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INTRINSIC AND EXTRINSIC STRUCTURES OF LAGRANGIAN SURFACES IN COMPLEX SPACE FORMS

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Abstract. Lagrangian H-umbilical submanifolds introduced in [1, 2] can be regarded as the simplest Lagrangian submanifolds in Kaehler manifolds next to totally geodesic ones. It was proved in [1] that Lagrangian H-umbilical submanifolds of dimension ≥ 3 in complex Euclidean spaces are complex extensors, Lagrangian pseudo-spheres, and flat Lagrangian H-umbilical submanifolds. Lagrangian H-umbilical submanifolds of dimension ≥ 3 in non-flat complex space forms are classified in [2]. In this paper we deal with the remaining case; namely, non-totally geodesic Lagrangian H-umbilical surfaces in complex space forms. Such Lagrangian surfaces are characterized by a very simple property; namely, JH is an eigenvector of the shape operator A_H, where H is the mean curvature vector field. The main purpose of this paper is to determine both the intrinsic and the extrinsic structures of Lagrangian H-umbilical surfaces.

1. Introduction

Let $f : M \to \tilde{M}^m$ be an isometric immersion of a Riemannian n-manifold M into a Kaehler manifold $\tilde{M}^m$ of complex dimension m. The submanifold M is called *totally real* (or *isotropic* in symplectic geometry) if the almost complex structure J of $\tilde{M}^m$ carries each tangent space of M into its corresponding normal space [5]. A totally real submanifold M of $\tilde{M}^m$ is called *Lagrangian* if $n = m$. From the symplectic point of view, a local classification of Lagrangian submanifolds is trivial, using local Darboux coordinates [9]. However, from the Riemannian point of view, Lagrangian submanifolds are far from trivial. In this

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respect, there exist a number of very interesting results, both local and global (cf. [8]). For instance it was proved in [5, 7] that a minimal Lagrangian submanifold with constant sectional curvature in a complex space form has to be totally geodesic or flat.

Totally umbilical submanifolds, if they exist, are the simplest submanifolds next to totally geodesic submanifolds in a Riemannian manifold. However, it was proved in [6] that a complex space form of complex dimension ≥ 2 admits no totally umbilical Lagrangian submanifolds except the totally geodesic ones. In views of above facts the author introduced in [1, 2] the notion of Lagrangian H-umbilical submanifolds.

According to [1, 2] a Lagrangian H-umbilical submanifold of Kaehler manifold $\tilde{M}^n$ is a non-totally geodesic Lagrangian submanifold whose second fundamental form takes the simple form:

$$\begin{aligned} &h(e_1,e_1) = \lambda Je_1, \quad h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu Je_1, \\ &h(e_1,e_j) = \mu Je_j, \quad h(e_j,e_k) = 0, \quad j \neq k, \quad j,k = 2,\dots,n \end{aligned} \tag{1.1}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field $e_1,\dots,e_n$.

A Lagrangian submanifold with nonzero mean curvature vector H is Lagrangian H-umbilical if and only if (a) JH is an eigenvector of the shape operator A_H and (b) the restriction of A_H to $(JH)^{\perp}$ is proportional to the identity map.

It is important to point out that condition (b) follows from condition (a) automatically for Lagrangian surfaces (cf. Lemma 3.1).

Lagrangian H-umbilical submanifolds M of dimension ≥ 3 in a complex space form of constant holomorphic sectional curvature $4c$ have an important property; namely, the integral curves of JH are geodesics of M whenever $H \neq 0$, unless M is a real space form of constant sectional curvature c. This important property does not hold for 2-dimensional Lagrangian H-umbilical submanifolds in general. Using this important property the author was able to classify in [1, 2] Lagrangian H-umbilical submanifolds of dimension ≥ 3 in complex space forms. In particular, he proved that, except the flat ones, Lagrangian H-umbilical submanifolds in C^n with $n \geq 3$ are either Lagrangian pseudo-spheres or complex extensors. Lagrangian H-umbilical submanifolds of dimension ≥ 3 in non-flat complex space forms were determined in [2] via Legendre curves and Hopf's fibration (see [4] for Lagrangian submanifolds of constant curvature c). The explicit description of flat Lagrangian H-umbilical submanifolds in C^n with $n \geq 2$ were established in [3].

In this paper we deal with the remaining case; namely, Lagrangian H-umbilical surfaces in complex space forms. Because the integral curves of JH are not longer geodesics in general, the method utilized in [1, 2] does not apply to this case.

We point out in section 3 that, except totally geodesic ones, minimal Lagrangian surfaces in any Kaehler surface are Lagrangian H-umbilical automatically. The main purpose of section 3 is to establish a general existence and uniqueness theorem for Lagrangian H-umbilical surfaces in complex space forms. As a by-product, we are able to determine the intrinsic and the extrinsic structures of minimal Lagrangian surfaces in complex space forms. The intrinsic and the extrinsic structures of Lagrangian H-umbilical surfaces with constant Gauss curvature or with constant mean curvature are established in sections 4 and 5, respectively. In section 6 we determine Lagrangian H-umbilical surfaces such that the functions λ and μ given in (1.1) are linearly dependent. The Lagrangian surfaces investigated in sections 4, 5 and 6 share the property that $e_2\mu = 0$. The last section determines completely the intrinsic and the extrinsic structures of Lagrangian H-umbilical surfaces satisfying $e_1\mu = 0$.

2. Preliminaries

Let $\tilde{M}^n(4c)$ denote a complete simply-connected Kaehler n-manifold with constant holomorphic sectional curvature $4c$. Let M be a Lagrangian submanifold in $\tilde{M}^n(4c)$. We denote the Levi-Civita connections of M and $\tilde{M}^n(4c)$ by ∇ and $\tilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for tangent vector fields X and Y and normal vector field ξ, where D is the connection on the normal bundle. The second fundamental form h is related to the shape operator A_ξ by $\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle$. The mean curvature vector H of M in $\tilde{M}^2(4c)$ is defined by $H = 1/n$ trace h, where $n = \dim M$. We put $H^2 = \langle H, H\rangle$ which is called the squared mean curvature.

For Lagrangian submanifolds we have [5]

$$D_X JY = J\nabla_X Y, \tag{2.3}$$

$$\langle h(X, Y), JZ\rangle = \langle h(Y, Z), JX\rangle = \langle h(Z, X), JY\rangle. \tag{2.4}$$

If we denote the curvature tensor of ∇ by R, then the equations of Gauss, Codazzi and Ricci are given respectively by

$$\langle R(X,Y)Z,W\rangle = \langle A_{h(Y,Z)}X,W\rangle - \langle A_{h(X,Z)}Y,W\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{2.5}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.6}$$

$$\langle R^D(X,Y)JZ,JW\rangle = \langle [A_{JZ},A_{JW}]X,Y\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{2.7}$$

where X, Y, Z, W are vector fields tangent to M and ∇h is defined by

$$(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{2.8}$$

We need the following existence and uniqueness theorems for Lagrangian immersions (cf. [1, 4]).

THEOREM 2.1. *Let (M^n,g) be a simply-connected Riemannian n-manifold. If σ is a symmetric bilinear vector-valued form on M satisfying*

(1) *$g(\sigma(X,Y),Z)$ is totally symmetric,*

(2) *$(\nabla\sigma)(X,Y,Z) = \nabla_X\sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z)$ is totally symmetric,*

(3) *$R(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y) + \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y)$,*

then there exists a Lagrangian isometric immersion $L:(M,g)\to\tilde M^n(4c)$ whose second fundamental form h is given by $h(X,Y) = J\sigma(X,Y)$.

THEOREM 2.2. *Let $L_1, L_2 : M \to \tilde M^n(4c)$ be two Lagrangian isometric immersions of a Riemannian n-manifold M with second fundamental forms h^1 and h^2, respectively. If*

$$\langle h^1(X,Y), JL_{1*}Z\rangle = \langle h^2(X,Y), JL_{2*}Z\rangle,$$

for all vector fields X Y, Z tangent to M, then there exists an isometry ϕ of $\tilde M^n(4c)$ such that $L_1 = L_2\circ\phi$.

3. Lagrangian H-umbilical surfaces in complex space forms

We provide some lemmas for later use.

LEMMA 3.1. *Let $L: M\to\tilde M^2$ be a Lagrangian surface in a Kaehler surface without totally geodesic points. We have*

(1) *L is Lagrangian H-umbilical if and only if JH is an eigenvector of the shape operator A_H.*

(2) *If L is minimal, then L is a Lagrangian H-umbilical surface satisfying* (1.1) *with $\lambda = -\mu$.*

PROOF. (1) follows from (2) and the definition of Lagrangian H-umbilical surfaces (cf. section 1).

(2) Let M be a minimal Lagrangian surface without totally geodesic points in a Kaehler surface. We define a function γ_p by

$$\gamma_p : UM_p \to \boldsymbol{R} : v \mapsto \gamma_p(v) = \langle h(v,v), Jv\rangle, \tag{3.1}$$

where $UM_p = \{v \in T_pM : \langle v, v\rangle = 1\}$. Since UM_p is a compact set, there exists a vector v in UM_p such that γ_p attains an absolute minimum at v. Since p is not totally geodesic, it follows from (2.4) that $\gamma_p \neq 0$. By linearity, we have $\gamma_p(v) < 0$. Because γ_p attains an absolute minimum at v, it follows from (2.4) that $\langle h(v,v), Jw\rangle = 0$ for all w orthogonal to v. So, using (2.4), v is an eigenvector of the symmetric operator A_{Jv}. By choosing an orthonormal basis $\{e_1, e_2\}$ of T_pM with $e_1 = v$, we obtain

$$h(e_1,e_1) = \lambda Je_1, \quad h(e_1,e_2) = -\lambda Je_2, \quad h(e_2,e_2) = -\lambda Je_1$$

for some λ. Thus M is a Lagrangian H-umbilical surface with $\mu = -\lambda$. □

LEMMA 3.2. *Except totally geodesic ones, a Lagrangian H-umbilical surface of constant Gauss curvature c in a complex space form $\tilde{M}^2(4c)$ is a Lagrangian H-umbilical surface satisfying* (1.1) *with $\mu = 0$ or with $\lambda = \mu$.*

Conversely, every Lagrangian H-umbilical surface in $\tilde{M}^2(4c)$ satisfying (1.1) *with $\mu = 0$ or with $\lambda = \mu$ has constant Gauss curvature c.*

PROOF. Let M be a Lagrangian H-umbilical surface in $\tilde{M}^2(4c)$. Then by (2.3) and (2.7) we have

$$\langle R(X,Y)Z,W\rangle = \langle [A_{JZ}, A_{JW}]X, Y\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{3.2}$$

for X, Y, Z, W tangent to M. If M has constant Gauss curvature c, (3.2) implies that the shape operators of M commute. Thus, at each point $p \in M$ there exists an orthonormal basis e_1, e_2 such that A_{Je_1}, A_{Je_2} are simultaneously diagonalizable. Hence, by (2.4) we obtain

$$h(e_1,e_1) = \lambda Je_1, \quad h(e_1,e_2) = h(e_2,e_2) = 0$$

for some λ with respect to some suitable orthonormal frame field e_1, e_2 unless $\lambda = \mu$.

The converse follows immediately from the equation of Gauss. □

Lagrangian H-umbilical isometric immersions of a real space form $M^n(c)$ of constant sectional curvature c into a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$ were classified in [1] and [2] for $c = 0$ and $c \neq 0$, respectively. The explicit description of such Lagrangian immersions was established in [3] for $c = 0$.

Given a real number $b > 0$, let $F : \boldsymbol{R} \to \boldsymbol{C}$ be the unit speed curve defined by

$$F(s) = \frac{e^{2bsi} + 1}{2bi}. \tag{3.3}$$

With respect to the induced metric the complex extensor $\phi = F \otimes \iota$ of the unit hypersphere of $\boldsymbol{E}^n$ via F is a Lagrangian isometric immersion of an open portion of an n-sphere $S^n(b^2)$ of sectional curvature b^2 into $\boldsymbol{C}^n$ which is called a *Lagrangian pseudo-sphere* (see [1] for details).

Lagrangian H-umbilical submanifolds in complex Euclidean spaces satisfying (1.1) with $\lambda = 2\mu$ were determined in [1] as follows.

THEOREM 3.3. *Up to rigid motions of $\boldsymbol{C}^n$, a Lagrangian isometric immersion $L : M \to \boldsymbol{C}^n$ is a Lagrangian pseudo-sphere if and only if it is a Lagrangian H-umbilical immersion satisfying* (1.1) *with $\lambda = 2\mu$.*

Lagrangian pseudo-spheres have both constant mean curvature and constant Gauss curvature.

REMARK 3.1. Lagrangian H-umbilical submanifolds satisfying (1.1) with $\lambda = 2\mu$ in nonflat complex space forms also have constant mean curvature and constant Gauss curvature [2]. Such Lagrangian H-umbilical submanifolds have been completely classified in [2] (see Theorems 5.1 and 6.1 of [2]).

The following lemma is easy to verify.

LEMMA 3.4. *Let $L : M \to \tilde{M}^2(4c)$ be a Lagrangian H-umbilical surface. Then the squared mean curvature and the Gauss curvature of M satisfy $4H^2 = 9(K - c)$ if and only if the second fundamental form of L takes the form*:

$$h(e_1, e_1) = 2\mu J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1$$

for some function $\mu \neq 0$, with respect to some orthonormal frame field e_1, e_2.

In views of Lemma 3.2, Theorem 3.3, Lemma 3.4 and Remark 3.1, we only need to consider Lagrangian H-umbilical surfaces in a complex space form $\tilde{M}^2(4c)$ such that $K \neq c,\ c + (4/9)H^2$.

Now, assume that M is a Lagrangian H-umbilical surface in $\tilde{M}^2(4c)$ satisfying the condition $K \neq c,\ c + (4/9)H^2$. Then the second fundamental form of M takes the form:

$$h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1 \tag{3.4}$$

for some functions λ, μ with $\mu \neq 0,\ \lambda/2$, with respect to some orthonormal frame field e_1, e_2.

Let ω^1, ω^2 denote the dual 1-forms of e_1, e_2 and let (ω_B^A) be the connection forms on M defined by

$$\tilde{\nabla} e_i = \sum_{j=1}^{2} \omega_i^j e_j + \sum_{j=1}^{2} \omega_i^{j^*} e_{j^*}, \quad \tilde{\nabla} e_{i^*} = \sum_{j=1}^{2} \omega_{i^*}^j e_j + \sum_{j=1}^{2} \omega_{i^*}^{j^*} e_{j^*}, \tag{3.5}$$

where $e_{i^*} = Je_i$, $\omega_i^j = -\omega_j^i$, $\omega_{i^*}^{j^*} = -\omega_{j^*}^{i^*}$, $i = 1, 2$.

For a Lagrangian surface M in $\tilde{M}^2(4c)$, we have [5]

$$\omega_j^{i^*} = \omega_i^{j^*}, \quad \omega_i^j = \omega_{i^*}^{j^*}, \quad \omega_j^{i^*} = \sum_{k=1}^{n} h_{jk}^i \omega^k. \tag{3.6}$$

From (3.4) and (3.6) we find

$$\omega_1^{1^*} = \lambda \omega^1, \quad \omega_2^{1^*} = \mu \omega^2, \quad \omega_2^{2^*} = \mu \omega^1. \tag{3.7}$$

By (3.4), (3.7) and the equation of Codazzi we obtain

$$e_1 \mu = (\lambda - 2\mu)\omega_1^2(e_2), \tag{3.8}$$

$$e_2 \lambda = (2\mu - \lambda)\omega_2^1(e_1), \tag{3.9}$$

$$e_2 \mu = 3\mu \omega_1^2(e_1), \tag{3.10}$$

Since Span $\{e_1\}$ and Span $\{e_2\}$ are one-dimensional distributions, there exists a local coordinate system $\{x, y\}$ on M such that $\partial/\partial x$ and $\partial/\partial y$ are parallel to e_1, e_2, respectively. Thus, the metric tensor g on M takes the form:

$$g = E^2\, dx^2 + G^2\, dy^2, \tag{3.11}$$

for some nonzero functions E and G. Without loss of generality we may assume

$$e_1 = \frac{1}{E} \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{G} \frac{\partial}{\partial y}. \tag{3.12}$$

From (3.11) we find

$$\omega_2^1(e_1) = \frac{E_y}{EG}, \quad \omega_1^2(e_2) = \frac{G_x}{EG}, \quad E_y = \frac{\partial E}{\partial y}, \quad G_x = \frac{\partial G}{\partial x}. \tag{3.13}$$

By (3.10), (3.12) and (3.13) we have

$$(\ln \mu)_y = -3(\ln E)_y. \tag{3.14}$$

Solving (3.14) yields

$$E = \frac{m(x)}{\mu^{1/3}}, \quad \mu = \frac{m^3(x)}{E^3} \tag{3.15}$$

for some function $m(x) \neq 0$.

By applying (3.9), (3.12), (3.13) and (3.15) we find

$$E\lambda_y = \frac{2}{E^3} m^3(x) E_y - \lambda E_y. \tag{3.16}$$

Solving (3.16) yields

$$\lambda = -\mu + \frac{f(x)}{E}, \tag{3.17}$$

for some function $f(x)$. From (3.15), (3.17) and the assumption $2\mu \neq \lambda$, we obtain $f(x) \neq 3m(x)\mu^{2/3}$.

Using (3.8), (3.12), (3.13), (3.15) and (3.17), we find

$$\mu_x = \left(\frac{f(x)}{m(x)}\mu^{1/3} - 3\mu\right)(\ln G)_x. \tag{3.18}$$

Solving (3.18) yields

$$G = q(y) \exp\left(\int^x k(x, y)\, dx\right), \quad k(x, y) = \frac{m(x)\mu_x}{f(x)\mu^{1/3} - 3m(x)\mu} \tag{3.19}$$

for some function $q(y) \neq 0$. Consequently, the metric tensor of M takes the following form:

$$g = E^2\, dx^2 + G^2\, dy^2, \quad E = \frac{m(x)}{\mu^{1/3}}, \quad G = q(y) \exp\left(\int^x k\, dx\right). \tag{3.20}$$

From (3.11) it follows that the Gauss curvature K of M is given by

$$K = -\frac{1}{EG}\left\{\frac{\partial}{\partial y}\left(\frac{E_y}{G}\right) + \frac{\partial}{\partial x}\left(\frac{G_x}{E}\right)\right\}. \tag{3.21}$$

By (3.15), (3.17), (3.19) and (3.21), we conclude that the functions $f(x)$, $m(x)$, $q(y)$ and $\mu(x, y)$ satisfy the following second order differential equation:

$$(3.22)\quad \left(\frac{f(x)}{m(x)}\mu - 2\mu^{5/3} + c\mu^{-1/3}\right)m(x)q(y)\exp\left(\int^x k\,dx\right)$$
$$= \frac{m(x)}{3}\left(\frac{\mu_y \exp(-\int^x k\,dx)}{\mu^{4/3}q(y)}\right)_y - q(y)\left(\frac{\mu^{1/3}k\exp(\int^x k\,dx)}{m(x)}\right)_x.$$

Conversely, suppose that $f(x)$, $m(x)$, $q(y)$ and $\mu(x,y)$ are functions defined on a simply-connected domain U of $\boldsymbol{R}^2$ such that $m(x)$, $q(y)$ and $\mu(x,y)$ and nowhere zero, $f(x) \neq 3m(x)\mu^{2/3}$, and they satisfy (3.22). We define a metric tensor g on U by

$$(3.23)\qquad g = E^2\,dx^2 + G^2\,dy^2, \quad E = \frac{m(x)}{\mu^{1/3}}, \quad G = q(y)\exp\left(\int^x k\,dx\right),$$

where $k = k(x,y)$ is defined by (3.19).

We define a symmetric bilinear form σ on (U,g) by

$$(3.24)\qquad \sigma(e_1,e_1) = \left(\frac{f(x)\mu^{1/3}}{m(x)} - \mu\right)e_1, \quad \sigma(e_1,e_2) = \mu e_2, \quad \sigma(e_2,e_2) = \mu e_1.$$

By applying (3.22)–(3.24) and a straight-forward computation, we know that $((U,g),\sigma)$ satisfies conditions (1), (2) and (3) of Theorem 2.1.

From the conditions $f(x) \neq 3m(x)\mu^{2/3}$ and $\mu \neq 0$, it follows that $K \neq c$, $c + (4/9)H^2$.

Consequently, by Theorem 2.1 and Theorem 2.2, we obtain the following.

THEOREM 3.5. *Let $L : M \to \tilde{M}^2(4c)$ be a Lagrangian H-umbilical surface such that $K \neq c$, $c + (4/9)H^2$. Then*

(1) *there exist functions $f(x)$, $m(x)$, $q(y)$ and $\mu(x,y)$ such that $m(x)$, $q(y)$ and $\mu(x,y)$ are nowhere zero, $f(x) \neq 3m(x)\mu^{2/3}$, and they satisfy* (3.22),

(2) *with respect to some coordinate system $\{x,y\}$ on M, the metric tensor of M is given by*

$$(3.25)\qquad g = E^2\,dx^2 + G^2\,dy^2, \quad E = m(x)\mu^{-1/3}, \quad G = q(y)\exp\left(\int^x k\,dx\right),$$

where

$$(3.26)\qquad k = \frac{m(x)\mu_x}{f(x)\mu^{1/3} - 3m(x)\mu},$$

(3) *the second fundamental form of L is given by*

$$(3.27)\qquad h(e_1,e_1)=\left(\frac{f(x)}{m(x)}\mu^{1/3}-\mu\right)Je_1,\quad h(e_1,e_2)=\mu Je_2,\quad h(e_2,e_2)=\mu Je_1,$$

where $e_1=E^{-1}\partial/\partial x$ *and* $e_2=G^{-1}\partial/\partial y$.

Conversely, suppose that $f(x)$, $m(x)$, $q(y)$ *and* $\mu(x,y)$ *are functions defined on a simply-connected domain* U *of* $\boldsymbol{R}^2$ *such that* $m(x)$, $q(y)$ *and* $\mu(x,y)$ *are nowhere zero,* $f(x)\neq 3m(x)\mu^{2/3}$, *and they satisfy* (3.22). *Let* g *be the metric tensor on* U *defined by* (3.25). *Then, up to rigid motions of* $\tilde{M}^2(4c)$, *there exists a unique Lagrangian H-umbilical isometric immersion of* (U,g) *into* $\tilde{M}^2(4c)$ *whose second fundamental form is given by* (3.27). *The Gauss curvature* K *and the squared mean curvature* H^2 *of such a Lagrangian surface satisfy the condition* $K\neq c$, $c+4/9H^2$.

Now, suppose that $L: M\to\tilde{M}^2(4c)$ is a minimal Lagrangian surface without totally geodesic points. Then, according to Lemma 3.1, the second fundamental form of L satisfies

$$(3.28)\qquad h(e_1,e_1)=-\hat{\mu}Je_1,\quad h(e_1,e_2)=\hat{\mu}Je_2,\quad h(e_2,e_2)=\hat{\mu}Je_1.$$

for some nonzero function $\hat{\mu}$ with respect to some orthonormal frame field e_1, e_2. Thus, by (3.15), (3.17), (3.18) and (3.19), we obtain

$$(3.29)\qquad g=\hat{\mu}^{-2/3}\{m^2(\bar{x})\,d\bar{x}^2+q^2(\bar{y})\,d\bar{y}^2\}$$

for some coordinate system $\{\bar{x},\bar{y}\}$ with $e_1=\bar{\mu}^{1/3}m(\bar{x})^{-1}\partial/\partial\bar{x}$, $e_2=\bar{\mu}^{1/3}q(\bar{y})^{-1}\partial/\partial\bar{y}$.

After applying the coordinate transformation:

$$(3.30)\qquad x=\int^x m(\bar{x})\,d\bar{x}\quad\text{and}\quad y=\int^y q(\bar{y})\,d\bar{y},$$

the metric tensor of M takes the simple form:

$$(3.31)\qquad g=\mu^{-2/3}(dx^2+dy^2)$$

where $\mu(x,y)=\hat{\mu}(\bar{x}(x),\bar{y}(y))$. With respect the coordinate system $\{x,y\}$, equation (3.22) becomes

$$(3.32)\qquad \Delta(\ln\mu)=3(c-2\mu^2)\mu^{-2/3},$$

where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$. With respect to x and y, (3.28) becomes

$$(3.33)\qquad \begin{gathered} h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right)=-\mu^{2/3}J\left(\frac{\partial}{\partial x}\right),\quad h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=\mu^{2/3}J\left(\frac{\partial}{\partial y}\right),\\ h\left(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right)=\mu^{2/3}J\left(\frac{\partial}{\partial x}\right). \end{gathered}$$

Conversely, if μ is a nowhere zero function defined on a simply-connected domain U of $\mathbf{R}^2$ which satisfies (3.23). We define a metric tensor on U by

$$g = \mu^{-2/3}(dx^2 + dy^2)$$

and define a symmetric bilinear form σ on (U, g) by

$$\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = -\mu^{2/3}\frac{\partial}{\partial x}, \quad \sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \mu^{2/3}\frac{\partial}{\partial y}, \quad \sigma\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \mu^{2/3}\frac{\partial}{\partial x}.$$

Then, by a straight-forward computation, we know that $((U, g), \sigma)$ satisfies conditions (1), (2) and (3) of Theorem 2.1. Thus, by Lemma 3.1, Theorem 2.1 and 2.2, we obtain the following.

Corollary 3.6. *Let $L : M \to \tilde{M}^2(4c)$ be a minimal Lagrangian surface without totally geodesic points. Then, with respect to a suitable coordinate system $\{x, y\}$, we have*

(1) *the metric tensor of M takes the form of* (3.31) *for some nowhere zero function μ satisfying* (3.32) *and*

(2) *the second fundamental form of L is given by* (3.33).

Conversely, if μ is a nowhere zero function defined on a simply-connected domain U of $\mathbf{R}^2$ satisfying (3.32) *and $g = \mu^{-2/3}(dx^2 + dy^2)$ is the metric tensor on U, then, up to rigid motions of $\tilde{M}^2(4c)$, there is a unique minimal (U, g) into $\tilde{M}^2(4c)$ whose second fundamental form is given by* (3.33).

4. Lagrangian H-umbilical surfaces with constant Gauss curvature

The following result determines the intrinsic and the extrinsic structures of Lagrangian H-umbilical surfaces with constant Gauss curvature in complex space forms.

Theorem 4.1. *Let $L : M \to \tilde{M}^2(4c)$ be a Lagrangian H-umbilical surface. If M has constant Gauss curvature K such that $K \neq c$, $c + (4/9)H^2$, then*

(1) *with respect to some coordinate system $\{x, y\}$ on M, the metric tensor of M is given by*

$$g = dx^2 + G^2\,dy^2, \tag{4.1}$$

where

$$G = \begin{cases} \dfrac{1}{\sqrt{K}}\cos(\sqrt{K}x), & \text{if } K > 0; \\ x, & \text{if } K = 0; \\ \dfrac{1}{\sqrt{-K}}\cosh(\sqrt{-K}x), & \text{if } K < 0, \end{cases} \tag{4.2}$$

(2) *the second fundamental form of L is given by*

$$h(e_1,e_1)=\left(\frac{K-c+\mu^2}{\mu}\right)Je_1,\quad h(e_1,e_2)=\mu Je_2,\quad h(e_2,e_2)=\mu Je_1, \tag{4.3}$$

where $e_1=\partial/\partial x$, $e_2=G^{-1}\partial/\partial y$ and μ is a nonzero function satisfying

$$|K-c-\mu^2|=\begin{cases} K\sec^2(\sqrt{K}x), & \text{if } K>0;\\ x^{-2}, & \text{if } K=0;\\ -K\operatorname{sech}^2(\sqrt{-K}x), & \text{if } K<0.\end{cases} \tag{4.4}$$

Conversely, suppose that c, K are two unequal constants, U a simply-connected domain of $\boldsymbol{R}^2$ such that (4.1) *is a well-defined positive-definite metric on U and μ is a function satisfying* (4.4). *Then*

(3) (U,g) *has constant Gauss curvature K and*

(4) *up to rigid motions of $\tilde{M}^2(4c)$, there exists a unique Lagrangian H-umbilical isometric immersion of (U,g) into $\tilde{M}^2(4c)$ whose second fundamental form is given by* (4.3).

PROOF. Assume that $L: M\to\tilde{M}^2(4c)$ is a Lagrangian H-umbilical surface such that $K\neq c,\ c+(4/9)H^2$. Then the second fundamental form of L takes the form:

$$h(e_1,e_1)=\lambda Je_1,\quad h(e_1,e_2)=\mu Je_2,\quad h(e_2,e_2)=\mu Je_1. \tag{4.5}$$

for some functions λ, μ with $\mu\neq 0,\ \lambda/2$, with respect to an orthonormal frame field e_1, e_2.

From the assumption $K\neq c+(4/9)H^2$, we obtain $\mu^2\neq K-c$. If the Gauss curvature K of M is constant, then

$$\lambda\mu-\mu^2+c=K=constant. \tag{4.6}$$

By applying (3.9), (3.10) and (4.5), we get $\omega_1^2(e_1)=0$ and $e_2\lambda=e_2\mu=0$.

From $\omega_1^2(e_1)=0$, it follows that the integral curves of e_1 are geodesics in M. Thus, there exists a local coordinate system $\{x,y\}$ on M such that the metric tensor of M takes the form:

$$g=dx^2+G^2\,dy^2 \tag{4.7}$$

and $e_1=\partial/\partial x$, $e_2=G^{-1}\partial/\partial y$. From $e_2\lambda=e_2\mu=0$, we obtain $\lambda=\lambda(x)$ and $\mu=\mu(x)$.

From (3.15), (3.17), (3.19) and (4.7), we get

$$(4.8)\qquad m(x)=\mu^{1/3},\quad f(x)=\lambda(x)+\mu(x),\quad G=q(y)\exp\left(\int^x k\,dx\right),$$

where k is defined by (3.19). Equations (3.19), (4.6) and (4.8) imply

$$(4.9)\qquad k=\frac{\mu\mu'}{K-c-\mu^2},\quad \mu'=\mu'(x).$$

Solving (4.9) yields

$$(4.10)\qquad k(x)=-\frac{1}{2}(\ln|K-c-\mu^2|)'(x).$$

Thus, the metric tensor of M takes the form:

$$(4.11)\qquad g=dx^2+\frac{q^2(y)}{|K-c-\mu^2|}dy^2.$$

After applying a suitable change of variable in y if necessary, we get

$$(4.12)\qquad g=dx^2+\frac{1}{|K-c-\mu^2|}dy^2.$$

From $\mu_y=0$, (4.6), (4.7), (4.9) and equation (3.21) of Gauss, we obtain

$$(4.12)\qquad k'(x)+k^2(x)=-K.$$

Solving (4.12) and using (4.9), we get

$$(4.13)\qquad |K-c-\mu^2|=\begin{cases}\dfrac{a}{\cos^2(\sqrt{K}(b-x))}, & \text{if } K>0;\\ \dfrac{a}{(x-b)^2}, & \text{if } K=0;\\ \dfrac{a}{\cosh^2(\sqrt{-K}(x-b))}, & \text{if } K<0,\end{cases}$$

where a, b are integration constants.

Therefore, by applying a translation in x and dilation in y if necessary, we obtain (4.4) and statement (1). (4.3) now follows from (4.5) and (4.6).

Conversely, assume that K, c are unequal constants, U is a simply-connected domain of $\boldsymbol{R}^2$ such that (4.1) is a well-defined positive-definite metric on U and μ is a function which satisfies (4.4). Then, by a direct computation, we obtain statement (3).

If we define a symmetric bilinear form σ on (U,g) by

$$(4.14)\qquad \sigma(e_1,e_1)=\left(\frac{K-c+\mu^2}{\mu}\right)e_1,\quad \sigma(e_1,e_2)=\mu e_2,\quad \sigma(e_2,e_2)=\mu e_1,$$

where $e_1 = \partial/\partial x$, $e_2 = G^{-1}\partial/\partial y$, then, by a straight-forward long computation, we conclude that $((U,g),\sigma)$ satisfies conditions (1), (2) and (3) of Theorem 2.1. Hence, according to Theorem 2.1, there is a Lagrangian isometric immersion of (U,g) into $\tilde{M}^2(4c)$ with second fundamental form given by $h = J\sigma$. Moreover, by (4.14), we obtain statement (5).

The uniqueness of the Lagrangian immersion now follows from Theorem 2.2. □

REMARK 4.1. Theorem 4.1 of [1] states that Lagrangian H-umbilical submanifolds of dimension ≥ 3 with constant sectional curvature in complex Euclidean spaces are either flat or open portions of Lagrangian pseudo-spheres. In contrast, Theorem 4.1 shows that there exist many Lagrangian H-umbilical surfaces with constant Gauss curvature in the complex Euclidean plane which are neither flat nor open portions of Lagrangian pseudo-spheres.

REMARK 4.2. The intrinsic and the extrinsic structures of Lagrangian H-umbilical surfaces in $\tilde{M}^2(4c)$ with constant Gauss curvature $K = c + (4/9)H^2$ have been completely determined in [1] and [2] for $c = 0$ and $c \neq 0$, respectively.

It is obvious that a Lagrangian H-umbilical surface in a complex space form has constant mean curvature and constant Gauss curvature if and only if both λ and μ are constant. However, Theorem 3.5 yields the following.

PROPOSITION 4.2. *Let $L : M \to \tilde{M}^2(4c)$ be a Lagrangian isometric immersion whose second fundamental form satisfies*

$$h(e_1,e_1) = \lambda J e_1, \quad h(e_1,e_2) = \mu J e_2, \quad h(e_2,e_2) = \mu J e_1. \tag{4.15}$$

with respect to an orthonormal frame field e_1, e_2. If μ is constant, then M has constant Gauss curvature. Moreover, M is flat unless $\mu = 0$ or $\mu = \lambda/2$.

PROOF. Let M be a Lagrangian surface in $\tilde{M}^2(4c)$ satisfies (4.15). If $\mu = 0$, then M has constant Gauss curvature c. If $\mu = \lambda/2$, then M also has constant Gauss curvature according to Theorem 3.1 of [1] and Theorems 5.1 and 6.1 of [2] for $c = 0$ and $c \neq 0$, respectively. Finally, if $\mu \neq 0, \lambda/2$, (3.23) implies that E and G are functions of x and y, respectively. In this case M is flat according to (3.21). □

REMARK 4.3. The converse of Corollary 3.6 is false. In fact, there exist

Lagrangian H-umbilical surfaces with constant Gauss curvature in a complex space form such that the function μ of (4.15) is non-constant.

The following result shows in particular that Lagrangian H-umbilical surfaces with λ being constant do not have Gauss curvature in general.

PROPOSITION 4.3. *Let $L : M \to \tilde{M}^2(4c)$ be a Lagrangian isometric immersion whose second fundamental form satisfies* (4.15) *for $\mu \neq 0$, $\lambda/2$, with respect to an orthonormal frame field e_1, e_2. If λ is constant, then*

(1) *there is a coordinate system $\{x, y\}$ on M such that the metric tensor of M is given by*

$$g = dx^2 + \frac{dy^2}{|\lambda - 2\mu|}, \tag{4.16}$$

and

(2) *μ is a function of x satisfying*

$$\mu'^2 = (\lambda - 2\mu)^3 \left\{ b + \frac{\mu}{2} - \frac{\lambda^2 + 4c}{4(\lambda - 2\mu)} \right\}, \tag{4.17}$$

for some constant b.

Conversely, suppose that b, c, λ are constants and $\mu(x)$ is a non-constant function satisfying (4.17) *on some open interval I. Let g be the metric tensor on $U = I \times \boldsymbol{R}$ defined by* (4.16). *Then, up to rigid motions of $\tilde{M}^2(4c)$, there is a unique Lagrangian H-umbilical isometric immersion of (U, g) into $\tilde{M}^2(4c)$ whose second fundamental form is given by* (4.15).

PROOF. Assume that M is a Lagrangian surface in $\tilde{M}^2(4c)$ satisfying (4.15) with $\mu \neq 0$, $\lambda/2$ for some constant λ. Then (3.9) and (3.10) yield $\nabla_{e_1} e_1 = 0$ and $e_1 \mu = 0$. Thus, it follows as before that the metric tensor of M takes the form:

$$g = dx^2 + G^2 \, dy^2 \tag{4.18}$$

with respect to some coordinate system $\{x, y\}$ with $e_1 = \partial/\partial x$, $e_2 = G^{-1}\partial/\partial y$.

From $e_2 \mu = 0$, we obtain $\mu = \mu(x)$. Moreover, from (3.17), (3.19), (3.20) and (4.18) we have

$$k = \frac{\mu'(x)}{\lambda - 2\mu} = -\frac{1}{2}(\ln|\lambda - 2\mu|)', \quad G = \frac{q(y)}{|\lambda - 2\mu|^{1/2}}. \tag{4.19}$$

Thus,

$$g = dx^2 + \frac{q^2(y)}{|\lambda - 2\mu|} dy^2. \tag{4.20}$$

After applying a suitable change of variable in y if necessary, we have

$$g = dx^2 + \frac{dy^2}{|\lambda - 2\mu|}. \tag{4.21}$$

From (4.15), (4.21), and the equation of Gauss we know that the function $\mu = \mu(x)$ satisfies the following differential equation:

$$k'(x) + k^2(x) = \mu^2 - \lambda\mu - c, \quad k(x) = \frac{\mu'(x)}{\lambda - 2\mu}. \tag{4.22}$$

Solving (4.22) for μ' yields equation (4.17) for some constant a.

Conversely, suppose that b, c, λ are constants and $\mu(x)$ is a non-constant function satisfying (4.17) on some open interval I. We define a metric tensor g on $U = I \times \boldsymbol{R}$ by (4.16) and define a symmetric bilinear map σ on (U, g) by

$$\sigma(e_1, e_1) = \lambda e_1, \quad \sigma(e_1, e_2) = \mu e_2, \quad \sigma(e_2, e_2) = \mu e_1, \tag{4.23}$$

where $e_1 = \partial/\partial x$ and $e_2 = |\lambda - 2\mu|^{1/2}\partial/\partial y$. Then by a straight-forward computation we conclude that $((U, g), \sigma)$ satisfies conditions (1), (2) and (3) of Theorem 2.1.

Consequently, by Theorems 2.1 and 2.2 we conclude that, up to rigid motions of $\tilde{M}^2(4c)$, there is a unique Lagrangian isometric immersion of (U, g) into $\tilde{M}^2(4c)$ whose second fundamental form is given by (4.15) with constant λ. □

Proposition 4.3 implies that Lagrangian H-umbilical surfaces with constant λ in a complex space form do not have constant Gauss curvature in general.

5. Lagrangian H-umbilical surfaces with constant mean curvature

Let $L : M \to \tilde{M}^2(4c)$ be a Lagrangian H-umbilical surface with $K \neq c$, $c + (4/9)H^2$. If M has constant mean curvature $\beta \neq 0$, then the second fundamental form of L takes the form:

$$h(e_1, e_1) = (2\beta - \mu)Je_1, \quad h(e_1, e_2) = \mu Je_2, \quad h(e_2, e_2) = \mu Je_1, \tag{5.1}$$

for $\mu \neq 0$, $2\beta/3$ with respect to some suitable orthonormal frame field e_1, e_2.

From (3.9) and (3.10) we get $0 = e_2(\beta) = \beta\omega_2^1(e_1)$ which yields $\nabla_{e_1}e_1 = 0$. Hence, by (3.9) and (3.10), we also have $e_2\lambda = e_2\mu = 0$.

From $\omega_1^2(e_1) = 0$, it follows as before that the metric tensor of M takes the form:

(5.2) $$g = dx^2 + G^2\, dy^2$$

with respect to some local coordinate system $\{x, y\}$ with $e_1 = \partial/\partial x$, $e_2 = G^{-1}\partial/\partial y$.

From $e_2\lambda = e_2\mu = 0$, we obtain $\lambda = \lambda(x)$ and $\mu = \mu(x)$. Thus, (3.17), (3.19), and (5.1) imply $k(x) = \mu'/(2\beta - 3\mu)$. Hence, after applying a suitable change of variable in y if necessary, the metric tensor of M takes the form:

(5.3) $$g = dx^2 + \frac{dy^2}{(2\beta - 3\mu)^{2/3}}.$$

From (5.1), (5.3), and the equation of Gauss we know that the function $\mu = \mu(x)$ satisfies the following differential equation:

(5.4) $$\mu''(x) + \frac{4\mu'^2}{2\beta - 3\mu} = (2\beta - 3\mu)(2\mu^2 - 2\beta\mu - c).$$

Solving (5.4) for μ' yields

(5.5) $$\mu'^2 = (3\mu - 2\beta)^2\{b(2\beta - 3\mu)^{2/3} - c - \mu^2\},$$

where b is an integration constant satisfying $b(2\beta - 3\mu)^{2/3} > c + \mu^2$. Such constant exists at least locally, since $(2\beta - 3\mu)^2 = (\lambda - 2\mu)^2 > 0$.

Conversely, suppose that b, c and $\beta \neq 0$ are constants and $\mu(x)$ is a function with $\mu \neq 0$, $2\beta/3$ which satisfy (5.5) on some open interval I. We define a metric tensor g on $U = I \times \boldsymbol{R}$ by (5.3) and define a symmetric bilinear map σ on (U, g) by

(5.6) $$\sigma(e_1, e_1) = (2\beta - \mu)e_1, \quad \sigma(e_1, e_2) = \mu e_2, \quad \sigma(e_2, e_2) = \mu e_1,$$

where $e_1 = \partial/\partial x$ and $e_2 = (2\beta - 3\mu)^{1/3}\partial/\partial y$. Then by a straight-forward computation we conclude that $((U, g), \sigma)$ satisfies conditions (1), (2) and (3) of Theorem 2.1.

Consequently, by applying Theorems 2.1 and 2.2 we obtain the following.

THEOREM 5.1. *Let $L : M \to \tilde{M}^2(4c)$ be a Lagrangian H-umbilical surface with $K \neq c$, $c + (4/9)H^2$. If M has constant mean curvature $\beta \neq 0$, then*

(1) *there exist a constant b and a nonzero function $\mu(x) \neq 2\beta/3$ satisfying* (5.5),

(2) *there exists a coordinate system $\{x, y\}$ on M such that the metric tensor of M is given by* (5.3), *and*

(3) *the second fundamental form of L is given by*

$$h(e_1, e_1) = (2\beta - \mu)Je_1, \quad h(e_1, e_2) = \mu Je_2, \quad h(e_2, e_2) = \mu Je_1, \tag{5.7}$$

where $e_1 = \partial/\partial x$, $e_2 = (2\beta - 3\mu)^{1/3}\partial/\partial y$.

Conversely, suppose that b, c and $\beta \neq 0$ are constants and $\mu(x)$ is a function satisfying (5.5) *and $\mu(x) \neq 0$, $2\beta/3$ on some open interval I. Let g be the metric tensor on $U = I \times \boldsymbol{R}$ defined by* (5.3). *Then, up to rigid motions of $\tilde{M}^2(4c)$, there is a unique Lagrangian H-umbilical isometric immersion of (U, g) into $\tilde{M}^2(4c)$ whose second fundamental form is given by* (5.7). *Such a Lagrangian H-umbilical surface has prescribed constant mean curvature $\beta \neq 0$.*

REMARK 5.1. If we put

$$\phi_b(\mu) = \int^{\mu} \frac{d\mu}{(3\mu - 2\beta)\sqrt{b(2\beta - 3\mu)^{2/3} - c - \mu^2}}, \tag{5.8}$$

then $\phi_b(\mu)$ is a monotonic function, since $3\mu - 2\beta = 2\mu - \lambda$ is assumed to be nowhere zero. Hence, ϕ_b has an inverse function which is denoted by ϕ_b^{-1}. In terms of ϕ_b^{-1}, the solutions of (5.5) is given either by $\mu(x) = \phi_b^{-1}(x + a)$ or by $\mu(x) = \phi_b^{-1}(-(x + a))$, where a is a constant.

Theorem 5.1 yields the following.

COROLLARY 5.2. *If M is a Lagrangian H-umbilical surface in $\boldsymbol{C}^2$ with constant mean curvature, then M is one of the following Lagrangian H-umbilical surfaces*:

(1) *a minimal Lagrangian surface*,

(2) *an open portion of Lagrangian circular cylinder: $S^1(r) \times \boldsymbol{R} \subset \boldsymbol{C}^1 \times \boldsymbol{C}^1 = \boldsymbol{C}^2$, on a Lagrangian Clifford torus: $S^1(r) \times S^1(r) \subset \boldsymbol{C}^2$*,

(3) *an open portion of a Lagrangian pseudo-sphere, or*

(4) *a complex extensor which is not an open portion of a Lagrangian pseudo-sphere.*

PROOF. Let M be a Lagrangian H-umbilical surface in $\boldsymbol{C}^2$ with constant mean curvature. If M is flat, then the second fundamental form of M takes the form:

$$h(e_1, e_1) = \beta J e_1, \quad h(e_1, e_2) = h(e_2, e_2) = 0, \tag{5.9}$$

for some constant $\beta \neq 0$, according to Lemma 3.2 unless $\lambda = \mu$. Thus, (3.8) and (3.9) imply $\omega_1^2 = 0$. Hence, by (2.3) we obtain $DH = 0$. These imply that M is a flat surface with parallel mean curvature vector. Hence, using (5.9), we may conclude that M is an open portion of a Lagrangian circular cylinder or a Lagrangian Clifford torus.

If M is a nonflat Lagrangian H-umbilical surface with nonzero constant mean curvature, then from the discussion given at the beginning of this section, we know that the integral curves of e_1 are geodesics in M. Therefore, by applying Theorem 4.3 of [1], M is either an open portion of a Lagrangian pseudo-sphere or a complex extensor. □

REMARK 5.2. If a Lagrangian H-umbilical surface M with constant mean curvature β is a complex extensor, then, up to rigid motions of $\boldsymbol{C}^2$, it is given by the tensor product $F \otimes G$, where G is the unit circle in $\boldsymbol{E}^2$ centered at the origin and F is the unit speed curve in the complex plane $\boldsymbol{C}$ defined by

$$F(s) = \gamma + \int^s \left(\exp\left(i \int^t (2\beta - \mu(x))\, dx \right) dt \right), \tag{5.10}$$

where γ is a complex number and $\mu(x)$ is given either by $\mu(x) = \phi_b^{-1}(x + a)$ or by $\mu(x) = \phi_b^{-1}(-(x + a))$, where ϕ^{-1} is defined in Remark 5.1.

6. Lagrangian H-umbilical surfaces with $\lambda = \alpha\mu$

First we give the following existence theorem.

THEOREM 6.1. *For any given constants c and α, there exists a Lagrangian H-umbilical surface in $\tilde{M}^2(4c)$ whose second fundamental form satisfies*

$$h(e_1, e_1) = \alpha\mu J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1, \tag{6.1}$$

for some nonzero function μ with respect to some orthonormal frame field e_1, e_2.

PROOF. When $\alpha = -1$, this follows from Corollary 3.6. When $\alpha = 2$, this follows from Theorems 5.1 and 6.1 of [2] and Theorem 3.1 of [1].

Now, suppose $\alpha \neq -1, 2$. If we choose a sufficiently large positive number b such that $b > (\alpha - 2)^2 (c + \mu^2) \mu^{2/(\alpha - 2)}$ on some open interval $\hat{I} \subset (0, \infty)$, then

$$\psi_b(\mu) = \int^{\mu} \frac{d\mu}{\mu^{(\alpha-3)/(\alpha-2)}\sqrt{b-(\alpha-2)^2(c+\mu^2)\mu^{2/(\alpha-2)}}} \tag{6.2}$$

is an increasing function on $\hat{I}$. Let $\mu(x) = \psi_b^{-1}(x)$ denote the inverse function of ψ_b defined on the corresponding open interval, say I.

We define a metric tensor g on $U = I \times \boldsymbol{R}$ by

$$g = dx^2 + \mu^{2/(\alpha-2)}\, dy^2 \tag{6.3}$$

and define a symmetric bilinear map σ on (U, g) by

$$\sigma(e_1, e_1) = \alpha\mu e_1, \quad \sigma(e_1, e_2) = \mu e_2, \quad \sigma(e_2, e_2) = \mu e_1, \tag{6.4}$$

where $e_1 = \partial/\partial x$, $e_2 = \mu^{-1/(\alpha-2)}\partial/\partial y$. Then, by a straight-forward computation we conclude that $((U, g), \sigma)$ satisfies conditions (1), (2) and (3) of Theorem 2.1. Thus, by Theorem 2.1, there exists a Lagrangian isometric immersion from (U, g) into $\tilde{M}^2(4c)$ whose second fundamental form is given by (6.1). □

THEOREM 6.2. *Let M be a nonflat Lagrangian H-umbilical surface in $\boldsymbol{C}^2$ whose Gauss curvature K and squared mean curvature H^2 are proportional. Then M is one of the following Lagrangian surfaces:*

(1) *a minimal Lagrangian surface,*

(2) *an open portion of a Lagrangian pseudo-sphere, or*

(3) *a complex extensor which is not an open portion of a Lagrangian pseudo-sphere.*

PROOF. Assume that M is a non-minimal Lagrangian H-umbilical surface in $\boldsymbol{C}^2$ whose Gauss curvature K and squared mean curvature H^2 are proportional, that is, $K = aH^2$ for some real number a. Since M is Lagrangian H-umbilical, the second fundamental form of M in $\boldsymbol{C}^2$ satisfies

$$h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1, \tag{6.5}$$

for some function λ, $\mu \neq 0$ with respect to some orthonormal frame field e_1, e_2.

From (6.5), the equation of Gauss and the definition of the squared mean curvature, we obtain

$$a\lambda^2 + 2(a-2)\mu\lambda + (a+4)\mu^2 = 0. \tag{6.6}$$

Solving (6.6) yields

$$\lambda = \frac{1}{a}((2-a)\mu \pm 2\sqrt{(1-2a)\mu^2}). \tag{6.7}$$

Since λ is real, (6.7) yields $a \leq 1/2$. Thus, there is real number α such that $a = 4(\alpha-1)/(\alpha^2+1)^2$. Thus, we get

$$(\alpha+1)^2 K = 4(\alpha-1)H^2. \tag{6.8}$$

From (6.5) and (6.8), we know that the second fundamental form of M in C^2 satisfies (6.1) for some nonzero function μ. Hence, by applying (3.9) and (3.10), we get $(1+\alpha)e_2\mu = 0$ which implies that either M is minimal or $e_2\mu = 0$. If $e_2\mu = 0$, (3.9) yields $(2-\alpha)\mu\omega_2^1(e_1) = 0$. Thus, we have either $\alpha = 2$ or $\nabla_{e_1}e_1 = 0$.

If $\alpha = 2$, M is an open portion of a Lagrangian pseudo-sphere according to Theorem 3.1 of [1].

If $\nabla_{e_1}e_1 = 0$, then, according to Theorem 4.3 of [1], M is either a flat surface or a complex extensor. However, the flat case cannot occurs. □

Remark 6.1. We are able to determine the intrinsic and the extrinsic structures of a Lagrangian surface in a complex space form $\tilde{M}^2(4c)$ which satisfies (6.1) for $\alpha \neq -1, 2$, too. In fact, by applying the same method utilized in section 5, we may prove that the function μ of such a Lagrangian surface is a function of x which is a solution of

$$u'(x)^2 = \mu^{2(\alpha-3)/(\alpha-2)}\{b - (\alpha-2)^2(c+\mu^2)\mu^{2/(\alpha-2)}\} \tag{6.9}$$

for some constant b and, moreover, the metric tensor of such a Lagrangian surface is given by

$$g = dx^2 + \mu^{2/(\alpha-2)}\, dy^2 \tag{6.10}$$

with respect to a coordinate system $\{x, y\}$ satisfying $e_1 = \partial/\partial x$, $e_2 = \mu^{1/(2-\alpha)}\partial/\partial y$.

Remark 6.2. If the Lagrangian H-umbilical surface M mentioned in Theorem 6.2 is a complex extensor, then, up to rigid motions of C^2, it is given by the tensor product $F \otimes G$, where G is the unit circle in E^2 centered at the origin and F is the unit speed curve in the complex plane C defined by

$$F(s) = \gamma + \int^s \left(\exp\left(i\int^t \alpha\mu(x)\,dx\right) dt\right), \tag{6.11}$$

where γ is a complex number, α a real number and $\mu(x)$ a solution of (6.9).

7. Lagrangian H-umbilical surfaces with $\mu = \mu(y)$

All of the Lagrangian H-umbilical surfaces studied in sections 4, 5 and 6 satisfy the condition $e_2\mu = 0$.

In this section we determine the intrinsic and the extrinsic structures of Lagrangian H-umbilical surfaces in $\tilde{M}^2(4c)$ whose second fundamental form satisfies

$$h(e_1, e_1) = \lambda Je_1, \quad h(e_1, e_2) = \mu Je_2, \quad h(e_2, e_2) = \mu Je_1, \quad e_1\mu = 0 \tag{7.1}$$

for $\mu \neq 0$, $\lambda/2$ with respect to some suitable orthonormal frame field e_1, e_2.

From section 3 we know that, with respect to some coordinate system $\{x, y\}$, the metric tensor of such a Lagrangian H-umbilical surface M takes the form:

$$g = E^2\,dx^2 + G^2\,dy^2, \quad E = \frac{m(x)}{\mu^{1/3}}, \quad G = q(y)\exp\left(\int^x k\,dx\right), \tag{7.2}$$

where $e_1 = E^{-1}\partial/\partial x$, $e_2 = G^{-1}\partial/\partial y$ and k is defined by

$$k(x, y) = \frac{m(x)\mu_x}{f(x)\mu^{1/3} - 3m(x)\mu} \tag{7.3}$$

for some function $f(x)$ and nonzero functions $m(x)$, $q(y)$. Moreover, from section 3 we also have

$$\lambda = -\mu + \frac{f(x)}{E}. \tag{7.4}$$

The assumption $e_1\mu = 0$ is equivalent to $\mu_x = 0$, that is, $\mu = \mu(y)$. Thus (7.3) yields $k = 0$. Hence, equation (3.22) reduces to

$$3\left(\frac{f(x)}{m(x)}\mu - 2\mu^{5/3} + c\mu^{-1/3}\right)q(y) = \left(\frac{\mu'(y)}{\mu^{4/3}q(y)}\right)' \tag{7.5}$$

which implies in particular that $f(x)/m(x)$ is a constant, which is denoted by b. Therefore, (7.5) can be rewritten as

$$\left(\frac{\mu'}{\mu^{4/3}}\right)q'(y) - \left(\frac{\mu'}{\mu^{4/3}}\right)' q(y) = -3(b\mu - 2\mu^{5/3} + c\mu^{-1/3})q^3(y). \tag{7.6}$$

Solving (7.6) yields

$$q(y)^2 = \mu'^2\{9(a + b\mu^{2/3} - \mu^{4/3} + c\mu^{-2/3})\}^{-1}, \tag{7.7}$$

where a is an integration constant.

Consequently, the metric tensor of M takes the form:

$$g = \frac{m^2(x)}{\mu^{2/3}} dx^2 + \frac{\mu'^2}{9(a + b\mu^{2/3} - \mu^{4/3} + c\mu^{-2/3})} dy^2. \tag{7.8}$$

Thus, by applying a suitable change of variable in x if necessary, we obtain

$$g = \mu^{-2/3} dx^2 + G^2 dy^2, \quad G = \frac{\mu'}{3}(a + b\mu^{2/3} - \mu^{4/3} + c\mu^{-2/3})^{-1/2}. \tag{7.9}$$

Using (7.1), (7.4) and (7.9) we conclude that the second fundamental satisfies

$$h(e_1, e_1) = (b\mu^{1/3} - \mu)Je_1, \quad h(e_1, e_2) = \mu Je_2, \quad h(e_2, e_2) = \mu Je_1. \tag{7.10}$$

Conversely, suppose that a, b are constants and $\mu = \mu(y)$ a nowhere zero function which satisfy $a > \mu^{-2/3}(\mu^2 - c - b\mu^{4/3})$ on some open interval I. We define a metric tensor g on $U = \mathbf{R} \times I$ by (7.9) and define a symmetric bilinear map σ on (U, g) by

$$\sigma(e_1, e_1) = (b\mu^{1/3} - \mu)e_1, \quad \sigma(e_1, e_2) = \mu e_2, \quad \sigma(e_2, e_2) = \mu e_1, \tag{7.11}$$

where $e_1 = \mu^{1/3}\partial/\partial x$, $e_2 = G^{-1}\partial/\partial x$. Then we can verify by a straight-forward computation that $\{(U, g), \sigma\}$ satisfies conditions (1), (2) and (3) of Theorem 2.1.

Consequently, by applying Theorems 2.1 and 2.2, we obtain the following.

THEOREM 7.1. *Let $L : M \to \tilde{M}^2(4c)$ be a Lagrangian H-umbilical surface whose second fundamental form satisfies*

$$h(e_1, e_1) = \lambda Je_1, \quad h(e_1, e_2) = \mu Je_2, \quad h(e_2, e_2) = \mu Je_1 \tag{7.12}$$

for $\mu \neq 0$, $\lambda/2$ with respect to an orthonormal frame field e_1, e_2. If $e_1\mu = 0$, then there exist constants a and b such that

(1) $\lambda = b\mu^{1/3} - \mu$ and

(2) the metric tensor of M is given by (7.9) with respect to a coordinate system $\{x, y\}$ such that $e_1 = \mu^{1/3}\partial/\partial x$, $e_2 = G^{-1}\partial/\partial y$.

Conversely, if $\mu = \mu(y)$ is a nowhere zero function and a, b are constants which satisfy $a > \mu^{-2/3}(\mu^2 - c - b\mu^{4/3})$ on some open interval I, then, up to rigid motions of $\tilde{M}^2(4c)$, there is a unique Lagrangian H-umbilical isometric immersion of (U, g) into $\tilde{M}^2(4c)$ whose second fundamental form is given by (7.10), where $U = \mathbf{R} \times I$ and g is the metric on U defined by (7.9).

Finally, we remark that, unless the function μ is constant, the integral curves of JH are not necessary geodesics for the Lagrangian H-umbilical surfaces given in Theorem 7.1. Consequently, these Lagrangian surfaces cannot be complex extensors in the complex Euclidean plane when $c = 0$.

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