha p^4)^4|\, d\alpha.$$

But $\mathfrak{M} = \mathfrak{W}(L)$, so that Lemma 5.1 yields

$$K \ll (L^{1/2+\varepsilon} Q^2)(P^{\varepsilon-1} Y Q^4).$$

The upper bound $K \ll n^{43/36-\delta}$ follows from (3.1) and (3.2) via a smidgeon of computation, and the proof of the lemma is completed on substituting the latter bound into (5.21).

Finally, we prune down from the arcs $\mathfrak{M}$ to the narrow set $\mathfrak{P}$.

LEMMA 5.4. *One has*

$$r_2(n) = \int_{\mathfrak{P}} \mathscr{F}_2(\alpha)e(-n\alpha)\,d\alpha + O(n^{43/36}(\log n)^{-1-\delta/5}).$$

PROOF. On recalling (5.9) and applying partial integration, one obtains the bound

$$w(\beta) \ll Q(1+Q^4|\beta|)^{-1/4}.$$

Thus we deduce from (5.6), (5.7) and (5.11) that when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, one has

$$g_p^*(\alpha) \ll Q(q+Q^4|p^4(q\alpha-a)|)^{-1/4},$$

whence for $\alpha \in \mathfrak{W}(2X)\backslash\mathfrak{W}(X)$ one has

$$g_p^*(\alpha) \ll QX^{-1/4}. \tag{5.22}$$

Next we observe that $\mathfrak{W}(V) \subseteq \mathfrak{P}$, and hence $\mathfrak{M}\backslash\mathfrak{P}$ is contained in the union of the sets

$$\mathfrak{V}(X) = \mathfrak{W}(2X)\backslash\mathfrak{W}(X),$$

where we put $X = 2^l V$, and take the union with $l \geq 0$ satisfying $2^l V \leq L$. On making use of a trivial bound for $f(p^4\alpha)$, we find from (5.20) and (5.22) that

$$\int_{\mathfrak{V}(X)} |\mathscr{F}_2(\alpha)|\,d\alpha \ll Q^3X^{-1/4}\int_{\mathfrak{W}(2X)} \sum_{\substack{Y<p\leq 2Y\\ p\equiv 2\ (\mathrm{mod}\ 3)}} |F_p^*(\alpha)^2 f(\alpha p^4)^4|\,d\alpha,$$

whence by Lemma 5.1,

$$\begin{aligned}\int_{\mathfrak{V}(X)} |\mathscr{F}_2(\alpha)|\,d\alpha &\ll (Q^3X^{-1/4})(X^{\varepsilon}P^{-1}YQ^4(\log Y)^{-1})\\ &\ll X^{-1/5}P^{-1}YQ^7(\log Y)^{-1}.\end{aligned}$$

On summing over the aforementioned values of l, we find that the total contribution arising from the union of the arcs $\mathfrak{V}(2^l V)$ yields the upper bound

$$\int_{\mathfrak{M}\setminus\mathfrak{P}} |\mathscr{F}_2(\alpha)|\,d\alpha \ll P^{-1}YQ^7(\log n)^{-1-\delta/5}.$$

The conclusion of the lemma therefore follows from (3.1) and (3.2) with a modest computation.

6. Major Arcs without a Difference, II

The remaining obstructions to the proof of Theorem 1 offer only token resistance, and we pause merely to introduce some additional notation. Write c_η for $\rho(\eta^{-1})$, where $\rho(t)$ is the Dickman function (see, for example, § 12.1 of Vaughan [16]). For our purposes here it suffices to note only that when $\eta > 0$ one has $c_\eta > 0$. Next define the function $f_p^*(\alpha)$ for $\alpha \in [0,1)$ by taking

$$f_p^*(\alpha) = c_\eta q^{-1} T(q, ap^4) w(p^4(\alpha - a/q))$$

when $\alpha \in \mathfrak{P}(q,a) \subseteq \mathfrak{P}$, and by taking $f_p^*(\alpha)$ to be zero otherwise. As a consequence of Lemma 8.5 of Wooley [17] (see also Lemma 5.4 of Vaughan [14] for a related conclusion), one has for $\alpha \in \mathfrak{P}$ that

$$f(\alpha p^4) - f_p^*(\alpha) \ll Q(\log n)^{-1/4}. \tag{6.1}$$

Suppose that $\alpha \in \mathfrak{P}(q,a) \subseteq \mathfrak{P}$. Then $q \le V < Y$, and hence for $p > Y$ one has $(p,q) = 1$. A change of variables therefore reveals that $T(q, ap^4) = T(q,a)$, and a further change of variables shows that

$$w(p^4(\alpha - a/q)) = p^{-1} w_p(\alpha - a/q),$$

where we write

$$w_p(\beta) = \int_0^{pQ} e(\beta\gamma^4)\,d\gamma. \tag{6.2}$$

Thus we obtain that for $\alpha \in \mathfrak{P}(q,a) \subseteq \mathfrak{P}$, one has

$$f_p^*(\alpha) = c_\eta (pq)^{-1} T(q,a) w_p(\alpha - a/q), \tag{6.3}$$

and in the same circumstances a similar argument leads from (5.11) to the relation

$$g_p^*(\alpha) = (pq)^{-1} T(q,a) w_p(\alpha - a/q). \tag{6.4}$$

LEMMA 6.1. *One has*

$$\int_{\mathfrak{P}} \mathscr{F}_2(\alpha) e(-n\alpha)\,d\alpha \gg n^{43/36}(\log n)^{-1}.$$

PROOF. We begin by replacing the exponential sum $f(\alpha p^4)$ in (5.20) by its approximant $f_p^*(\alpha)$. Since the measure of $\mathfrak{P}$ is $O(V^3n^{-1})$, on making trivial estimates for F_p^*, g_p^* and $f(\alpha p^4)$, it follows from (5.20) and (6.1) that

$$\int_{\mathfrak{P}} \mathscr{F}_2(\alpha)e(-n\alpha)\,d\alpha - \int_{\mathfrak{P}} \mathscr{F}_3(\alpha)e(-n\alpha)\,d\alpha \ll (V^3n^{-1})(Q(\log n)^{-1/4})P^2Q^6Y(\log Y)^{-1},$$

where

$$\mathscr{F}_3(\alpha) = \sum_{\substack{Y<p\leq 2Y \\ p\equiv 2 \pmod 3}} F_p^*(\alpha)^2 g_p^*(\alpha) f_p^*(\alpha)^6. \tag{6.5}$$

Hence, with a little computation one arrives at the conclusion

$$\int_{\mathfrak{P}} \mathscr{F}_2(\alpha)e(-n\alpha)\,d\alpha - \int_{\mathfrak{P}} \mathscr{F}_3(\alpha)e(-n\alpha)\,d\alpha \ll n^{43/36}(\log n)^{-1-\delta}. \tag{6.6}$$

Observing next that when $q < Y$ and $p > Y$ one has $S(q, ap^3) = S(q, a)$, it follows from (5.2) that $S(q, a, p) = (1 - p^{-1})S(q, a)$, and hence on substituting from (5.10), (6.3) and (6.4) into (6.5), we obtain

$$\int_{\mathfrak{P}} \mathscr{F}_3(\alpha)e(-n\alpha)\,d\alpha = c_\eta^6 \sum_{\substack{Y<p\leq 2Y \\ p\equiv 2 \pmod 3}} (1-p^{-1})^2 p^{-7} J_0(n,p) \sum_{1\leq q\leq V} A(q,n), \tag{6.7}$$

where

$$J_0(n,p) = \int_{-Vn^{-1}}^{Vn^{-1}} v(\beta)^2 w_p(\beta)^7 e(-n\beta)\,d\beta \tag{6.8}$$

and

$$A(q,n) = q^{-9} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(q,a)^2 T(q,a)^7 e(-na/q). \tag{6.9}$$

In order to make further progress we complete the singular integral $J_0(n,p)$ to obtain a new one

$$J(n,p) = \int_{-\infty}^{\infty} v(\beta)^2 w_p(\beta)^7 e(-n\beta)\,d\beta. \tag{6.10}$$

On recalling (5.9) and (6.2), a partial integration yields the bounds

$$v(\beta) \ll P(1+P^3|\beta|)^{-1} \quad \text{and} \quad w_p(\beta) \ll pQ(1+(pQ)^4|\beta|)^{-1/4}.$$

On substituting these bounds into (6.8) and (6.10), we deduce that for $Y < p \leq 2Y$, one has

$$J(n,p) - J_0(n,p) \ll \int_{Vn^{-1}}^{\infty} P^2(pQ)^7(1+n\beta)^{-15/4}\,d\beta \ll P^2(pQ)^7 V^{-2}n^{-1}. \tag{6.11}$$

We may rewrite (6.10) in the form

$$J(n,p) = \int_{-\infty}^{\infty}\int_{\mathfrak{B}(p)} e(\beta(\gamma_1^3+\gamma_2^3+\gamma_3^4+\cdots+\gamma_9^4-n))\,d\gamma d\beta,$$

where

$$\mathfrak{B}(p) = [P,2P]^2 \times [0,pQ]^7.$$

Since for $Y < p \leq 2Y$ it is immediate from (3.1) that

$$[(\tfrac{1}{4}n)^{1/3}, n^{1/3}]^2 \times [0, n^{1/4}]^7 \subseteq \mathfrak{B}(p),$$

an application of Fourier's integral formula rapidly establishes that

$$J(n,p) \gg n^{2/3+7/4-1}. \tag{6.12}$$

Next we consider the singular series, which we complete to obtain

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} A(q,n). \tag{6.13}$$

Recalling first (5.5)–(5.7), it is immediate that

$$S(q,a) \ll q^{2/3} \quad \text{and} \quad T(q,a) \ll q^{3/4},$$

whence we deduce from (6.9) that

$$A(q,n) \ll q^{-8}(q^{2/3})^2(q^{3/4})^7 \ll q^{-17/12}. \tag{6.14}$$

Thus it follows from (6.13) that the series $\mathfrak{S}(n)$ is absolutely convergent, and moreover that

$$\mathfrak{S}(n) - \sum_{1\leq q\leq V} A(q,n) \ll \sum_{q>V} q^{-17/12} \ll V^{-1/3}. \tag{6.15}$$

Next write

$$\omega_\pi(n) = \sum_{h=0}^{\infty} A(\pi^h, n). \tag{6.16}$$

Then by (6.14) it follows that for each prime π one has

$$\omega_\pi(n) - 1 \ll \pi^{-17/12}. \tag{6.17}$$

But the standard theory of exponential sums shows that $A(q,n)$ is a multiplicative function of q (see, for example, §2.6 of Vaughan [16]), and hence we may rewrite $\mathfrak{S}(n)$ as an absolutely convergent product

$$\mathfrak{S}(n) = \prod_\pi \omega_\pi(n).$$

Next we observe that the argument underlying the proof of Lemma 2.12 of Vaughan [16] shows that when $h \geq 1$, one has

$$\sum_{l=0}^{h} A(\pi^l, n) = \pi^{-8h}\Omega(\pi^h, n),$$

where $\Omega(\pi^h, n)$ denotes the number of incongruent solutions of the congruence

$$x_1^3 + x_2^3 + x_3^4 + \cdots + x_9^4 \equiv n \pmod{\pi^h}. \tag{6.18}$$

In particular, it follows from (6.16) that $\omega_\pi(n)$ is real and non-negative. When π is not equal to 3 and $h = 1$, it follows from the Cauchy-Davenport Theorem (see Lemma 2.14 of [16]) that the congruence (6.18) is soluble with $\pi \nmid x_1$. When $\pi^h = 9$, on the other hand, an instant computation shows that the congruence (6.18) is again soluble with $\pi \nmid x_1$. Then the methods of §2.6 of Vaughan [16], in combination with (6.17), therefore show that for a sufficiently large but fixed positive number C, one has

$$\mathfrak{S}(n) \gg \prod_{\pi > C}(1 - \pi^{-4/3}) \gg 1, \tag{6.19}$$

uniformly in n.

We may now swiftly overwhelm the final defenses of Lemma 6.1. First we substitute (6.19) into (6.15), and also substitute (6.12) into (6.11), to deduce that

$$\sum_{1 \leq q \leq V} A(q,n) \gg 1 \quad \text{and} \quad J_0(n,p) \gg n^{2/3+7/4-1}.$$

Then on recalling (6.7), we deduce that

$$\int_{\mathfrak{P}} \mathscr{F}_3(\alpha)e(-n\alpha)\,d\alpha \gg n^{17/12} \sum_{\substack{Y < p \leq 2Y \\ p \equiv 2 \pmod 3}} (1 - p^{-1})^2 p^{-7},$$

whence an elementary estimate for the number of primes in arithmetic progressions leads to the lower bound

$$\int_{\mathfrak{P}} \mathscr{F}_3(\alpha)e(-n\alpha)\,d\alpha \gg n^{17/12}Y^{-6}(\log Y)^{-1}.$$

Then by (3.1) and (6.6), we finally obtain

$$\int_{\mathfrak{P}} \mathscr{F}_2(\alpha)e(-n\alpha)\,d\alpha \gg n^{43/36}(\log n)^{-1}(1 + O((\log n)^{-\delta})),$$

and the conclusion of the lemma follows immediately.

Theorem 1 now follows almost instantaneously. Collecting together the conclusions of Lemmata 5.4 and 6.1, one obtains

$$r_2(n) \gg n^{43/36}(\log n)^{-1},$$

and yet (4.24) provides the estimate

$$r_1(n) \ll n^{43/36-\delta/26}.$$

Then in view of (3.7), (4.1) and (5.1), it follows that

$$r(n) = r_1(n) + r_2(n) \gg n^{43/36}(\log n)^{-1},$$

and thus the lower bound $v(n) \gg r(n)$ discussed in § 3 confirms the conclusion of Theorem 1.

References

[1] R. C. Baker, J. Brüdern and T. D. Wooley, Cubic diophantine inequalities, Mathematika **42** (1995), 264–277.

[2] J. Brüdern, A problem in additive number theory, Math. Proc. Cambridge Philos. Soc. **103** (1988), 27–33.

[3] J. Brüdern, On Waring's problem for cubes and biquadrates, J. London Math. Soc. (2) **37** (1988), 25–42.

[4] J. Brüdern, On Waring's problem for cubes, Math. Proc. Cambridge Philos. Soc. **109** (1991), 229–256.

[5] J. Brüdern, K. Kawada and T. D. Wooley, Additive representation in thin sequences, I: Waring's problem for cubes, Ann. Sci. École Norm. Sup. (4) (to appear).

[6] J. Brüdern and T. D. Wooley, On Waring's problem for cubes and smooth Weyl sums, Proc. London Math. Soc. (3) 82 (2001), 89–109.

[7] J. Brüdern and T. D. Wooley, On Waring's problem: three cubes and a sixth power. Nagoya Math. J. (to appear).

[8] K. B. Ford, The representation of numbers as sums of unlike powers. II, J. Amer. Math. Soc. **9** (1996), 919–940.

[9] K. B. Ford, Addendum and Corrigendum to "The representation of numbers as sums of unlike powers, II", J. Amer. Math. Soc. 12 (1999), 1213.

[10] M. I. Israilov and I. A. Allakov, On the sum of kth powers of natural numbers, Number theory and analysis, Trudy Mat. Inst. Steklov **207** (1994), 172–179. (Russian).

[11] K. Kawada and T. D. Wooley, Sums of fourth powers and related topics, J. reine angew. Math. **512** (1999), 173–223.
[12] J. V. Linnik, On the representation of large numbers as sums of seven cubes, Mat. Sbornik **12** (1943), 218–224.
[13] R. C. Vaughan, On Waring's problem for cubes, J. reine angew. Math. **365** (1986), 122–170.
[14] R. C. Vaughan, A new iterative method in Waring's problem, Acta Math. **162** (1989), 1–71.
[15] R. C. Vaughan, A new iterative method in Waring's problem II, J. London Math. Soc. (2) **39** (1989), 219–230.
[16] R. C. Vaughan, The Hardy-Littlewood method, 2nd edition, Cambridge University Press, Cambridge, 1997.
[17] T. D. Wooley, On simultaneous additive equations, II, J. reine angew. Math. **418** (1991), 141–198.
[18] T. D. Wooley, Large improvements in Waring's problem, Ann. of Math. (2) **135** (1992), 131–164.
[19] T. D. Wooley, Breaking classical convexity in Waring's problem: sums of cubes and quasi-diagonal behaviour, Invent. Math. **122** (1995), 421–451.

Jörg Brüdern
Mathematisches Institut A
Universität Stuttgart, Postfach 80 11 40
D-70511 Stuttgart, Germany
E-mail address: bruedern@mathematik.uni-stuttgart.de

Trevor D. Wooley
Department of Mathematics
University of Michigan, East Hall,
525, East University Avenue, Ann Arbor,
Michigan 48109-1109, U.S.A.
E-mail address: wooley@math.lsa.umich.edu