

A CLASS OF REAL-ANALYTIC SURFACES IN THE 3-EUCLIDEAN SPACE

By

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Abstract. A smooth surface S in \mathbf{R}^3 is called *parallel curved* if there exists a plane in \mathbf{R}^3 such that at each point of S , there exists a principal direction parallel to the plane. For example, a plane, a cylinder and a round sphere are parallel curved. More generally, a surface of revolution is also parallel curved. The purposes of this paper are to study the behavior of the principal distributions on a real-analytic, parallel curved surface and to classify the connected, complete, real-analytic, embedded, parallel curved surfaces.

1. Introduction

Let S be a smooth surface in \mathbf{R}^3 and $\text{Umb}(S)$ the set of the umbilical points of S . If $S \setminus \text{Umb}(S) \neq \emptyset$, then there exists a one-dimensional continuous distribution on $S \setminus \text{Umb}(S)$ which gives a principal direction at each points of $S \setminus \text{Umb}(S)$. Such a distribution is called a *principal distribution* on S . Let p_0 be an isolated umbilical point of S . Then the indices of p_0 with respect to two principal distributions coincide with each other. The common number is called the *index* of p_0 on S and denoted by $\text{ind}_{p_0}(S)$. Let (x, y) be local coordinates around p_0 such that p_0 corresponds to $(0, 0)$ and r_0 a positive number such that p_0 is the only umbilical point on $\{x^2 + y^2 < r_0^2\}$, and let $\phi_{S;p_0}$ denote a continuous function on $(0, r_0) \times \mathbf{R}$ such that for any $(r, \theta) \in (0, r_0) \times \mathbf{R}$, a tangent vector $\cos \phi_{S;p_0}(r, \theta) \partial / \partial x + \sin \phi_{S;p_0}(r, \theta) \partial / \partial y$ is in a principal direction at $(r \cos \theta, r \sin \theta)$. Then the index $\text{ind}_{p_0}(S)$ is represented as follows:

$$\text{ind}_{p_0}(S) = \frac{\phi_{S;p_0}(r, \theta + 2\pi) - \phi_{S;p_0}(r, \theta)}{2\pi}. \quad (1)$$

Let \mathcal{P}^k be the set of the homogeneous polynomials in two variables of degree $k \geq 2$ and \mathcal{P}_o^k the set of the elements of \mathcal{P}^k such that on each of their graphs, the origin $o := (0, 0, 0)$ of \mathbf{R}^3 is an isolated umbilical point. For $g \in \mathcal{P}^k$ and for $\theta \in \mathbf{R}$, set $\tilde{g}(\theta) := g(\cos \theta, \sin \theta)$. In [1], we studied the behavior of the principal distributions around o on the graph \mathbf{G}_g of $g \in \mathcal{P}_o^k$. Then we divided the study into two cases: $d\tilde{g}/d\theta \equiv 0$ and $d\tilde{g}/d\theta \not\equiv 0$. If $g \in \mathcal{P}_o^k$ satisfies $d\tilde{g}/d\theta \equiv 0$, then the ‘‘position vector field’’ $x\partial/\partial x + y\partial/\partial y$ is in a principal direction at each point of \mathbf{G}_g , and from this together with formula (1), $\text{ind}_o(\mathbf{G}_g) = 1$ follows. For $g \in \mathcal{P}_o^k$ satisfying $d\tilde{g}/d\theta \not\equiv 0$, we mainly paid attention to the relation between the behavior of the principal distributions and the behavior of the position vector field around a point at which the position vector field is in a principal direction, and we presented a way of computing $\text{ind}_o(\mathbf{G}_g)$ and proved $\text{ind}_o(\mathbf{G}_g) \in \{1 - k/2 + i\}_{i=0}^{\lfloor k/2 \rfloor}$. In [2], we have further studied the behavior of the principal distributions in relation to the existence of other umbilical points than o , around a point at which the position vector field is in a principal direction. We may find such a point, because Euler’s identity holds for any homogeneous polynomial. In order to study the behavior of the principal distributions around an isolated umbilical point on a general surface by a similar method, we need some other vector field than the position vector field.

For a smooth function f of two variables x, y , we set

$$p_f := \frac{\partial f}{\partial x}, \quad q_f := \frac{\partial f}{\partial y}, \quad r_f := \frac{\partial^2 f}{\partial x^2}, \quad s_f := \frac{\partial^2 f}{\partial x \partial y}, \quad t_f := \frac{\partial^2 f}{\partial y^2},$$

$$\text{grad}_f := \begin{pmatrix} p_f \\ q_f \end{pmatrix}, \quad \text{grad}_f^\perp := \begin{pmatrix} -q_f \\ p_f \end{pmatrix}, \quad \text{Hess}_f := \begin{pmatrix} r_f & s_f \\ s_f & t_f \end{pmatrix}.$$

Let $\langle \cdot, \cdot \rangle$ be the scalar product in \mathbf{R}^2 and set

$$\varpi_f := \langle \text{Hess}_f \text{grad}_f, \text{grad}_f^\perp \rangle.$$

In Section 2, we shall prove the following:

PROPOSITION 1.1. *Let f be a smooth function of two variables and \mathbf{G}_f the graph of f . Then at a point of \mathbf{G}_f , the gradient vector field of f is in a principal direction if and only if $\varpi_f = 0$ holds.*

For $g \in \mathcal{P}^k$, we see by Euler’s identity $(k-1) \text{grad}_g = \text{Hess}_g^t(x, y)$ that

$$(k-1)\varpi_g = \det(\text{Hess}_g) \frac{d\tilde{g}}{d\theta}(\theta) \quad (2)$$

holds at $(\cos \theta, \sin \theta)$ for any $\theta \in \mathbf{R}$. Therefore $\varpi_g \equiv 0$ holds if and only if $\det(\text{Hess}_g) \equiv 0$ or $d\tilde{g}/d\theta \equiv 0$ holds. If $g \in \mathcal{P}^k$ satisfies $\det(\text{Hess}_g) \equiv 0$, then there exists a vector ${}^t(\alpha, \beta) \in \mathbf{R}^2$ satisfying $g = (\alpha x + \beta y)^k$, which implies $g \notin \mathcal{P}_o^k$. Therefore we see that for $g \in \mathcal{P}_o^k$, $\varpi_g \equiv 0$ (resp. $\neq 0$) is equivalent to $d\tilde{g}/d\theta \equiv 0$ (resp. $\neq 0$) and this leads us to study the behavior of the principal distributions in relation to the behavior of the gradient vector field. In [2], we have carried out this on \mathbf{G}_g for $g \in \mathcal{P}_o^k$.

Let $\mathcal{A}_o^{(2)}$ be the set of the real-analytic functions defined on a connected neighborhood of $(0, 0)$ in \mathbf{R}^2 such that for each $F \in \mathcal{A}_o^{(2)}$, $F(0, 0) = p_F(0, 0) = q_F(0, 0) = 0$ hold, and \mathcal{A}_o^2 the set of the elements of $\mathcal{A}_o^{(2)}$ such that on each of their graphs, o is an isolated umbilical point. One of the purposes of this paper is to study the behavior of the principal distributions around o on the graph \mathbf{G}_F of $F \in \mathcal{A}_o^{(2)}$ satisfying $\varpi_F \equiv 0$ and the index $\text{ind}_o(\mathbf{G}_F)$ of o for $F \in \mathcal{A}_o^2$ satisfying $\varpi_F \equiv 0$. In Section 5, we shall prove the following:

THEOREM 1.2. *Let F be an element of \mathcal{A}_o^2 satisfying $\varpi_F \equiv 0$. Then \mathbf{G}_F is part of a surface of revolution such that o lies on the axis of rotation; at any point of \mathbf{G}_F , the position vector field is in a principal direction and $\text{ind}_o(\mathbf{G}_F) = 1$ holds.*

THEOREM 1.3. *Let F be an element of $\mathcal{A}_o^{(2)} \setminus \mathcal{A}_o^2$ satisfying $\varpi_F \equiv 0$. Then one of the following holds:*

- (1) \mathbf{G}_F is part of a plane or a round sphere;
- (2) There exist a neighborhood U_o of $(0, 0)$ in \mathbf{R}^2 and a real-analytic curve C_0 in U_o satisfying the following:
 - (a) $C_0 = \{(x, y) \in U_o; F(x, y) = 0\}$,
 - (b) $C_0 = \text{Umb}(\mathbf{G}_{F|_{U_o}})$ or $\text{Umb}(\mathbf{G}_{F|_{U_o}}) = \emptyset$ holds,
 - (c) For any point $q \in C_0$ and for the plane P_q^\perp in \mathbf{R}^3 normal to C_0 at q , the set $C_q^\perp := P_q^\perp \cap \mathbf{G}_{F|_{U_o}}$ is a real-analytic curve such that at each point of C_q^\perp , a tangent vector to C_q^\perp is in a principal direction of \mathbf{G}_F .

REMARK. For an integer $l \geq 3$, let $\mathcal{A}_o^{(l)}$ be the subset of $\mathcal{A}_o^{(2)}$ such that for any $F \in \mathcal{A}_o^{(l)}$ and for non-negative integers $m, n \geq 0$ satisfying $0 \leq m + n < l$, $(\partial^{m+n} F / \partial x^m \partial y^n)(0, 0) = 0$ holds. For each $F \in \mathcal{A}_o^2$, there exists an element $f_F \in \mathcal{A}_o^{(3)}$ satisfying $\text{Umb}(\mathbf{G}_{F-f_F}) = \mathbf{G}_{F-f_F}$, and there exists a homogeneous polynomial g_F of degree k_F satisfying $f_F - g_F \in \mathcal{A}_o^{(k_F+1)}$. Let \mathcal{A}_{oo}^2 be the subset of \mathcal{A}_o^2 such that each $F \in \mathcal{A}_{oo}^2$ satisfies $g_F \in \mathcal{P}_o^{k_F}$. In [3], we have mainly studied the behavior of the principal distributions around o on \mathbf{G}_F for $F \in \mathcal{A}_{oo}^2$ satisfying $\varpi_F \neq 0$ and proved $\text{ind}_o(\mathbf{G}_{g_F}) \leq \text{ind}_o(\mathbf{G}_F) \leq 1$ for $F \in \mathcal{A}_{oo}^2$.

The gradient vector field of a smooth function f is in a principal direction at a point of \mathbf{G}_f if and only if there exists a principal direction at the same point parallel to the xy -plane. A smooth surface S in \mathbf{R}^3 is called *parallel curved* if there exists a plane P in \mathbf{R}^3 such that at each point of S , there exists a principal direction parallel to P ; if S is parallel curved, then such a plane as P is called a *base plane* of S and the set of the base planes of S is denoted by \mathcal{B}_S . A plane, a cylinder and a round sphere are examples of parallel curved surfaces. More generally, a surface of revolution is also parallel curved. We see by Proposition 1.1 that a smooth function f satisfies $\varpi_f \equiv 0$ if and only if \mathbf{G}_f is a parallel curved surface such that the xy -plane is an element of $\mathcal{B}_{\mathbf{G}_f}$. A surface does not have to be entirely represented as the graph of a function so that the surface is parallel curved. The other of the purposes of this paper is to classify the connected, complete, real-analytic, embedded, parallel curved surfaces.

Let C_b, C_g be real-analytic, simple curves in \mathbf{R}^3 with the unique intersection $p_{(C_b, C_g)}$ and contained in planes P_b, P_g , respectively. Then a pair (C_b, C_g) is called *generating* if we may choose as P_g the plane normal to C_b at $p_{(C_b, C_g)}$; if (C_b, C_g) is generating, then C_b and C_g are called the *base curve* and the *generating curve* of (C_b, C_g) , respectively. In Section 4, we shall prove the following:

PROPOSITION 1.4. *Let (C_b, C_g) be a generating pair of which C_b (resp. C_g) is the base (resp. generating) curve. Then there exists a connected, real-analytic, parallel curved surface S_0 which contains a neighborhood of $p_{(C_b, C_g)}$ in $C_b \cup C_g$ and satisfies $P_b \in \mathcal{B}_{S_0}$. In addition, if $S_0^{(1)}$ and $S_0^{(2)}$ are such surfaces as S_0 , then $S_0^{(1)} \cap S_0^{(2)}$ is also such a surface.*

For a generating pair (C_b, C_g) , the maximum of such surfaces as S_0 in Proposition 1.4 is denoted by $S_{(C_b, C_g)}$. In Section 6, we shall prove the following:

THEOREM 1.5. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface. Then S is homeomorphic to a sphere, a plane, a cylinder, or to a torus. In addition,*

- (1) *if S is homeomorphic to a sphere, then S is a surface of revolution which crosses its axis of rotation at just two points;*
- (2) *if S is homeomorphic to a plane, then one of the following holds:*
 - (a) *S is a surface of revolution which crosses its axis of rotation at just one point,*
 - (b) *$S = S_{(C_b, C_g)}$ holds, where (C_b, C_g) is a generating pair each element of which is isometric to \mathbf{R} ;*

- (3) if S is homeomorphic to a cylinder, then $S = S_{(C_b, C_g)}$ holds, where (C_b, C_g) is a generating pair such that one of C_b and C_g is isometric to \mathbf{R} and the other a simple closed curve;
- (4) if S is homeomorphic to a torus, then $S = S_{(C_b, C_g)}$ holds, where (C_b, C_g) is a generating pair each element of which is isometric to a simple closed curve.

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2. Preliminaries

Let f be a smooth function of two variables x, y , and \mathbf{G}_f the graph of f . We set

$$E_f := 1 + p_f^2, \quad F_f := p_f q_f, \quad G_f := 1 + q_f^2,$$

$$L_f := \frac{r_f}{\sqrt{\det(\mathbf{I}_f)}}, \quad M_f := \frac{s_f}{\sqrt{\det(\mathbf{I}_f)}}, \quad N_f := \frac{t_f}{\sqrt{\det(\mathbf{I}_f)}},$$

where $\det(\mathbf{I}_f) := E_f G_f - F_f^2$. The Weingarten map of \mathbf{G}_f is a tensor field \mathbf{W}_f on \mathbf{G}_f of type $(1, 1)$ satisfying

$$\left[\mathbf{W}_f \left(\frac{\partial}{\partial x} \right), \mathbf{W}_f \left(\frac{\partial}{\partial y} \right) \right] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] \mathbf{W}_f,$$

where

$$\mathbf{W}_f := \begin{pmatrix} E_f & F_f \\ F_f & G_f \end{pmatrix}^{-1} \begin{pmatrix} L_f & M_f \\ M_f & N_f \end{pmatrix}.$$

A principal direction of \mathbf{G}_f at (x_0, y_0) is a one-dimensional eigenspace of $\mathbf{W}_f, (x_0, y_0)$. Let PD_f be a symmetric tensor field on \mathbf{G}_f of type $(0, 2)$ represented in terms of the coordinates (x, y) as

$$\text{PD}_f := \frac{1}{\sqrt{\det(\mathbf{I}_f)}} \{ A_f dx^2 + 2B_f dx dy + C_f dy^2 \},$$

where

$$A_f := E_f M_f - F_f L_f, \quad 2B_f := E_f N_f - G_f L_f, \quad C_f := F_f N_f - G_f M_f,$$

$$dx^2 := dx \otimes dx, \quad dx dy := \frac{1}{2}(dx \otimes dy + dy \otimes dx), \quad dy^2 := dy \otimes dy.$$

For vector fields V_1, V_2 on \mathbf{G}_f , the following holds:

$$\frac{1}{2} \sum_{\{i,j\}=\{1,2\}} V_i \wedge W_f(V_j) = \frac{\text{PD}_f(V_1, V_2)}{\sqrt{\det(\mathbf{I}_f)}} \left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \right).$$

Therefore we obtain

PROPOSITION 2.1. *A tangent vector v_0 to \mathbf{G}_f at (x_0, y_0) is in a principal direction if and only if $\text{PD}_f(x_0, y_0)(v_0, v_0) = 0$ holds.*

Let D_f, N_f be symmetric tensor fields on \mathbf{G}_f of type $(0, 2)$ represented in terms of the coordinates (x, y) as

$$D_f := s_f dx^2 + (t_f - r_f) dx dy - s_f dy^2,$$

$$N_f := (s_f p_f^2 - p_f q_f r_f) dx^2 + (t_f p_f^2 - r_f q_f^2) dx dy + (p_f q_f t_f - s_f q_f^2) dy^2.$$

Then we obtain $\det(\mathbf{I}_f)\text{PD}_f = D_f + N_f$. For a vector field V on \mathbf{G}_f , we set

$$\tilde{D}_f(V) := D_f(V, V), \quad \tilde{N}_f(V) := N_f(V, V),$$

$$\widetilde{\text{PD}}_f(V) := \text{PD}_f(V, V).$$

For $\phi \in \mathbf{R}$, we set

$$u_\phi := \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad U_\phi := \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y}.$$

Then we obtain

LEMMA 2.2. *For any $\phi \in \mathbf{R}$, the following hold:*

$$\tilde{D}_f(U_\phi) = \langle \text{Hess}_f u_\phi, u_{\phi+\pi/2} \rangle,$$

$$\tilde{N}_f(U_\phi) = \langle \text{grad}_f, u_\phi \rangle \langle \text{grad}_f^\perp, \text{Hess}_f u_\phi \rangle.$$

We set

$$\mathbf{Grad}_f := p_f \frac{\partial}{\partial x} + q_f \frac{\partial}{\partial y}, \quad \mathbf{Grad}_f^\perp := -q_f \frac{\partial}{\partial x} + p_f \frac{\partial}{\partial y}.$$

We shall prove

PROPOSITION 2.3. *At each point of \mathbf{G}_f , the following conditions are mutually equivalent:*

- (1) $\varpi_f = 0$;
- (2) $A_f + C_f = 0$;
- (3) \mathbf{Grad}_f is in a principal direction of \mathbf{G}_f ;
- (4) \mathbf{Grad}_f^\perp is in a principal direction of \mathbf{G}_f ;
- (5) there exists a principal direction parallel to the xy -plane.

PROOF. The following holds:

$$\varpi_f = (A_f + C_f)\sqrt{\det(\mathbf{I}_f)}.$$

Therefore we see that (1) is equivalent to (2). By Lemma 2.2, we obtain

$$\varpi_f = \widetilde{\mathbf{PD}}_f(\mathbf{Grad}_f) = -\det(\mathbf{I}_f)\widetilde{\mathbf{PD}}_f(\mathbf{Grad}_f^\perp).$$

Therefore we see by Proposition 2.1 that (1), (3) and (4) are mutually equivalent. It is easily seen that (4) is equivalent to (5). \square

From Proposition 2.3, we obtain Proposition 1.1.

3. Parallel Curved Surfaces

Let S be a connected, real-analytic, embedded, parallel curved surface and for $P \in \mathcal{B}_S$, let $\Xi_{S,P}$ be the subset of S such that for any $q \in \Xi_{S,P}$, the tangent plane $T_q(S)$ to S at q is not parallel to P . We see that $\Xi_{S,P}$ is an open set of S . If there exists an element P_0 of \mathcal{B}_S satisfying $\Xi_{S,P_0} = \emptyset$, then we see that S is part of a plane in \mathbf{R}^3 . In the following, suppose $\Xi_{S,P} \neq \emptyset$ for any $P \in \mathcal{B}_S$.

For $P_0 \in \mathcal{B}_S$ and for $q \in \Xi_{S,P_0}$, let $P_{P_0,q}^\perp$ be the plane in \mathbf{R}^3 through q perpendicular to P_0 and to $T_q(S)$, and $C_{P_0,q}^\perp$ the connected component of $P_{P_0,q}^\perp \cap \Xi_{S,P_0}$ containing q . We shall prove

PROPOSITION 3.1. *The plane $P_{P_0,q}^\perp$ is perpendicular to $T_p(S)$ for each $p \in C_{P_0,q}^\perp$.*

PROOF. For each $q \in \Xi_{S,P_0}$, there exist orthogonal coordinates (ξ, v, ζ) on \mathbf{R}^3 satisfying the following:

- (1) the point q corresponds to $(0, 0, 0)$;
- (2) the $\xi\zeta$ -plane $P_{\xi\zeta}$ is parallel to P_0 ;
- (3) the $v\zeta$ -plane $P_{v\zeta}$ is equal to $P_{P_0,q}^\perp$.

Then we see that the ξv -plane $P_{\xi v}$ is not perpendicular to $T_q(\Xi_{S,P_0})$. Therefore there

exist two positive numbers $\xi_0, v_0 > 0$ and a real-analytic function F^\perp defined on a neighborhood $U_{\xi_0, v_0} := (-\xi_0, \xi_0) \times (-v_0, v_0)$ of q in $P_{\xi v}$ such that the graph \mathbf{G}_{F^\perp} of F^\perp is a neighborhood of q in Ξ_{S, P_0} . The function F^\perp satisfies $F^\perp(0, 0) = (\partial F^\perp / \partial \xi)(0, 0) = 0$. We see that at each point of \mathbf{G}_{F^\perp} , the tangent vector $\partial / \partial \xi$ is in a principal direction. Therefore by Proposition 2.1, we obtain

$$\frac{\partial^2 F^\perp}{\partial \xi \partial v} \left\{ 1 + \left(\frac{\partial F^\perp}{\partial \xi} \right)^2 \right\} = \frac{\partial F^\perp}{\partial \xi} \frac{\partial F^\perp}{\partial v} \frac{\partial^2 F^\perp}{\partial \xi^2} \tag{3}$$

on U_{ξ_0, v_0} . We may represent F^\perp as

$$F^\perp(\xi, v) := \sum_{i, j=0}^{\infty} \alpha_{ij} \xi^i v^j,$$

where $\alpha_{ij} \in \mathbf{R}$ and where $\alpha_{00} = \alpha_{10} = 0$. Then at $(0, v) \in U_{\xi_0, v_0}$, we may rewrite (3) into

$$\begin{aligned} & \left(\sum_{j=0}^{\infty} (j+1) \alpha_{1j+1} v^j \right) \times \left(1 + \left(\sum_{j=0}^{\infty} \alpha_{1j} v^j \right)^2 \right) \\ &= 2 \left(\sum_{j=0}^{\infty} \alpha_{1j} v^j \right) \times \left(\sum_{j=0}^{\infty} (j+1) \alpha_{0j+1} v^j \right) \times \left(\sum_{j=0}^{\infty} \alpha_{2j} v^j \right). \end{aligned} \tag{4}$$

Since $\alpha_{10} = 0$, we obtain $\alpha_{11} = 0$. Generally, we see by (4) that if each element of $\{\alpha_{1k}\}_{k=0}^{j-1}$ for $j \in \mathbf{N}$ is equal to zero, then α_{1j} is also equal to zero. Therefore we obtain $\alpha_{1j} = 0$ for any $j \in \mathbf{N} \cup \{0\}$. Then for any $v \in (-v_0, v_0)$, $(\partial F^\perp / \partial \xi)(0, v) = 0$ holds. This implies that $T_{(0,v)}(\mathbf{G}_{F^\perp})$ is perpendicular to $P_{v\xi}$. Noticing $P_{v\xi} = P_{P_0, q}^\perp$, we obtain Proposition 3.1. □

COROLLARY 3.2. *The following hold:*

- (1) $C_{P_0, q}^\perp$ is a real-analytic curve;
- (2) A principal direction of S at each point of $C_{P_0, q}^\perp$ parallel to P_0 is perpendicular to $P_{P_0, q}^\perp$;
- (3) A nonzero tangent vector to $C_{P_0, q}^\perp$ at each point of $C_{P_0, q}^\perp$ is in a principal direction of S and not parallel to P_0 .

We shall prove

PROPOSITION 3.3. *Let F be an element of $\mathcal{A}_o^{(2)}$ satisfying $\varpi_F \equiv 0$. Then one of the following holds:*

- (1) \mathbf{G}_F is part of a surface of revolution such that o lies on an axis of rotation;
- (2) There exist a neighborhood V_o of o in the xy -plane P_{xy} and a positive number $\varepsilon_0 > 0$ and a real-analytic curve γ_ε in V_o for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ satisfying the following:
 - (a) $V_o = \bigcup_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)} \gamma_\varepsilon$
 - (b) for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and for any $(x, y) \in \gamma_\varepsilon$, $|F(x, y)| = |\varepsilon|$ holds,
 - (c) if a line l^\perp in P_{xy} is normal to γ_ε at a point of $l^\perp \cap \gamma_\varepsilon$ for some $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, then for any $\varepsilon' \in (-\varepsilon_0, \varepsilon_0)$, l^\perp is normal to $\gamma_{\varepsilon'}$ at any point of $l^\perp \cap \gamma_{\varepsilon'}$.

To prove Proposition 3.3, we need lemmas.

For any $\phi \in \mathbf{R}$, we set $u_\phi(x, y) := (\cos \phi)x + (\sin \phi)y$. For an element $F \in \mathcal{A}_o^{(2)}$, it is said that F is of one-variable if there exist a number $\phi_0 \in \mathbf{R}$ and a real-analytic function $f_{F,1}$ defined on a neighborhood of 0 in \mathbf{R} satisfying $F = f_{F,1} \circ u_{\phi_0}$ around $(0, 0)$, and it is said that F is radial if there exists a real-analytic function $f_{F,2}$ defined on a neighborhood of 0 in \mathbf{R} satisfying $F = f_{F,2} \circ r^2$ around $(0, 0)$, where $r(x, y) := \sqrt{x^2 + y^2}$. We shall prove

LEMMA 3.4. *Let g be an element of \mathcal{P}^k . Then $\varpi_g \equiv 0$ holds if and only if g is of one-variable or radial.*

PROOF. We see from equation (2) that $\varpi_g \equiv 0$ holds if and only if $\det(\text{Hess}_g) \equiv 0$ or $d\tilde{g}/d\theta \equiv 0$ holds.

If $d\tilde{g}/d\theta \equiv 0$, then g is radial (see [1]). Suppose $\det(\text{Hess}_g) \equiv 0$ and $d\tilde{g}/d\theta \not\equiv 0$, and let \tilde{g} attain a nonzero extremum at $\theta_0 \in \mathbf{R}$. If we represent g as

$$g := \sum_{i=0}^k a_i u_{\theta_0}(x, y)^{k-i} u_{\theta_0+\pi/2}(x, y)^i,$$

then by $(d\tilde{g}/d\theta)(\theta_0) = 0$, we obtain $a_1 = 0$. In addition, by $\det(\text{Hess}_g) \equiv 0$, we obtain $a_i = 0$ for each $i \in \{2, \dots, k\}$. Therefore we see that g is of one-variable.

If g is of one-variable (resp. radial), then by direct computation, we obtain $\det(\text{Hess}_g) \equiv 0$ (resp. $d\tilde{g}/d\theta \equiv 0$).

Hence we have proved Lemma 3.4. □

For integers $k_1, k_2, k_3 \geq 2$, let g_1, g_2, g_3 be elements of $\mathcal{P}^{k_1}, \mathcal{P}^{k_2}, \mathcal{P}^{k_3}$, respectively. We set

$$t_{g_1, g_2, g_3} := \langle \text{Hess}_{g_1} \text{grad}_{g_2}, \text{grad}_{g_3}^\perp \rangle,$$

$$T_{g_1, g_2, g_3} := \sum_{\{j_1, j_2, j_3\}=\{1, 2, 3\}} t_{g_{j_1}, g_{j_2}, g_{j_3}}.$$

We shall prove

LEMMA 3.5. *Suppose $k_3 \geq k_2 \geq k_1$ and that g_1 and g_2 are radial. Then g_3 is also radial if and only if $T_{g_1, g_2, g_3} \equiv 0$ holds.*

PROOF. If g_1 and g_2 are radial, then k_1 and k_2 are even. If we set $g_j = r^{k_j}$ for $j = 1, 2$, then we obtain

$$t_{g_{j_1}, g_{j_2}, g_3} = -k_1 k_2 (k_{j_1} - 1) r^{k_1 + k_2 - 4} (xq_{g_3} - yp_{g_3}),$$

$$t_{g_{j_1}, g_3, g_{j_2}} = k_1 k_2 r^{k_1 + k_2 - 4} (xq_{g_3} - yp_{g_3}),$$

$$t_{g_3, g_{j_1}, g_{j_2}} = k_1 k_2 (k_3 - 1) r^{k_1 + k_2 - 4} (xq_{g_3} - yp_{g_3}),$$

where $\{j_1, j_2\} = \{1, 2\}$. Therefore we obtain

$$T_{g_1, g_2, g_3} = k_1 k_2 (2k_3 - k_1 - k_2 + 2) r^{k_1 + k_2 - 4} (xq_{g_3} - yp_{g_3}).$$

Since $k_3 \geq k_2 \geq k_1$, we see that $T_{g_1, g_2, g_3} \equiv 0$ is equivalent to $xq_{g_3} \equiv yp_{g_3}$. In addition, noticing that $xq_{g_3} \equiv yp_{g_3}$ is equivalent to $d\tilde{g}_3/d\theta \equiv 0$, we see that $T_{g_1, g_2, g_3} \equiv 0$ holds if and only if g_3 is radial. Hence we have proved Lemma 3.5. \square

PROOF OF PROPOSITION 3.3. We may represent $F \in \mathcal{A}_o^{(2)}$ as $F := \sum_{i \geq 2} F^{(i)}$, where $F^{(i)} \in \mathcal{P}^i$. We suppose $F \neq 0$ and set

$$I_F := \{i_0 \in \mathbb{N}; F^{(i_0)} \neq 0\}, \quad m_F := \min I_F.$$

Then we may represent ϖ_F as

$$\varpi_F = \sum_{j_1, j_2, j_3 \in I_F} t_{F^{(j_1)}, F^{(j_2)}, F^{(j_3)}},$$

and we obtain $\varpi_F^{(3m_F - 4)} = \varpi_{F^{(m_F)}}$. Therefore by Lemma 3.4, we see that if $F \in \mathcal{A}_o^{(2)}$ satisfies $\varpi_F \equiv 0$, then $F^{(m_F)}$ is of one-variable or radial. If $I_F = \{m_F\}$, then we obtain Proposition 3.3.

Suppose $I_F \neq \{m_F\}$ and that I_F is a finite set. Then set $n := \#I_F$ and let i_1, \dots, i_n be the integers satisfying $i_1 < \dots < i_n$ and $I_F = \{i_j\}_{j=1}^n$. If $F^{(i_1)}, \dots, F^{(i_n)}$

are radial for $j \in \{1, \dots, n-1\}$, then we see by Lemma 3.5 that $F^{(j+1)}$ is also radial. Therefore we see that if $F^{(m_F)}$ is radial, then F is also radial. If I_F is an infinite set, then we obtain the same result. Hence we see that if $F^{(m_F)}$ is radial, then G_F is part of a surface of revolution such that o lies on an axis of rotation.

Suppose $I_F \neq \{m_F\}$ and that $F^{(m_F)}$ is of one-variable. Then we may suppose $F^{(m_F)} = x^{m_F}$. For each $q \in G_F$, let Π_q^\perp be the set of the planes in R^3 through q such that each $P^\perp \in \Pi_q^\perp$ is perpendicular to P_{xy} and to $T_p(G_F)$ for any point p of the connected component of $P^\perp \cap G_F$ containing q . By Proposition 3.1, we obtain $\sharp \Pi_q^\perp = 1$ for any $q \in \Xi_{G_F, P_{xy}}$. In addition, we shall prove

LEMMA 3.6. *If F is not of one-variable, then the following hold:*

- (1) *For each $q \in G_F$, $\sharp \Pi_q^\perp = 1$ holds;*
- (2) *the xz -plane P_{xz} is the only one element of Π_o^\perp .*

PROOF. By $\varpi_F \equiv 0$, we obtain $q_{F^{(i)}}(x, 0) = 0$ for any $x \in R$ and for any $i \in I_F$. Therefore we obtain $P_{xz} \in \Pi_o^\perp$ and $P_{xz} = P_{P_{xy}, q}^\perp$ for any $q \in P_{xz} \cap \Xi_{G_F, P_{xy}}$. We easily see that for any $\phi \in (-\pi/2, \pi/2) \setminus \{0\}$, the plane perpendicular to P_{xy} and determined by u_ϕ is not an element of Π_o^\perp . Therefore we see that for each $q \in G_F$, $\sharp \Pi_q^\perp = 1$ or $= 2$ holds and that if $\sharp \Pi_q^\perp = 2$, then the two elements of Π_q^\perp are perpendicular to each other. Suppose that there exists a point $q_0 \in G_F$ satisfying $\sharp \Pi_{q_0}^\perp = 2$. Then we see that for any $q \in G_F$, an element of Π_q^\perp is parallel or perpendicular to P_{xz} . Therefore by Proposition 2.3 and by Corollary 3.2, we see that each of $\partial/\partial x$ and $\partial/\partial y$ is in a principal direction at each point of G_F and that F is of one-variable. Therefore we obtain $\sharp \Pi_q^\perp = 1$ for any $q \in G_F$. Particularly, $\Pi_o^\perp = \{P_{xz}\}$ holds and we have proved Lemma 3.6. □

Suppose that $F^{(m_F)}$ is of one-variable and that F is not of one-variable. Then for each $q \in G_F$, we denote by P_q^\perp the only one element of Π_q^\perp . Then we may find a positive number $y_0 > 0$ and an open line segment l_y in P_{xy} through $(0, y)$ for each $y \in (-y_0, y_0)$ satisfying the following:

- (1) $l_y \subset P_{(0, y, F(0, y))}^\perp$ holds for any $y \in (-y_0, y_0)$;
- (2) $\tilde{V}_o := \bigcup_{y \in (-y_0, y_0)} l_y$ is a neighborhood of o in P_{xy} .

In addition, we may find a real-analytic vector field on \tilde{V}_o nonzero and tangent to l_y for some $y \in (-y_0, y_0)$ at each point of \tilde{V}_o . Therefore we may find a neighborhood V_o of o in P_{xy} and a positive number $\varepsilon_0 > 0$ and a real-analytic curve γ_ε in V_o for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ satisfying (a)~(c) of (2) of Proposition 3.3. If F is of one-variable, then we may easily obtain the same result. Hence we have proved Proposition 3.3. □

4. Generating Pairs

Let P be a plane in \mathbf{R}^3 and Γ_P the set of the real-analytic, simple curves in P , and for $C \in \Gamma_P$ and for a point $p \in C$, let L_p^\perp be the line in P normal to C at p . Then for $C, \tilde{C} \in \Gamma_P$, we write $C \sim \tilde{C}$ if and only if there exists a continuous bijective map $\alpha_{C, \tilde{C}}$ from C onto \tilde{C} satisfying $L_{\alpha_{C, \tilde{C}}(p)}^\perp = L_p^\perp$ for any $p \in C$. It is seen that \sim is an equivalence relation in Γ_P . We denote by Γ_C the equivalence class of $C \in \Gamma_P$, and by $\Sigma(C)$ the connected component of the set $\bigcup_{C' \in \Gamma_C} C'$ containing C . We immediately obtain

LEMMA 4.1. *For $C \in \Gamma_P$ and for each $p \in C$, there exists a neighborhood O_p of p in C such that $\Sigma(O_p)$ is a domain of P .*

For each plane P in \mathbf{R}^3 , we denote by Proj_P the map from \mathbf{R}^3 onto P such that if a line L is perpendicular to P , then $\text{Proj}_P(L)$ consists of the only one point of $P \cap L$. Then by Corollary 3.2 and by Proposition 3.3, we obtain

PROPOSITION 4.2. *Let S be a connected, real-analytic, embedded, parallel curved surface and P_0 an element of \mathcal{B}_S . Then for any $q \in S$, just one of the following holds:*

- (1) *S is part of a surface of revolution such that the line through q perpendicular to P_0 is an axis of rotation of S ;*
- (2) *There exists a neighborhood U_q of q in S such that if P_1 and P_2 are base planes of S parallel to P_0 and satisfying $P_i \cap U_q \neq \emptyset$ for $i = 1, 2$, then each connected component C_i of $\text{Proj}_{P_0}(P_i \cap U_q)$ is an element of Γ_{P_0} satisfying $C_1 \sim C_2$.*

COROLLARY 4.3. *Let S be a real-analytic, embedded, parallel curved surface and P_0 an element of \mathcal{B}_S and q a point of S for which (2) of Proposition 4.2 holds. Then there exists a generating pair (C_b, C_g) of which C_b (resp. C_g) is the base (resp. generating) curve and which satisfies $q = p_{(C_b, C_g)}$, $C_b, C_g \subset S$ and that P_b is parallel to P_0 .*

PROOF OF PROPOSITION 1.4. Let (C_b, C_g) be a generating pair and P_b, P_g planes satisfying $C_b \subset P_b$, $C_g \subset P_g$ and that P_g is normal to C_b at $p_{(C_b, C_g)}$, and P^\perp the plane through $p_{(C_b, C_g)}$ perpendicular to P_b and to P_g . If $C_g \subset P_b$, then we see that a connected, real-analytic, parallel curved surface S_0 which contains a neighborhood of $p_{(C_b, C_g)}$ in $C_b \cup C_g$ and satisfies $P_b \in \mathcal{B}_{S_0}$ is part of P_b . In

the following, suppose $C_g \not\subseteq P_b$. Then by Lemma 4.1, we see that there exist neighborhoods O_b, O_g of $p_{(C_b, C_g)}$ in C_b, C_g , respectively satisfying $\text{Proj}_{P_b}(O_g) \subset \Sigma(O_b)$ and the condition that Proj_{P^\perp} embeds each connected component of $O_g \setminus \{p_{(C_b, C_g)}\}$ into P^\perp . For O_b, O_g , there exists a real-analytic surface S satisfying $O_b, O_g \subset S$ and the condition that if P is a plane parallel to P_b and satisfying $P \cap O_g \neq \emptyset$, then each connected component of $\text{Proj}_{P_b}(P \cap S)$ is an element of Γ_{O_b} . The minimum of such surfaces as S is denoted by S_{O_b, O_g} . Then we see that P_b is not parallel to $T_q(S_{O_b, O_g})$ for any $q \in S_{O_b, O_g} \setminus O_b$. For each $q \in S_{O_b, O_g} \setminus O_b$, let (ξ, v, ζ) be orthogonal coordinates on \mathbf{R}^3 satisfying the following:

- (1) the point q corresponds to $(0, 0, 0)$;
- (2) $P_{\xi\zeta}$ is parallel to P_b ;
- (3) $P_{v\zeta}$ is perpendicular to P_b and to $T_q(S_{O_b, O_g})$.

Then there exist two positive numbers $\xi_0, v_0 > 0$ and a real-analytic function F^\perp defined on a neighborhood $U_{\xi_0, v_0} := (-\xi_0, \xi_0) \times (-v_0, v_0)$ of q in $P_{\xi v}$ such that the graph \mathbf{G}_{F^\perp} of F^\perp is a neighborhood of q in $S_{O_b, O_g} \setminus O_b$. Then we obtain

$$\frac{\partial F^\perp}{\partial \xi}(0, v) = \frac{\partial^2 F^\perp}{\partial \xi \partial v}(0, v) = 0$$

for any $v \in (-v_0, v_0)$. Therefore by Proposition 2.1, we see that each of $\partial/\partial \xi$ and $\partial/\partial v$ is in a principal direction at $(0, v, F^\perp(0, v))$ for any $v \in (-v_0, v_0)$. Since $\partial/\partial \xi$ is parallel to P_b , we see that $S_{O_b, O_g} \setminus O_b$ is a parallel curved surface satisfying $P_b \in \mathcal{B}_{S_{O_b, O_g} \setminus O_b}$. Then we see that a tangent vector to O_b at each point of O_b is in a principal direction of S_{O_b, O_g} . Therefore $S_0 := S_{O_b, O_g}$ is a parallel curved surface which contains a neighborhood $O_b \cup O_g$ of $p_{(C_b, C_g)}$ in $C_b \cup C_g$ and satisfies $P_b \in \mathcal{B}_{S_0}$. It is clear that if $S_0^{(1)}$ and $S_0^{(2)}$ are parallel curved surfaces which contain a neighborhood of $p_{(C_b, C_g)}$ in $C_b \cup C_g$ and satisfy $P_b \in \mathcal{B}_{S_0^{(i)}}$ for $i = 1, 2$, then $S_0^{(1)} \cap S_0^{(2)}$ is also such a surface as $S_0^{(i)}$. Hence we have proved Proposition 1.4. □

5. Proof of Theorem 1.2 and Theorem 1.3

Suppose that $F \in \mathcal{A}_0^{(2)}$ satisfies $\varpi_F \equiv 0$ and (1) of Proposition 3.3. Then F is radial. Then the following hold:

$$\text{grad}_F = 2 \frac{df_{F,2}}{d\rho} \circ r^2 \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\text{Hess}_F = 2 \frac{df_{F,2}}{d\rho} \circ r^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4 \frac{d^2 f_{F,2}}{d\rho^2} \circ r^2 \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}.$$

Therefore by Lemma 2.2, we obtain

$$\det(\mathbf{I}_F)\widetilde{\mathbf{PD}}_F(\mathbf{U}_\phi) = 4 \left\{ -\frac{d^2 f_{F,2}}{d\rho^2} \circ r^2 + 2 \left[\frac{df_{F,2}}{d\rho} \circ r^2 \right]^3 \right\} u_\phi(x, y) u_\phi(-y, x).$$

This implies that the position vector field $x\partial/\partial x + y\partial/\partial y$ is in a principal direction at any $(x, y) \in \mathbf{G}_F$. If $d^2 f_{F,2}/d\rho^2 \not\equiv 2(df_{F,2}/d\rho)^3$, then we obtain $F \in \mathcal{A}_o^2$ and by formula (1), we obtain $\text{ind}_o(\mathbf{G}_F) = 1$. If $d^2 f_{F,2}/d\rho^2 \equiv 2(df_{F,2}/d\rho)^3$, then $f_{F,2} \equiv 0$ holds or there exists a positive number $a_F > 0$ satisfying $f_{F,2} = \sqrt{a_F} - \sqrt{a_F - \rho}$ or $= -\sqrt{a_F} + \sqrt{a_F - \rho}$. Therefore we see that \mathbf{G}_F is part of a plane or a round sphere.

Suppose that $F \in \mathcal{A}_o^{(2)}$ satisfies $\varpi_F \equiv 0$ and (2) of Proposition 3.3. Then we see that there exist a neighborhood V_o of o in P_{xy} and a real-analytic curve γ_o in V_o satisfying $\gamma_o = \{(x, y) \in V_o; F(x, y) = 0\}$. For each $F_0 \in \mathcal{A}_o^{(2)}$ and for each $q_0 := (x_0, y_0) \in \gamma_o$, we set $f_{F_0, q_0}(x, y) := F_0(x - x_0, y - y_0)$. The function f_{F_0, q_0} is defined on a neighborhood of q_0 in P_{xy} . We shall prove

LEMMA 5.1. *For each $q_0 \in \gamma_o$, there exists an element F_{q_0} of $\mathcal{A}_o^{(2)}$ satisfying $\mathbf{G}_{f_{F_{q_0}, q_0}} \subset \mathbf{G}_F$ and $m_{F_{q_0}} = m_F$.*

PROOF. There exist positive numbers $u_0, v_0 > 0$ and a real-analytic map Φ from $U_{u_0, v_0} := (-u_0, u_0) \times (-v_0, v_0)$ into V_o satisfying the following:

- (1) The Jacobian of Φ is nonsingular at each point of U_{u_0, v_0} ;
- (2) for any $u' \in (-u_0, u_0)$, Φ maps the open line segment $\{u = u'\}$ in U_{u_0, v_0} into γ_ε for some $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$;
- (3) for any $v' \in (-v_0, v_0)$, Φ maps the open line segment $\{v = v'\}$ in U_{u_0, v_0} into l_y for some $y \in (-y_0, y_0)$.

Then the function $F \circ \Phi$ on U_{u_0, v_0} is of one-variable. This implies Lemma 5.1. □

Suppose $m_F = 2$. Then $\text{Umb}(\mathbf{G}_F) \cap \gamma_o = \emptyset$ holds. Therefore by Corollary 3.2, Proposition 3.3 and by Lemma 3.6, we may find a neighborhood U_o of $(0, 0)$ in \mathbf{R}^2 and a real-analytic curve C_o in U_o satisfying (a), (c) of (2) of Theorem 1.3 and $\text{Umb}(\mathbf{G}_{F|_{C_o}}) = \emptyset$.

Suppose $m_F \geq 3$. Then $\gamma_o \subset \text{Umb}(\mathbf{G}_F)$ holds. There exist real-analytic functions θ_F, c_F on γ_o such that an element $\tilde{F}_{q_0} \in \mathcal{A}_o^{(2)}$ defined for each $q_0 \in \gamma_o$ by

$$\tilde{F}_{q_0}(x, y) := F_{q_0}(x \cos \theta_F(q_0) - y \sin \theta_F(q_0), x \sin \theta_F(q_0) + y \cos \theta_F(q_0))$$

satisfies $m_{\tilde{F}_{q_0}} = m_F$ and $\tilde{F}_{q_0}^{(m_F)} = c_F(q_0)x^{m_F}$. We may suppose that there exist a neighborhood V'_o of o in V_o and a neighborhood γ'_o of o in γ_o such that for any $(x, y) \in V'_o$ and for any $q_0 \in \gamma'_o$, $\Psi_F(x, y, q_0) := \tilde{F}_{q_0}(x, y)$ makes sense. Then we see that the function Ψ_F is real-analytic on $V'_o \times \gamma'_o$. Therefore we may find a continuous function \tilde{x} on γ'_o satisfying $\tilde{x}(q_0) > 0$ and $(x, 0, \tilde{F}_{q_0}(x, 0)) \notin \text{Umb}(\mathbf{G}_{\tilde{F}_{q_0}})$ for any $x \in (-\tilde{x}(q_0), \tilde{x}(q_0)) \setminus \{0\}$ and for any $q_0 \in \gamma'_o$. Then by Corollary 3.2, Proposition 3.3 and by Lemma 3.6, we may find a neighborhood U_o of $(0, 0)$ in \mathbf{R}^2 and a real-analytic curve C_0 in U_o satisfying (a), (c) of (2) of Theorem 1.3 and $C_0 = \text{Umb}(\mathbf{G}_{F|_{U_o}})$. Hence we have proved Theorem 1.2 and Theorem 1.3.

6. Classification

In this section, let S be a connected, complete, real-analytic, embedded, parallel curved surface.

Suppose that there exists an element P_0 of \mathcal{B}_S satisfying $\Xi_{S, P_0} = S$. Then for each $q \in S$, we see by Corollary 3.2 that $C_{P_0, q}^\perp$ is isometric to \mathbf{R} . There exists the element $P_{P_0, q} \in \mathcal{B}_S$ satisfying $q \in P_{P_0, q}$ and the condition that $P_{P_0, q}$ is parallel to P_0 . Then by Proposition 4.2, we see that $P_{P_0, q} \cap S$ is a real-analytic curve isometric to \mathbf{R} or to a simple closed curve. Therefore we obtain

PROPOSITION 6.1. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface satisfying $\Xi_{S, P_0} = S$ for some $P_0 \in \mathcal{B}_S$. Then there exists a generating pair (C_b, C_g) of which C_b (resp. C_g) is the base (resp. generating) curve and which satisfies the following:*

- (1) P_b is parallel to P_0 ;
- (2) C_b is isometric to \mathbf{R} or to a simple closed curve;
- (3) C_g is isometric to \mathbf{R} ;
- (4) $S = S_{(C_b, C_g)}$.

Then S is homeomorphic to a plane or to a cylinder.

Suppose $\Xi_{S, P_0} \neq S$ and $\Xi_{S, P_0} \neq \emptyset$ for $P_0 \in \mathcal{B}_S$. Then for $P_0 \in \mathcal{B}_S$ and for $q \in \Xi_{S, P_0}$, we see by Corollary 3.2 and by Proposition 4.2 that the connected component of $P_{P_0, q}^\perp \cap S$ containing q is a real-analytic curve isometric to \mathbf{R} or to a simple closed curve. There exists the element $P_{P_0, q} \in \mathcal{B}_S$ satisfying $q \in P_{P_0, q}$ and that $P_{P_0, q}$ is parallel to P_0 . Then by Proposition 4.2, we see that the connected

component of $P_{P_0, q} \cap S$ containing q is a real-analytic curve isometric to \mathbf{R} or to a simple closed curve. We shall prove

LEMMA 6.2. *Let P_0 be an element of \mathcal{B}_S and q_0 a point of Ξ_{S, P_0} such that some connected component of $P_{P_0, q_0} \cap S$ shares plural points with some connected component of $P_{P_0, q_0}^\perp \cap S$. Then S is a surface of revolution such that a line perpendicular to P_0 is an axis of rotation of S .*

PROOF. Let $O_{q_0}, O_{q_0}^\perp$ be domains in $P_{P_0, q_0} \cap S, P_{P_0, q_0}^\perp \cap S$, respectively satisfying $O_{q_0} \cap O_{q_0}^\perp = \emptyset$ and $\#(\bar{O}_{q_0} \cap \bar{O}_{q_0}^\perp) = 2$, and q_1, q_2 two points of S satisfying $\bar{O}_{q_0} \cap \bar{O}_{q_0}^\perp = \{q_1, q_2\}$. Then by Proposition 3.1, we see that there exists the only one point p_0 of $S \setminus \Xi_{S, P_0}$ satisfying $P_{P_0, q}^\perp \cap O_{q_0}^\perp = \{p_0\}$ for any $q \in O_{q_0}$. By Proposition 4.2, we see that S is a surface of revolution such that the line through p_0 perpendicular to P_0 is an axis of rotation of S . Hence we have proved Lemma 6.2. \square

By Lemma 6.2, we obtain

PROPOSITION 6.3. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface satisfying $\Xi_{S, P_0} \neq S$ and $\Xi_{S, P_0} \neq \emptyset$ for any $P_0 \in \mathcal{B}_S$. Then one of the following holds:*

- (1) *S is a surface of revolution such that the number of the intersections of S with its axis of rotation is equal to one or two, and then S is homeomorphic to a plane or to a sphere,*
- (2) *There exists a generating pair (C_b, C_g) of which C_b (resp. C_g) is the base (resp. generating) curve and which satisfies the following:*
 - (a) *each of C_b and C_g is isometric to \mathbf{R} or to a simple closed curve,*
 - (b) *$S = S_{(C_b, C_g)}$,**and then S is homeomorphic to a plane, a cylinder or to a torus.*

Using Proposition 6.1 and Proposition 6.3, we obtain Theorem 1.5.

REMARK. If C_b is a circumference in each of Proposition 6.1 and Proposition 6.3, then S is a surface of revolution and its axis of rotation is perpendicular to P_b .

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