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REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE SATISFYING A POINTWISE NULLITY CONDITION

By

Jong Taek CHO* and U-Hang KI**

Abstract. In this paper, we give a classification of real hypersurfaces of a complex projective space CP^n satisfying a pointwise nullity condition for the structure vector field ξ i.e., $R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\}$, k is a function, and further we prove a local structure theorem of real hypersurfaces of CP^n which satisfies $R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}$. The motivation of the present paper is a well-known fact that CP^n does not admit a real hypersurface of constant curvature.

0. Introduction

Let $CP^n = (CP^n, J, \tilde{g})$ be an n-dimensional complex projective space with Fubinistudy metric $\tilde{g}$ of constant holomorphic sectional curvature 4, and let M be an orientable real hypersurface of CP^n and N be a unit normal vector field on M. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kählerian structure $(J, \tilde{g})$ of CP^n (see Section 1). One of the typical examples of M is a geodesic hypersphere. R. Takagi ([8]) classified homogeneous hypersurfaces of CP^n into six types. T. E. Cecil and P. J. Ryan ([1]) extensively investigated hypersurfaces which are realized as tubes of constant radius r over a complex submanifold of CP^n on which the structure vector field ξ is a principal curvature vector field with principal curvature $\alpha_1 = 2\cot 2r$ and corresponding focal map $\varphi_r : M \to CP^n$ (defined by $\varphi_r(p) = \exp_p(rN)$) has constant rank. We denote by ∇ the Levi-Civita connection with respect to g. The curvature tensor

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field R on M is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ where X and Y are vector fields on M. It is well-known that CP^n does not admit a real hypersurface with constant sectional curvature (cf. [2]).

On the other hand, S. Tanno ([10]) defined for $k \in \boldsymbol{R}$ the k-nullity distribution $N(k)$ of a Riemannian manifold by $N(k) : p \to N_p(k) = \{z \in T_pM : R(x, y)z = k(g(y,z)x - g(x,z)y)$ for any $x, y \in T_pM\}$. If $T_pM = N_p(k)$ for any point $p \in M$, then we see that M is of constant curvature k. In the present paper, we consider a real hypersurface of CP^n whose structure vector field ξ satisfies a pointwise nullity condition, namely, in Section 2, we give a classification of a real hypersurface M of CP^n which satisfies $R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\}$, where k is a function on M. Moreover in Section 3, we investigate a real hypersurface of CP^n which satisfies $R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}$, where k is a function on M. In Section 4, we determine real hypersurfaces of CP^n which satisfies $A^2\xi = \lambda A\xi$ and $(\phi \cdot R)(X, A\xi)\xi = 0$, where $\phi \cdot R$ means ϕ operates on R as a derivation. In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

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1. Preliminaries

At first, we review the fundamental facts on a real hypersurface of CP^n. Let M be a real hypersurface of CP^n and N be a unit normal vector field on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric of CP^n. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$. An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi. \tag{1.1}$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \end{aligned} \tag{1.2}$$

From (1.2), we get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{1.3}$$

From the fact $\tilde{\nabla} J = 0$ and (1.1), making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y)\xi, \tag{1.4}$$

$$\nabla_X \xi = \phi AX. \tag{1.5}$$

Since the ambient space is of constant holomorphic sectional curvature 4, we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{1.7}$$

From (1.6), using (1.2), (1.3), then the Ricci tensor S is given by

$$SX = (2n+1)X - 3\eta(X)\xi + hAX - A^2 X, \tag{1.8}$$

where $h =$ the trace of A. We recall the following

Proposition 1 ([6]). *If ξ is a principal curvature vector field, then the corresponding principal curvature α_1 is constant.*

Proposition 2 ([1]). *Let M be a real hypersurface of CP^n on which ξ is principal with principal curvature $\alpha_1 = 2\cot 2r$ and the focal map φ_r has constant rank on M. Then the following hold:*

(i) *M lies on a tube (in the direction of $C = \gamma'(r)$ where $\gamma(r) = \exp_p(rN)$ and p is a base point of the normal vector N) of radius r over a certain Kählerian submanifold in CP^n.*

(ii) *Let $\cot\theta$ be a principal curvature of the shape operator A_C at $q = \gamma(r)$ of the Kählerian submanifold. Then the real hypersurface M has a principal curvature $\cot(r - \theta)$ at $p = \gamma(0)$.*

Theorem 1 ([7]). *Let M be a real hypersurface of CP^n. Then the followings are equivalent:*

(i) *M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius r over totally geodesic* CP^k $(0 \leq k \leq n-1)$, *where* $0 < r < \pi/2$

(ii) $\phi A = A\phi$.

A ruled real hypersurface of CP^n is defined by a foliated one by complex hyperplanes CP^{n-1} and its shape operator is written down in [3]. Namely,

$$
\begin{aligned}
A\xi &= \alpha_1\xi + \mu W \quad (\mu \neq 0), \\
AW &= \mu\xi, \\
AZ &= 0
\end{aligned}
\tag{1.9}
$$

for any $Z \perp \xi$, W, where W is unit vector orthogonal to ξ, α_1 and μ are functions on M. For more details about a ruled real hypersurface of CP^n, we refer to [4]. The ϕ-holomorphic sectional curvature is defined by a sectional curvature of $span\{X, \phi X\}$. In [3] it was proved that

THEOREM 2. *Let M be a real hypersurface of* $CP^n (n \geq 3)$ *with constant* ϕ-*holomorphic sectional curvature. Then M is locally congruent to the following spaces*:

(1) *a geodesic hypersphere* (*that is, a homogeneous real hypersurface which lies on a tube of radius r over a hyperplane* CP^{n-1}, *where* $0 < r < \pi/2$);

(2) *a ruled real hypersurface*;

(3) *a real hypersurface on which there is a foliation of codimension two such that each leaf of the foliation is contained in some complex hyperplane* CP^{n-1} *as a ruled hypersurface.*

We define a vector field U on M by $U = \nabla_\xi \xi$ and denote $\alpha_m = \eta(A^m\xi)$. Then from (1.2) and (1.5) we easily observe that

$$
\begin{aligned}
g(U, \xi) = 0, \quad g(U, A\xi) = 0, \\
\|U\|^2 = g(U, U) = \alpha_2 - \alpha_1^2.
\end{aligned}
\tag{1.10}
$$

From (1.2), (1.5) and (1.10) we have at once

LEMMA 1. *Let M be a real hypersurface of* CP^n. *Then* ξ *is a principal curvature vector field if and only if M satisfies* $\alpha_2 - \alpha_1^2 = 0$.

Now we recall that ([10]) the k-nullity distribution of a Riemannian manifold, for a real number k, is a distribution

$$N(k) : p \to N_p(k) = \{z \in T_pM : R(x, y)z = k\{g(y, z)x - g(x, z)y\} \text{ for any } x, y \in T_pM\}.$$

If $T_pM = N_p(k)$ for any point $p \in M$, then we see that M is of constant curvature k. In Section 2, we consider a pointwise nullity condition for the structure vector field ξ.

2. Real Hypersurfaces Satisfying a Pointwise Nullity Condition

In this section, we give a classification of a real hypersurface whose structure vector field ξ satisfying

(2.1) $$R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\}$$

for a function k, where X, Y are any vector fields tangent to M. First we prove

LEMMA 2. *Let M be a real hypersurface of CP^n. If M satisfies* (2.1), *then ξ is principal.*

PROOF. From (1.6) and (2.1) we have

(2.2) $$(k-1)\{\eta(Y)X - \eta(X)Y\} = \eta(AY)AX - \eta(AX)AY$$

for any vector field X and Y. We may put

$$A\xi = \alpha_1\xi + Z$$

where Z is orthogonal to ξ. For any vector field X orthogonal to ξ, let X_1 be the component of AX orthogonal to ξ, that is, $X_1 = AX - g(AX, \xi)\xi$. Putting $Y = \xi$ in (2.2), then for X orthogonal to ξ we have

(2.3) $$(k-1)X = \alpha_1X_1 - g(X, Z)Z.$$

First we consider where $\alpha_1 = 0$. Then by taking $X(\neq 0)$ orthogonal to Z in (2.3), we have $k = 1$, and hence, we have $Z = 0$ (by putting $X = Z$), that is, ξ is principal. If there exist a point p such that $\alpha_1(p) \neq 0$, by the continuity of α_1 we see that $\alpha_1 \neq 0$ sufficiently small neighborhood of p. Next we discuss on the neighborhood. If we put $X = Z$, then we see that $span\{\xi, Z\}$ is A-invariant. Here, if we put $Y = Z$ in (2.2) and we take $X(\neq 0)$ orthogonal to ξ and Z, then we have

(2.4) $$\eta(AZ)AX = \eta(AX)AZ.$$

Since $span\{\xi, Z\}$ is A-invariant, from (2.4) we have

$$g(Z, Z)X = 0,$$

and hence $Z = 0$, that is, ξ is principal. At last, we conclude that ξ is principal on M. (Q.E.D.)

Since $A\xi = \alpha_1 \xi$, taking account of Proposition 1 we may set $\alpha_1 = 2\cot 2r$ for some constant $0 < r < \pi/2$. Thus we have

Theorem 3. *Under the same assumption as that of Lemma 2 and in addition that $n \geq 3$ and the rank of the focal map φ_r is constant, then M is locally congruent to one of the following spaces*:

(1) *a geodesic hypersphere (that is, a homogeneous real hypersurface which lies on a tube of radius r over a hyperplane $\boldsymbol{C}P^{n-1}$ $(0 < r < \pi/2)$*;

(2) *a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic $\boldsymbol{C}P^l$ $(1 \leq l \leq n-2)$*;

(3) *a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kählerian submanifold with non-zero principal curvatures $\neq \pm 1$.*

Proof. It follows from $A\xi = \alpha_1 \xi$ and (2.2) that

$$(k-1)\{\eta(Y)X - \eta(X)Y\} = \alpha_1\{\eta(Y)AX - \eta(X)AY\}. \tag{2.8}$$

Since α_1 is constant (by Proposition 1) we divide our arguments into two cases, (i) $\alpha_1 = 0$, (ii) $\alpha_1 \neq 0$:

(i) $\alpha_1 = 0$. From (2.8) we see that $k = 1$, and from (1.6) we see that M satisfies $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$. Since the rank of corresponding focal map $\varphi_{\pi/4}$ is constant, by virtue of Proposition 2 we see that M is locally congruent to (2) or (3).

(ii) $\alpha_1 \neq 0$. Assume $Y = \xi$, $X \perp \xi$ in (2.8). Then we get

$$AX = (k-1)/\alpha_1 \cdot X \tag{2.9}$$

for any vector field X orthogonal to ξ, hence from (2.9) we see that M has at most two distinct principal curvatures. So, Theorem 3 in [1] implies that M is locally congruent to a geodesic hypersphere. (Q.E.D.)

Remark 1. In the case (3) in Theorem 3, the condition "Kählerian submanifold with principal curvatures $\neq \pm 1$" is necessary. In general, Proposition 2 (ii) shows that the point $p(=\gamma(0))$ is a singular point of M when $r = \theta$.

REMARK 2. In particular, for $k \in \boldsymbol{R}$, if ξ belongs to the k-nullity distribution, then in the case (ii) in the proof of Theorem 3, from (2.9) by using the result in [8], we conclude that M is locally congruent to geodesic hypersphere when the dimension $n = 2$, and thus we have same result as Theorem 3 when $n \geq 2$.

We denote $h^{(m)} = \operatorname{trace} A^m$, then in particular $h^{(1)} = h$ in (1.8). We also prove

PROPOSITION 3. *Let M be a real hypersurface of $\boldsymbol{C}P^n$. Then M always satisfies*

$$H_1^2 \leq 2(n-1)H_2,$$

where we put $H_m = h^{(m)}\alpha_m - \alpha_{2m}$. If the equality holds, then ξ is principal ($\alpha_1 = 2\cot 2r$). *Moreover, if we suppose that $n \geq 3$ and the rank of the focal map φ_r is constant, then M is locally congruent to one of* (1), (2), (3) *in Theorem* 3.

PROOF. We put

$$T(X, Y) = R(X, Y)\xi - k\{\eta(Y)X - \eta(X)Y\}$$

for any vector fields X and Y on M, where k is a function. Then T is a $(1,2)$-tensor field on M. We calculate $\|T\|^2$, then we have

$$\begin{aligned} (2.10) \qquad \|T\|^2 &= \sum_{i,j} g(R(e_i, e_j)\xi - k\{\eta(e_j)e_i - \eta(e_i)e_j\}, R(e_i, e_j)\xi - k\{\eta(e_j)e_i - \eta(e_i)e_j\}) \\ &= \|R(\cdot,\cdot)\xi\|^2 - 4k\eta(S\xi) + 4(n-1)k^2, \end{aligned}$$

where $\{e_i\}$ $(i = 1, 2, \ldots, 2n-1)$ is an orthonormal basis of the tangent space. From (1.6) and (1.8) a direct calculation yields

$$(2.11) \qquad \|R(\cdot,\cdot)\xi\|^2 = 4(n-1) + 4H_1 + 2H_2,$$

$$(2.12) \qquad \eta(S\xi) = (2n-2) + H_1.$$

From (2.10), (2.11) and (2.12) we have

$$(2.13) \qquad \|T\|^2 = 4(n-1)(1-k)^2 + 4H_1(1-k) + 2H_2 \geq 0.$$

Since (2.13) holds for any k at any point on M, we see that

$$(2.14) \qquad H_1^2 \leq 2(n-1)H_2.$$

Further we see that the equality holds in (2.14) if and only if $\|T\|^2 = 0$. Thus by using Theorem 3, we have our conclusion. (Q.E.D.)

3. Real Hypersurfaces of CP^n Satisfying $R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}$

In [2] we investigate a real hypersurface of CP^n which satisfies $R(X,\xi)\xi = k\{X - \eta(X)\xi\}$, where k is a function on M. In this section, we prove

THEOREM 4. *Let M be a real hypersurface of CP^n $(n \geq 3)$. Suppose that M satisfies*

$$R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}, \tag{3.1}$$

where k is a function on M. If ξ is principal with the associated principal curvature $\alpha_1 = 2\cot 2r$ and the rank of corresponding focal map φ_r is constant, then M is locally congruent to one of the following spaces:

(1) a geodesic hypersphere;

(2) a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic CP^l $(1 \leq l \leq n-2)$;

(3) a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kählerian submanifold with non-zero principal curvatures $\neq \pm 1$.

PROOF. From (1.6) and (3.1) we have

$$(k-1)\{\eta(X)A\xi - \alpha_1 X\} = \eta(AX)A^2\xi - \alpha_2 AX. \tag{3.2}$$

Taking the transpose of A, then we have

$$(k-1)\{\eta(AX)\xi - \alpha_1 X\} = \eta(A^2X)A\xi - \alpha_2 AX, \tag{3.3}$$

for any vector field X on M. Since ξ is principal, that is, $A\xi = \alpha_1\xi$, for any vector field Y orthogonal to ξ (3.3) yields

$$\alpha_2 AY = (k-1)\alpha_1 Y. \tag{3.4}$$

Since $\alpha_2 = \alpha_1^2$ is constant (cf. Proposition 1), we divide our arguments into two cases, (i) $\alpha_2 = 0$, (ii) $\alpha_2 \neq 0$:

(i) $\alpha_2 = 0$. We see that $A\xi = 0$ and M satisfies $R(X,\xi)A\xi = k\{\eta(A\xi)X - \eta(AX)\xi\} = 0$. Since the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant, by the same arguments in the proof of the Theorem 3 in Section 2, we see that M is locally congruent to (2) or (3).

(ii) $\alpha_2 \neq 0$. From (3.4) we see that M has at most two distinct principal curvatures. So, Theorem 3 in [1] implies that M is locally congruent to a geodesic hypersphere of CP^n. (Q.E.D.)

Here, we consider the case that ξ is not principal and M satisfies (3.1). Then we may assume that

$$A\xi = \alpha_1\xi + \mu W, \quad \mu \neq 0 \tag{3.5}$$

and

$$AW = \mu\xi + \nu W + \delta Z_1, \tag{3.6}$$

where $Z_1 \perp \xi, W$, W is a unit vector orthogonal to ξ, and μ, ν, δ are functions on M. Then from (3.3) we have

$$\alpha_2 AW = \{\alpha_1\mu(\alpha_1 + \nu) - \mu(k-1)\}\xi + \{\mu^2(\alpha_1 + \nu) + \alpha_1(k-1)\}W. \tag{3.7}$$

So from (3.6) and (3.7) we get

$$\alpha_2\mu = \alpha_1\mu(\alpha_1 + \nu) - \mu(k-1), \quad \alpha_2\nu = \mu^2(\alpha_1 + \nu) + \alpha_1(k-1) \text{ and } \alpha_2\delta = 0. \tag{3.8}$$

Further from (3.2) we have

$$\alpha_2 AZ = \alpha_1(k-1)Z. \tag{3.9}$$

for any vector field Z orthogonal to ξ and W. Therefore from (3.5), (3.7), (3.8) and (3.9) we have

$$A\xi = \alpha_1\xi + \mu W$$

$$AW = \mu\xi + \nu W$$

$$AZ = \alpha_1/\alpha_2 \cdot (k-1)Z,$$

$\alpha_2 = \alpha_1(\alpha_1 + \nu) - (k-1)$ and $\alpha_2\nu = \mu^2(\alpha_1 + \nu) + \alpha_1(k-1)$ for any $Z \perp \xi, W$, where W is a unit vector orthogonal to $\xi, \mu(\neq 0), \alpha_2$ and ν are functions on M.

Let M be a real hypersurface of CP^n which satisfies $R(X, A\xi)\xi = \eta(A\xi)X - \eta(X)A\xi$, i.e., $k = 1$. Then from (3.3) it follows that

$$\alpha_2 AX = \eta(A^2X)A\xi \tag{3.10}$$

for any vector field X on M. If there exist a point p in M such that $\alpha_2(p) \neq 0$, then (3.10) implies that the rank of A at p is at most 1. However it is seen (cf. [11]) that the point p is geodesic. So it is contradictory to the assumption that $\alpha_2(p) \neq 0$. Thus $\alpha_2 = 0$ on M. Therefore by Lemma 1, we see that $A\xi = 0$ on M.

REMARK 3. The above arguments together with (1.9) and (20) in [3] imply that neither ruled real hypersurface nor the case (3) in Theorem 3 satisfy the condition (3.1).

It is well-known that a geodesic hypersphere in CP^n is η-umbilical, that is, $A = aI + b\eta \otimes \xi$, where a, b are constants (cf. [1], [9], etc.). Thus, due to Theorems 2, 4 and Remark 3, we characterize a geodesic hypersphere of CP^n by following

THEOREM 5. *Let M be a real hypersurface of CP^n $(n \geq 3)$. Then M is of constant ϕ-holomorphic sectional curvature and M satisfies $R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}$, where k is a constant along M if and only if M is locally congruent to a geodesic hypersphere.*

REMARK 4. The above Theorem 5 is a slight improvement of Theorem 4 in [2].

4. Real Hypersurfaces of CP^n Satisfying $\phi \cdot R = 0$

In [6], Y. Maeda investigated a real hypersurface M of CP^n which satisfies

$$(C_1) \qquad A\xi = \alpha_1 \xi,$$

$$(C_2) \qquad \phi \cdot R = 0,$$

where $\cdot$ means that a (1,1)-tensor field ϕ operates on R as a derivation, i.e., for any vector fields X, Y and Z on M

$$(\phi \cdot R)(X, Y)Z = \phi R(X, Y)Z - R(\phi X, Y)Z - R(X, \phi Y)Z - R(X, Y)\phi Z.$$

Under the conditions $(C_1), (C_2)$ and $n \geq 3$, he proved that M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius r over totally geodesic CP^k $(0 \leq k \leq n-1)$, where $0 < r < \pi/2$ (Theorem 5.4 in [6]).

In this section, we consider the following two conditions (4.1) and (4.2) weaker than (C_1) and (C_2), respectively:

$$A^2\xi = \lambda A\xi, \tag{4.1}$$

$$(\phi \cdot R)(X, A\xi)\xi = 0 \tag{4.2}$$

for a function λ and for any vector field X on M. We prove

THEOREM 6. *Let M be a real hypersurface of CP^n, and suppose that M satisfies* (4.1) *and* (4.2). *Then ξ is a principal curvature vector field on M. Further assume that $\alpha_1 = 2\cot 2r$ and the rank of the focal map φ_r is constant, then M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius r over totally geodesic CP^k $(0 \leq k \leq n-1)$, where $0 < r < \pi/2$, or a non-homogeneous tube of radius $\pi/4$ of the case* (3) *in Theorem* 4.

PROOF. From the assumption (4.2), we get

$$\phi R(X, A\xi)\xi - R(\phi X, A\xi)\xi - R(X, U)\xi = 0. \tag{4.3}$$

From (1.6), (4.1) and (4.3), we have

$$\alpha_2(\phi A - A\phi)X - \lambda g(X, U)A\xi - \lambda g(X, A\xi)U + g(X, A\xi)AU = 0. \tag{4.4}$$

If we put $X = \xi$ in (4.4), then, since $\alpha_2 = \lambda\alpha_1$, we have

$$\alpha_1 AU = 0. \tag{4.5}$$

If there exists a point $p \in M$ such that $\alpha_1(p) = 0$, then we see that $\alpha_2 = 0$, and hence by Lemma 1, we have ξ is principal at p. So, from now we discuss on open subset where $\alpha_1 \neq 0$. Then from (4.5) it follows that

$$AU = 0. \tag{4.6}$$

With (4.6) we easily obtain

$$g((\nabla_X A)\xi, \xi) = d\alpha_1(X),$$

where d denotes the exterior differential. Since $U = \phi A\xi$, from (1.4), (1.7) and (4.6) we have

$$\nabla_\xi U = \alpha_1 A\xi - \alpha_2\xi + \phi\nabla\alpha_1, \tag{4.7}$$

where $\nabla\alpha_1$ denotes the gradient vector field of α_1. Differentiating (4.6) covariantly along M, then by using (1.7) and (4.7) we have

$$(\nabla_U A)\xi = -\phi U - \alpha_1 A^2\xi + \alpha_2 A\xi - A\phi\nabla\alpha_1. \tag{4.8}$$

Also, if we differentiate $A^2\xi = \lambda A\xi$ covariantly along M, then together with (1.5) we have

$$\begin{aligned} &g(A\xi, (\nabla_X A)Y) + g((\nabla_X A)\xi, AY) + g(\phi AX, A^2 Y) \\ &= d\lambda(X)g(A\xi, Y) + \lambda g((\nabla_X A)\xi, Y) + \lambda g(\phi AX, AY). \end{aligned} \tag{4.9}$$

From (1.7) and (4.9) we have

$$\begin{aligned} &\eta(X)g(A\xi, \phi Y) - \eta(Y)g(A\xi, \phi X) - 2\alpha g(\phi X, Y) \\ &\quad + g((\nabla_X A)\xi, AY) - g((\nabla_Y A)\xi, AX) + g(\phi AX, A^2 Y) - g(\phi AY, A^2 X) \\ &= d\lambda(X)g(A\xi, Y) - d\lambda(Y)g(A\xi, X) + \lambda g((\nabla_X A)\xi, Y) \\ &\quad - \lambda g((\nabla_Y A)\xi, X) + 2\lambda g(\phi AX, AY) \end{aligned}$$

for any vector fields X and Y on M. We put $X = U$ and making use of (1.7), (4.6) and (4.8), then we have

$$g((\nabla_U A)\xi, AY) = 2(\alpha - \lambda)g(\phi U, Y) - \eta(Y)g(U, U) + d\lambda(U)g(A\xi, Y). \tag{4.10}$$

Thus, from (4.8) and (4.10) we have

$$\begin{aligned} 2(\alpha - \lambda)g(\phi U, Y) - \eta(Y)g(U, U) + d\lambda(U)g(A\xi, Y) \\ = -g(\phi U, AY) - \alpha_1 g(A^2\xi, AY) + \alpha_2 g(A\xi, AY) + d\alpha_1(\phi A^2 Y). \end{aligned} \tag{4.11}$$

Putting $Y = \xi$ in (4.11), then together with (4.1) we get

$$\alpha_1 d\lambda(U) - \lambda d\alpha_1(U) = 2(\alpha_2 - \alpha_1^2).$$

Further we put $Y = A\xi$ in (4.11), then we get

$$\lambda\{\alpha_1 d\lambda(U) - \lambda d\alpha_1(U)\} = (\alpha_2 - \alpha_1^2)(3\alpha_1 - \lambda).$$

Thus, we have $\alpha_2 - \alpha_1^2 = \alpha_1(\lambda - \alpha_1) = 0$, from which using Lemma 1 we see that $A\xi = \alpha_1\xi$ on M. From (4.4) and Lemma 1, it follows that

$$\alpha_1(\phi A - A\phi)X = 0.$$

Since α_1 is constant, by a similar way as in the proof of Theorem 4 and using Theorem 1, we have our assertions. (Q.E.D.)

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First author.
Department of Mathematics,
Chonnam University,
Kwangju 500-757, Korea.

Second author.
Topology and Geometry Research Center,
Kyungpook University,
Taegu 702-701, Korea.