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NEUTRAL HYPERKÄHLER STRUCTURES ON PRIMARY KODAIRA SURFACES

By

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1. Introduction

In the present paper, we study neutral hyperkähler structures on four-dimensional manifolds, which draw attention recently in differential geometry and especially in mathematical physics (cf. Hull [14], Ooguri-Vafa [20]). A neutral hyperkähler structure on a pseudo-Riemannian four-manifold M of metric signature (2, 2) consists of a neutral metric g and three endomorphisms $(I, 'J, 'K)$ on the tangent bundle TM of M such that

$$I^2 = -\mathrm{Id}, \quad 'J^2 = 'K^2 = \mathrm{Id} \quad \text{and} \quad I'J = -'JI = 'K; \tag{1}$$

$$g(V_1, V_2) = g(IV_1, IV_2) = -g('JV_1, 'JV_2) = -g('KV_1, 'KV_2) \tag{2}$$

for arbitrary vector fields V_1, V_2 on M, and that these structures enjoy some desired properties similar to the Kähler condition. We shall call a triple $(I, 'J, 'K)$ satisfying (1) a split-quaternion structure (or a paraquaternionic structure in some literature (cf. Blažić [4], García-Río et al. [10])), g satisfying (2) a compatible metric with $(I, 'J, 'K)$, and $(g, I, 'J, 'K)$ a neutral almost hyperhermitian structure. For a four-manifold M endowed with such a structure $(g, I, 'J, 'K)$, the invariance of g by I and the skew-invariance by $'J$ and $'K$ allow us to define three nondegenerate 2-forms $\Omega_I, \Omega_{'J}, \Omega_{'K}$, called the fundamental forms, as follows:

$$\Omega_I(V_1, V_2) := g(IV_1, V_2), \Omega_{'J}(V_1, V_2) := g('JV_1, V_2), \Omega_{'K}(V_1, V_2) := g('KV_1, V_2),$$

where V_1, V_2 are vector fields on M.

DEFINITION. A neutral almost hyperhermitian four-manifold $(M, g, I, 'J, 'K)$

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is called a *neutral hyperkähler surface* if $I, 'J, 'K$ are integrable and if $\Omega_I, \Omega_{'J}, \Omega_{'K}$ are closed.

Here, $'J$ (resp. $'K$) is said to be integrable if the tangential distributions corresponding to the (± 1)-eigenspaces of $'J$ (resp. $'K$) are integrable, and a neutral almost hyperhermitian four-manifold with integrable split-quaternion structure is called a neutral hyperhermitian surface. The fundamental 2-forms on a neutral hyperkäher surface define three symplectic structures, with certain algebraic identities, which is called a hypersymplectic structure in Hitchin [13].

For the Riemannian analogue, it is known that a compact hyperkähler surface is biholomorphic and isometric to either a complex torus with the flat metric or a $K3$ surface with the Ricci-flat Calabi-Yau metric. Taking account of the Euler characteristics and the Hirzebruch signatures, we see that any hyperkähler metric on a torus (resp. a $K3$ surface) must be flat (resp. nonflat), since it is Ricci-flat and anti-self-dual (cf. Besse [3]).

We will focus our attention on the indefinite case, i.e., neutral hyperkähler structures on compact four-manifolds. Any complex torus has a flat neutral hyperkähler structure associated with the standard one on the complex plane. Moreover, we will observe in §2 that any neutral hyperkähler surface must be biholomorphic to either a complex torus or a primary Kodaira surface. We are then interested in seeking a compact complex surface with nonflat neutral hyperkähler structure, like a $K3$ surface in the Riemannian case.

Concerning the issue, we see that neutral hyperkähler structures on compact four-manifolds exhibit quite different properties to those of hyperkähler structures in the Riemannian case. We will give a characterization of neutral hyperkähler structures, in terms of a partial differential equation for the Kähler potentials, and show that any primary Kodaira surface admits neutral hyperkähler structures, whose compatible neutral metrics can be chosen to be flat or nonflat, according as some particular functions are constant or not. Our main results will be stated in Theorems 1 and 2.

It should be pointed out that J. Petean [21] has independently studied indefinite Ricci-flat Kähler metrics on compact complex surfaces. Furthermore, he has successfully obtained a classification of compact complex surfaces which admit indefinite Ricci-flat Kähler metrics.

2. Neutral Hyperkähler Structures

Let $\Omega_I, \Omega_{'J}, \Omega_{'K}$ be the fundamental forms of a neutral almost hyperhermitian four-manifold $(M, g, I, 'J, 'K)$. Then, $\wedge\Omega_l : \wedge^1 T^*M \to \wedge^3 T^*M$ are isomorphisms, and hence there exist three kinds of Lee forms $\beta_I, \beta_{'J}, \beta_{'K}$ such that $d\Omega_l = \beta_l \wedge \Omega_l$ $(l \in \{I, 'J, 'K\})$.

For the later convenience, we must recognize the following (see [16], cf. Boyer [6]):

PROPOSITION 1. *$I, 'J$ and $'K$ are integrable if and only if $\beta_I = \beta_{'J} = \beta_{'K}$ $(=: \beta)$.*

For a neutral hyperhermitian surface $(M, g, I, 'J, 'K)$, we call β the Lee form. If β is exact (resp. closed), then the surface is globally (resp. locally), conformally neutral hyperkähler.

Recall that, given a neutral hyperkähler surface $(M, g, I, 'J, 'K)$, the 2-form $\Omega_{'J} + \sqrt{-1}\Omega_{'K}$ is not only a nonvanishing holomorphic 2-form on (M, I) but also a parallel section of the canonical bundle $K_{(M,I)}$. Then the Ricci curvature of (M, g) is identically zero, and therefore the first Chern class $c_1(M, I)$ vanishes. On the other hand, any neutral hyperhermitian metric is (conformally) self-dual (cf. Akivis and Goldberg [1], [14], [16]). Summarizing these, we have the following:

PROPOSITION 2. *Any neutral hyperkähler surface is Ricci-flat and self-dual.*

Remark that any scalar-flat neutral Kähler metric is self-dual (cf. [17], for the Riemannian analogue, see Derziński [7], Itoh [15]).

We now recall that the fundamental 2-forms $\Omega_I, \Omega_{'J}, \Omega_{'K}$ of a neutral hyperkähler surface $(M, g, I, 'J, 'K)$ satisfy the following: $d\Omega_l \equiv 0$ and

$$(3) \qquad -\Omega_I^2 = \Omega_{'J}^2 = \Omega_{'K}^2, \quad \Omega_l \wedge \Omega_m \equiv 0, \quad (l \neq m; l, m \in \{I, 'J, 'K\}).$$

In particular, we note that Ω_I is compatible with the opposite orientation and that $(\Omega_{'J}, \Omega_{'K})$ is a conformal symplectic couple on M in the sense of Geiges [11].

By making use of analogous arguments in Geiges-Gonzalo [12] and Geiges [11] (see [16]), we can obtain the following:

PROPOSITION 3. *If a four-manifold M admits three nondegenerate 2-forms $\Omega_1, \Omega_2, \Omega_3$ satisfying the same relations as (3), then there exists a unique neutral almost hyperhermitian structure $(g, I, 'J, 'K)$ on M such that $\Omega_1 = \Omega_I, \Omega_2 = \Omega_{'J}$, $\Omega_3 = \Omega_{'K}$. If these 2-forms $\Omega_1, \Omega_2, \Omega_3$ are closed, then $(g, I, 'J, 'K)$ defines a neutral hyperkähler structure.*

By Proposition 1, the closedness of $\Omega_1, \Omega_2, \Omega_3$ leads the integrability of $I, 'J, 'K$.

In the rest of this section, we discuss compact neutral hyperkähler surfaces. According to the Enriques-Kodaira classification of compact complex surfaces (cf. Barth et al. [2]), we see that for any compact neutral hyperkähler surface $(M, g, I, 'J, 'K)$, the underlying complex surface (M, I) must be biholomorphic to one of the following possibilities:

(a) a complex torus, (b) a $K3$ surface, (c) a primary Kodaira surface, since $\Omega_{'J} + \sqrt{-1}\Omega_{'K}$ defines a nonvanishing holomorphic 2-form on (M, I). Among these candidates, it is already noted that a complex torus has the standard flat neutral hyperkähler structure.

For the second candidate, Matsushita [19] showed that $K3$ surfaces admit many neutral metrics (see also Bonome et al. [5]). However, Draghici [8] recently showed that $K3$ surfaces admit no symplectic structures compatible with the opposite orientation. Noting that one of the fundamental forms of a neutral hyperkähler surface defines a symplectic structure compatible with the opposite orientation, we see that $K3$ surfaces admit no neutral (hyper)kähler structures. Therefore, we must consider the other candidates.

3. Primary Kodaira Surfaces

In this section, we devote ourselves to a primary Kodaira surface X, which is a compact complex surface, with $b_1(X) = 3$, $c_1(X) = 0$, $c_2(X) = 0$, obtained as the total space of an elliptic fibre bundle over an elliptic curve. Moreover the other numerical characters of X are given as follows:

$$h^{1,0}(X) = 1, \quad q(X) = 2, \quad p_g(X) = 1, \quad b_2^+(X) = b_2^-(X) = 2,$$

where we denote respectively by $h^{1,0}(X)$, $q(X)$ and $p_g(X)$ the complex dimension of the space of holomorphic 1-forms, the irregularity and the geometric genus of X (see Barth et al. [2]). Any primary Kodaira surface cannot be Kähler, since its first Betti number is three. Fernández et al. [9] constructed examples of (flat) neutral Kähler structures on primary Kodaira surfaces of particular type.

It is well-known that every primary Kodaira surface X is covered by the complex plane $\boldsymbol{C}^2$ and its fundamental group $\pi_1(X)$ can be represented injectively into Affine$(\boldsymbol{C}^2)$, the complex affine transformation group on $\boldsymbol{C}^2$:

$$\rho : \pi_1(X) \to \mathrm{Affine}(\boldsymbol{C}^2), \quad \rho(\gamma)(z_1, z_2) = (z_1 + \alpha_\gamma, z_2 + \overline{\alpha_\gamma} z_1 + \beta_\gamma),$$

where (z_1, z_2) is the standard complex coordinates of $\boldsymbol{C}^2$ and $\alpha_\gamma, \beta_\gamma$ are constants in C depending only on γ. If we put $G := \rho(\pi_1(X))$, then we can identify X with C^2/G, as complex surface (see Kodaira [18]).

At this stage, we can state our main result:

Theorem 1. *Let $X = C^2/G$ be a primary Kodaira surface. Then the following 2-forms $\Omega_1, \Omega_2, \Omega_3$ define a neutral hyperkähler structure on X:*

$$\Omega_1 = \mathrm{Im}(dw_1 \wedge d\overline{w_2}) + \sqrt{-1}\,\mathrm{Re}(w_1)\, dw_1 \wedge d\overline{w_1} + (\sqrt{-1}/2)\partial\bar{\partial}\varphi, \tag{4}$$

$$\Omega_2 = \mathrm{Re}(e^{\sqrt{-1}\theta} dw_1 \wedge dw_2), \quad \Omega_3 = \mathrm{Im}(e^{\sqrt{-1}\theta} dw_1 \wedge dw_2),$$

where (w_1, w_2) is the standard complex coordinate system of C^2, θ is a real constant and φ is a solution to the equation:

$$4\sqrt{-1}(\mathrm{Im}(dw_1 \wedge d\overline{w_2}) + \sqrt{-1}\,\mathrm{Re}(w_1)\, dw_1 \wedge d\overline{w_1}) \wedge \partial\bar{\partial}\varphi = \partial\bar{\partial}\varphi \wedge \partial\bar{\partial}\varphi. \tag{5}$$

In particular, any primary Kodaira surface admits neutral hyperkähler structures. Conversely, under suitable complex coordinates (w_1, w_2) of C^2, the fundamental forms of any neutral hyperkähler structure on X can be expressed as (4).

Proof. Let $\Psi : X \to \Delta$ be an elliptic fibre bundle structure over the base elliptic curve Δ. Then we can verify the following commutative diagram:

$$\begin{array}{ccc} C^2 & \xrightarrow{\tilde{\varpi}} & X \\ {\scriptstyle\tilde{\Psi}}\downarrow & & \downarrow{\scriptstyle\Psi} \\ C & \xrightarrow[\varpi]{} & \Delta \end{array}$$

where $\tilde{\Psi}$ is the projection from C^2 to the first factor C, and $\tilde{\varpi}, \varpi$ are the covering maps. In this picture, if we denote by (z_1, z_2) the standard complex coordinate system of C^2, then $\phi := dz_1$ is a nonvanishing holomorphic 1-form on X, which generates the cohomology group $H^0(X, \Omega_X^1) \cong H_{\bar{\partial}}^{1,0}(X)$, and moreover, $\sigma^{0,1} := d\overline{z_2} - z_1 d\overline{z_1}$ is a $\bar{\partial}$-closed (0, 1)-form on X. Hence $H^1(X, \mathcal{O}_X) \cong H_{\bar{\partial}}^{0,1}(X)$, the Dolbeault cohomology group, is generated by the $\bar{\partial}$-cohomology classes of $\bar{\phi}$ and $\sigma^{0,1}$. Since $d\sigma^{0,1} = -dz_1 \wedge d\overline{z_1}$, a real 1-form $\sigma := \overline{\sigma^{0,1}} + \sigma^{0,1}$ is a d-closed 1-form on X. Furthermore, we see that the cohomology classes of $\phi, \bar{\phi}, \sigma$ generate $H^1(X, C)$.

Note that $\psi := dz_1 \wedge (dz_2 - \overline{z_1} dz_1)$ gives a nonvanishing holomorphic 2-form on X.

We now define real d-closed 2-forms $\Omega_2, \Omega_3, \Omega_2^-, \Omega_3^-$ respectively by

$$\Omega_2 + \sqrt{-1}\Omega_3 := dz_1 \wedge dz_2, \quad \Omega_2^- + \sqrt{-1}\Omega_3^- := \sqrt{-1} d\overline{z_1} \wedge (dz_2 - \overline{z_1} dz_1).$$

Then the cohomology classes of Ω_2, Ω_3 (resp. Ω_2^-, Ω_3^-) in $H^2(X, \boldsymbol{R})$ generate the cohomology group $H_+^2(X, \boldsymbol{R})$ (resp. $H_-^2(X, \boldsymbol{R})$). Moreover it is easy to see that $(\Omega_2^-, \Omega_2, \Omega_3)$ and $(\Omega_3^-, \Omega_2, \Omega_3)$ give neutral hyperkähler structures on X, respectively.

We next consider arbitrary neutral hyperkähler structures on X. Suppose that three symplectic structures $\Omega_1, \Omega_2, \Omega_3$ define a neutral hyperkähler structure. Then Ω_1 is a real (1, 1)-form on X and defines an element in $H_-^2(X, \boldsymbol{R})$. Thus there exists a real 1-form η and real constants a, b such that $\Omega_1 = a\Omega_2^- + b\Omega_3^- + d\eta$. It then follows from $\Omega_1^2 = (\Omega_2^-)^2 = (\Omega_3^-)^2$ that

$$(a^2 + b^2 - 1)\Omega_1^2 = d(\eta \wedge (2(a\Omega_2^- + b\Omega_3^-) + d\eta)).$$

By integrating the equation above, we obtain $a^2 + b^2 = 1$, so we may put $a = \cos\theta$, $b = \sin\theta$, where θ is a real constant.

Recalling the decomposition $\eta = \eta^{1,0} + \eta^{0,1} (\overline{\eta^{0,1}} = \eta^{1,0})$, we see that $\eta^{0,1}$ is $\bar{\partial}$-closed, since $\Omega_1, \Omega_2^-, \Omega_3^-$ are real (1, 1)-forms, and hence that

$$\eta^{0,1} = k\bar{\phi} + l\sigma^{0,1} + \bar{\partial}\mu, \quad d\eta = (\bar{l} - l)\, dz_1 \wedge d\overline{z_1} + \partial\bar{\partial}(\mu - \bar{\mu}),$$

where k, l are constants and μ is a complex-valued function on X. Setting $\sqrt{-1}c := \bar{l} - l (c \in \boldsymbol{R})$ and $\sqrt{-1}\varphi := 2(\mu - \bar{\mu})$, we then see that

$$\Omega_1 = \cos\theta\,\Omega_2^- + \sin\theta\,\Omega_3^- + \sqrt{-1}c dz_1 \wedge d\overline{z_1} + (\sqrt{-1}/2)\partial\bar{\partial}\varphi.$$

By making use of the coordinates $(w_1, w_2) := (e^{\sqrt{-1}\theta}z_1 + c, z_2)$, we can express $\Omega_1, \Omega_2, \Omega_3$ as

$$\Omega_1 = \Omega_0 + (\sqrt{-1}/2)\partial\bar{\partial}\varphi (=: \Omega_\varphi), \quad \Omega_2 + \sqrt{-1}\Omega_3 = e^{\sqrt{-1}\theta} dw_1 \wedge dw_2,$$

where Ω_0 is given by

$$\Omega_0 := (\sqrt{-1}/2)(d\overline{w_1} \wedge dw_2 - dw_1 \wedge d\overline{w_2} + (w_1 + \overline{w_1})\, dw_1 \wedge d\overline{w_1}).$$

Therefore we see that $(\Omega_1, \Omega_2, \Omega_3)$ defines a neutral hyperkähler structure on X if and only if φ satisfies the following equation:

$$4\sqrt{-1}\Omega_0 \wedge \partial\bar{\partial}\varphi = \partial\bar{\partial}\varphi \wedge \partial\bar{\partial}\varphi.$$

This concludes the proof. □

We note that the corresponding metric $g = g_\varphi$ is explicitly given by

$$g_\varphi = (w_1 + \overline{w_1})|dw_1|^2 - (dw_1 d\overline{w_2} + d\overline{w_1} dw_2) + D^2\varphi,$$

where D_φ^2 denotes the complex Hessian of φ. Clearly, the pullback of an arbitrary function on the base torus Δ is a solution to (5).

4. Flat Neutral Hyperkähler Structure on Primary Kodaira Surfaces

In this section, we shall prove the following:

THEOREM 2. *Let g_φ be the neutral hyperkähler metric on a primary Kodaira surface X defined by* (4). *Then g_φ is flat if and only if φ is constant.*

This shows that each nonconstant function φ on the base torus of any primary Kodaira surface defines a nonflat neutral hyperkähler metric g_φ (cf. Petean [21]).

PROOF. We first recall some preliminaries for a neutral Kähler surface (X, g, I). Let (w_1, w_2) be local holomorphic coordinates on X. For simplicity, we set

$$\partial_\alpha := \partial/\partial w_\alpha, \quad \partial_{\bar{\alpha}} := \partial/\partial \overline{w_\alpha} \text{ and } g_{\alpha\bar{\beta}} := 2g(\partial_\alpha, \partial_{\bar{\beta}})$$

$(\alpha, \beta = 1, 2)$. Let ∇ be the Levi-Civita connection of (X, g) and $\{\omega_B^A\}$ the connection form of ∇ with respect to $\{\partial_A\}$ $(A, B = 1, 2, \bar{1}, \bar{2})$. Then $\omega_\beta^{\bar{\alpha}} = \omega_{\bar{\beta}}^{\alpha} \equiv 0$, since $\nabla I \equiv 0$. Moreover ω_β^α (resp. $\omega_{\bar{\beta}}^{\bar{\alpha}}$) is a local $(1,0)$-(resp. $(0,1)$-) form, since ∇ is torsion-free. Hence the components of $\{\omega_B^A\}$, except for $\{\omega_\beta^\alpha(\partial_\gamma)\}$ and $\{\omega_{\bar{\beta}}^{\bar{\alpha}}(\partial_{\bar{\gamma}})\}$, must vanish. Since ∇ preserves the metric g, we have

$$\omega_\beta^\alpha = \sum_\varepsilon g^{\bar{\varepsilon}\alpha} \partial g_{\beta\bar{\varepsilon}}, \quad \omega_{\bar{\beta}}^{\bar{\alpha}} = \sum_\varepsilon g^{\bar{\alpha}\varepsilon} \bar{\partial} g_{\varepsilon\bar{\beta}}, \tag{6}$$

where $g^{\bar{\alpha}\beta}$ is given by $\sum_\varepsilon g_{\alpha\bar{\varepsilon}} g^{\bar{\varepsilon}\beta} = \sum_\varepsilon g^{\bar{\beta}\varepsilon} g_{\varepsilon\bar{\alpha}} = \delta_\alpha^\beta$. The curvature form $\{R_B^A\}$ of ∇ is given as follows:

$$R_\beta^\alpha = \bar{\partial}\omega_\beta^\alpha, \quad R_{\bar{\beta}}^{\bar{\alpha}} = \partial\omega_{\bar{\beta}}^{\bar{\alpha}}. \tag{7}$$

Therefore we see that g is flat (i.e., $R \equiv 0$) if and only if ω_β^α is a local holomorphic 1-form on X.

Let $X = \boldsymbol{C}^2/G$ be a primary Kodaira surface, g a neutral hyperkähler metric on X and $\Omega_1, \Omega_2, \Omega_3$ the fundamental forms. By making use of complex coordinates (w_1, w_2) satisfying $\Omega_2 + \sqrt{-1}\Omega_3 = e^{\sqrt{-1}\theta} dw_1 \wedge dw_2$ (θ is a real constant), the condition $-\Omega_1^2 = \Omega_2^2 = \Omega_3^2$ can be written as follows:

$$g_{1\bar{1}} g_{2\bar{2}} - g_{1\bar{2}} g_{2\bar{1}} \equiv -1. \tag{8}$$

Thus the components $g^{\bar{\alpha}\beta}$ satisfy

$$g^{\bar{1}1} = -g_{2\bar{2}}, \quad g^{\bar{1}2} = g_{1\bar{2}}, \quad g^{\bar{2}1} = g_{2\bar{1}}, \quad g^{\bar{2}2} = -g_{1\bar{1}}.$$

The connection form $\{\omega_\beta^\alpha\}$ is given by

$$\omega_1^1 = -g_{2\bar{2}}\partial g_{1\bar{1}} + g_{2\bar{1}}\partial g_{1\bar{2}}, \quad \omega_2^1 = -g_{2\bar{2}}\partial g_{2\bar{1}} + g_{2\bar{1}}\partial g_{2\bar{2}},$$
$$\omega_1^2 = g_{1\bar{2}}\partial g_{1\bar{1}} - g_{1\bar{1}}\partial g_{1\bar{2}}, \quad \omega_2^2 = g_{1\bar{2}}\partial g_{2\bar{1}} - g_{1\bar{1}}\partial g_{2\bar{2}}. \tag{9}$$

In particular, it follows from (8) that

$$\omega_1^1 + \omega_2^2 \equiv 0. \tag{10}$$

Recall that the fundamental form Ω_1 may be written as

$$\Omega_1 = (\sqrt{-1}/2)(-dw_1 \wedge d\overline{w_2} - dw_2 \wedge d\overline{w_1} + (w_1 + \overline{w_1})\, dw_1 \wedge d\overline{w_1} + \partial\bar{\partial}\varphi),$$

where φ is a smooth function on X. The components $g_{\alpha\bar{\beta}}$ are given explicitly in the following fashion:

$$g_{1\bar{1}} = w_1 + \overline{w_1} + \frac{\partial^2\varphi}{\partial w_1 \partial \overline{w_1}}, \quad g_{1\bar{2}} = -1 + \frac{\partial^2\varphi}{\partial w_1 \partial \overline{w_2}} (= \overline{g_{2\bar{1}}}), \quad g_{2\bar{2}} = \frac{\partial^2\varphi}{\partial w_2 \partial \overline{w_2}}.$$

From (9) and (7), if φ is constant, then g is flat.

For any $\gamma \in G$, we define $\rho_\gamma : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ by

$$\rho_\gamma(w_1, w_2) = (w_1 + \alpha_\gamma, w_2 + \overline{\alpha_\gamma} w_1 + \beta_\gamma).$$

It then follows that

$$\rho_\gamma^*(dw_1) = dw_1, \quad \rho_\gamma^*(dw_2) = dw_2 + \overline{\alpha_\gamma} dw_1, \tag{11}$$

$$\rho_{\gamma^*}(\partial_1) = \partial_1 + \overline{\alpha_\gamma}\partial_2, \quad \rho_{\gamma^*}(\partial_2) = \partial_2. \tag{12}$$

Then we can verify the following relations:

$$g_{1\bar{1}} \circ \rho_\gamma = g_{1\bar{1}} - \alpha_\gamma g_{1\bar{2}} - \overline{\alpha_\gamma} g_{2\bar{1}} + |\alpha_\gamma|^2 g_{2\bar{2}}$$
$$g_{1\bar{2}} \circ \rho_\gamma = g_{1\bar{2}} - \overline{\alpha_\gamma} g_{2\bar{2}}, \quad g_{2\bar{1}} \circ \rho_\gamma = g_{2\bar{1}} - \alpha_\gamma g_{2\bar{2}}, \quad g_{2\bar{2}} \circ \rho_\gamma = g_{2\bar{2}}. \tag{13}$$

By making use of the relations above, we can also verify the following

$$\rho_\gamma^*\omega_1^1 = \omega_1^1 - \overline{\alpha_\gamma}\omega_2^1, \quad \rho_\gamma^*\omega_2^1 = \omega_2^1, \quad \rho_\gamma^*\omega_1^2 = \omega_1^2 + 2\overline{\alpha_\gamma}\omega_1^1 - \overline{\alpha_\gamma}^2\omega_2^1. \tag{14}$$

If we set

$$\eta_1 := \omega_1^1 + \overline{w_1}\omega_2^1, \quad \eta_2 := \omega_2^1, \quad \eta^3 := \omega_1^2 - 2\overline{w_1}\omega_1^1 - \overline{w_1}^2\omega_2^1,$$

then η_1, η_2, η_3 may be regarded as 1-forms on $X = \boldsymbol{C}^2/G$.

In what follows, we suppose that g is flat. Then η_2 is a holomorphic 1-form on X. Since $h^{1,0}(X) = 1$, we can write η_2 as

$$\eta_2 = A dw_1,$$

where A is a constant. In particular,

$$d\eta_2 = \partial\eta_2 = \bar{\partial}\eta_2 \equiv 0.$$

Lemma. $\eta_2 \equiv 0$.

Proof. From the flatness of g and (10), we have

$$0 \equiv d\eta_2 = d\omega_2^1 = -(\omega_1^1 \wedge \omega_2^1 + \omega_2^1 \wedge \omega_2^2) = -2\omega_1^1 \wedge \omega_2^1.$$

Thus we also have

$$\eta_1 \wedge \eta_2 = (\omega_1^1 + \overline{w_1}\omega_2^1) \wedge \omega_2^1 \equiv 0.$$

If $A \neq 0$, then $\eta_1 \wedge dw_1 \equiv 0$. Since η_1 is a $(1,0)$-form on X, we can find a function F on X such that

$$\eta_1 = F dw_1, \quad \text{i.e.,} \quad \omega_1^1 = (F - A\overline{w_1})\, dw_1.$$

By the flatness of g again, we obtain

$$0 \equiv \bar{\partial}\omega_1^1 = (\bar{\partial}F - Ad\overline{w_1}) \wedge dw_1.$$

Namely, we see that $\bar{\partial}F = Ad\overline{w_1}$, and hence that $\partial\bar{\partial}F \equiv 0$. From the mean value property for the operator $\partial\bar{\partial}$, we conclude that F must be constant. Thus $Ad\overline{w_1} = \bar{\partial}F \equiv 0$, i.e., $A = 0$. This contradicts the assumption $A \neq 0$. □

From the lemma above and (14), $\eta_1 = \omega_1^1$ is a holomorphic 1-form on X. Hence there exists a constant B such that

$$\eta_1 = B dw_1.$$

It is easy to see that

$$\bar{\partial}\eta_3 = 2B dw_1 \wedge d\overline{w_1}, \quad \partial\eta_3 = 2B dw_1 \wedge \eta_3. \tag{15}$$

We may assume that η_3 is expressed as

$$\eta_3 = f_1 dw_1 + f_2(dw_2 - \overline{w_1} dw_1)$$

for smooth functions f_1, f_2 on X. It then follows from (15) that

$$\bar{\partial}(f_1 - \overline{w_1} f_2) + 2Bd\overline{w_1} \equiv 0, \quad \bar{\partial} f_2 \equiv 0. \tag{16}$$

In particular, f_2 is a holomorphic function on X. Namely, f_2 must be a constant, say C. It follows from (16) that

$$\partial\bar{\partial} f_1 = \partial((-2B + C)\, dw_1) \equiv 0.$$

From the mean value property for $\partial\bar{\partial}$ again, we see that f_1 is also constant, say K. It is easy to see from (15) that

$$2Bdw_1 \wedge d\overline{w_1} = \bar{\partial}\eta_3 = \bar{\partial}(Kdw_1 + C(dw_2 - \overline{w_1} dw_1)) = Cdw_1 \wedge d\overline{w_1},$$

$$2BCdw_1 \wedge dw_2 = \partial\eta_3 = \partial(Kdw_1 + C(dw_2 - \overline{w_1} dw_1)) \equiv 0,$$

and hence $B = C = 0$. Thus we obtain

$$\eta_1 = \eta_2 \equiv 0, \quad \eta_3 = Kdw_1.$$

Using (8) and (9), we have the following:

$$\partial g_{2\bar{2}} = -\partial g_{2\bar{1}} \equiv 0, \quad \partial g_{1\bar{2}} = -Kg_{2\bar{2}} dw_1, \quad \partial g_{1\bar{1}} = -Kg_{2\bar{1}} dw_1.$$

In particular, $g_{2\bar{2}}$ is a constant, since $\partial g_{2\bar{2}} = \bar{\partial} g_{2\bar{2}} \equiv 0$. By integrating $g_{2\bar{2}} = \partial^2\varphi / \partial w_2 \partial\overline{w_2}$ on each fibre T of $\Psi : X \to \Delta$, we obtain

$$g_{2\bar{2}} \int_T dw_2 \wedge d\overline{w_2} = \int_T \frac{\partial^2\varphi}{\partial w_2 \partial\overline{w_2}} dw_2 \wedge d\overline{w_2} = 0,$$

i.e., $g_{2\bar{2}} \equiv 0$. Then φ is depending only on the variable w_1, so φ may be regarded as a function on Δ. In particular, $g_{1\bar{2}} = g_{2\bar{1}} \equiv -1$. On the other hand, we note that $g_{1\bar{1}} - (w_1 + \overline{w_1})$ can be regarded as a function on X, and moreover that

$$\partial\bar{\partial}(g_{1\bar{1}} - (w_1 + \overline{w_1})) = -\bar{\partial}(\partial g_{1\bar{1}} - dw_1) = -\bar{\partial}(K - 1)\, dw_1 \equiv 0.$$

Then $g_{1\bar{1}} - (w_1 + \overline{w_1})$ must be constant, say L. Integrating $L = \partial^2\varphi / \partial w_1 \partial\overline{w_1}$ on Δ, we also have $L = 0$. Therefore φ is constant. Namely, g must coincide with g_0. □

We finally remark on neutral hyperkähler structures on complex tori. By using arguments similar to those in §§3–4, we can obtain an analogous result for flat neutral hyperkähler structures on complex tori. We thus see that complex tori of particular type (e.g., the product of elliptic curves) admit nonflat neutral hyperkähler structures. Such an example was given in Petean [21]. We are then led to the question whether all of complex tori admit nonflat neutral hyperkäher structures. This will be the future problem.

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