Gauge Invariance of Fractionally Charged Quasiparticles and Hidden Topological Z_n Symmetry

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Using the braid-group formalism we study the consequences of gauge invariance for fractionally charged anyonic quasiparticles in a two-dimensional multiply connected system. It is shown that gauge invariance requires multicomponent wave functions, and leads to the emergence of a hidden topological Z_n symmetry with associated quantum number and unavoidable occurrence of level crossings for many-body eigenstates. In certain situations, it relates the fractional charge to anyon statistics. The implications for the fractional quantum Hall effect are also discussed.

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Gauge invariance is known to play a fundamental role in macroscopic quantum phenomena, such as flux quantization[1] in a superconducting ring and the integral quantum Hall effect.[2] However, there is little understanding of its role in the fractional quantum Hall effect (FQHE),[3] though some suggestions have been made in the literature.[4-6] Recently, direct experimental evidence[7] has been reported for a quasiparticle of fractional charge $e^*=e/3$ in the FQHE with the Landau-level filling factor $\nu=\frac{1}{3}$, as predicted by Laughlin's theory.[8,9] This makes urgent the need for a better theoretical understanding of gauge invariance for fractionally charged quasiparticles.

Consider a cylindrical (or toroidal) system with a magnetic flux Φ through the hole. Gauge invariance[1,10] implies that all physical properties of the system are periodic functions of Φ with the period (flux quantum) $\Phi_0=hc/e$, where e is the constituent (electron) charge. The problem is under what conditions can a system of quasiparticle excitations with fractional charge e^* have, as required by gauge invariance, a (q times, if $e^*=e/q$) *smaller period* Φ_0 than the naturally expected $\Phi_0^*=hc/e^*$.

To study this problem, we use the braid-group formalism[11] on a cylinder[12] (or on a torus[13]), appropriate for anyonic[14,15] quasiparticles in FQHE.[16-18] We will show that if the anyon system is described by a one-component wave function, there is no period smaller than Φ_0^*. However, with the use of a multicomponent wave function, we can derive the condition for the existence of a smaller period, which lends to, in the cylinder case, the emergence of a hidden topological Z_n symmetry with associated quantum number, the n-ality, in the spectrum for the *many-body* eigenstates. Here n is the smallest integer satisfying $n(e^*/e)=$integer. On a torus, gauge invariance implies a relation between e^* and the statistics θ for an irreducible braid-group representation (BGR), and in the thermodynamic limit there are two noncommuting topological symmetries which lead to the ground-state degeneracy. The revelation of topological Z_n symmetry lends strong support to a "broken-symmetry" scenario for the FQHE, proposed by Tao and Wu[4] some time ago and refined recently by Thouless.[6]

According to the path-integral formalism,[11] it is the braid group which plays the same basic role for anyons as the permutation group for usual bosons and fermions. The braid group is nothing but the first homotopy group of the N-anyon configuration space; and the anyon wave function forms a representation of it. On a cylinder, the braid-group generators consist of not only the usual σ_i ($i=1,\ldots,N-1$), which interchanges the ith and $(i+1)$th particles, but also of additional ρ_j ($j=1,\ldots,N$) which represents moving a particle along a simple loop around the hole once with $j-1$ particles to its left (see Fig. 1). These generators satisfy

$$\sigma_i\sigma_j=\sigma_j\sigma_i \quad (i\neq j\pm 1)\,, \quad \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}\,, \tag{1}$$

$$\rho_i\rho_j=\rho_j\rho_i, \quad \sigma_i\rho_j=\rho_j\sigma_i \quad (i\neq j,j-1)\,, \tag{2}$$

$$\rho_{j+1}=\sigma_j\rho_j\sigma_j\,. \tag{3}$$

1D unitary BGR's are characterized[12] by two parameters θ and Φ:

$$\sigma_j=\exp(i\theta)\,, \qquad \rho_j=\exp\{i[2\theta(j-1)+2\pi e^*\Phi/hc]\}\,. \tag{4}$$

It is easy to see that the minimal change in Φ that leads to the same BGR is Φ_0^*. This is compatible with gauge invariance (the existence of another period Φ_0), if and

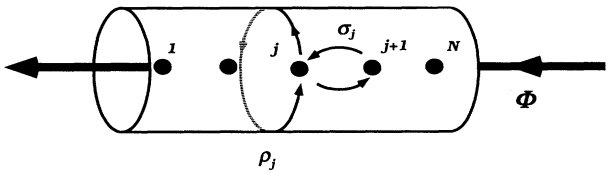

FIG. 1. Braid-group generators on a cylinder. Φ is the central flux through the hole.

only if e^* is an integral multiple of the constituent charge e. Conversely, when the anyon quasiparticles carry fractional charge e^*, for the system to possess a period smaller than Φ_0^*, the wave function must have more than one component. So we are led to consider an M-dimensional representation for σ_i and ρ_j. We assume the anyons obey *scalar statistics:* $\sigma_i = e^{i\theta}\mathbf{I}_M$ with $\mathbf{I}_M$ being the $M\times M$ unit matrix. The representation of ρ_j has a factorized Φ dependence:

$$\rho_j(\Phi) = \exp\{i2\pi e^*\Phi/hc\}T_j\,, \tag{5}$$

where T_j are Φ-independent $M\times M$ matrices satisfying $T_{j+1} = T_j e^{2i\theta}$. If $e^*/e = m/n$ with m,n mutually prime, a smaller period Φ_0 implies the minimal period Φ_0^*/n and the unitary equivalence of the two BGR's,

$$\rho_j(\Phi+\Phi_0^*/n) \simeq \rho_j(\Phi) \quad (j=1,\ldots,N)\,. \tag{6}$$

It is sufficient to have this only for $j=1$. Multiplying the eigenvalues of T_1 by $\exp(i2\pi/n)$ should just shuffle them, so M must be divisible by n. If $M=n$, the eigenvalues of T_1 must consist of n phases $e^{i\lambda_k}$ cyclic under the multiplication of $\exp(i2\pi/n)$: $\lambda_k = \lambda_0 + 2\pi k/n$ $(k=0,1,\ldots,n-1)$ with λ_0 a constant determined by the underlying microscopic physics. While reducible as a BGR, this situation is irreducible under the large gauge transformation which shifts Φ by Φ_0^*/n. The form of $\rho_j(\Phi)$ is thus determined as

$$\rho_j(\Phi) = \exp\{i\lambda_0 + i2\theta(j-1) + i2\pi\Phi e^*/hc\}W\,, \tag{7}$$

with W a diagonal $n\times n$ matrix given by

$$W = \mathrm{diag}\{1,\exp(i2\pi/n),\ldots,\exp[i2\pi(n-1)/n]\}\,, \tag{8}$$

or in an appropriate basis,

$$W = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ 0 & & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{9}$$

In such a basis, each base state changes into another if one moves one anyon around the hole once, and returns to itself after n rounds (up to a possible phase). For a reducible case with $M = nM'$, there are M' copies like this.

Similarly, we can discuss the torus case, in which anyons require a multicomponent wave function.[13,19] If the latter forms an irreducible BGR, gauge invariance relates the anyon charge to its (scalar) statistics θ. In fact, for anyons on a torus with $\sigma_i = e^{i\theta}\mathbf{I}_M$, $(\theta/\pi)\Phi_0^*$ is always a period for Φ. Note that on a torus, besides σ_i there are generators τ_i,ρ_i $(i=1,\ldots,N)$ corresponding to moving the ith particle along one of the fundamental noncontractible loops, as shown in Fig. 2. They satisfy,[13,20] among other relations,

$$\tau_{i+1} = \tau_i e^{-2i\theta}, \quad \rho_{i+1} = \rho_i e^{2i\theta}\,, \tag{10}$$

and τ_i (ρ_i) have a factorized Φ_x (Φ_y) dependence as in (5). Changing Φ_x by $(\theta/\pi)\Phi_0^*$ just gives rise to a phase factor $\exp(2i\theta)$, which shifts τ_i to τ_{i-1}. This only turns the BGR into an equivalent one and therefore does not alter any physical properties. For an irreducible BGR, $M=q$ for $\theta=\pi(p/q)$ with p and q mutually prime, there are no other periods and the minimal period is Φ_0^*/q. So the period Φ_0 required by gauge invariance must be an integral multiple of the latter, and therefore $e^*/e = m/q$ with integer m. If the anyon charge does not satisfy this condition, one has to use a collection of copies of the same q-dimensional irreducible BGR. For the case irreducible under gauge transformations, the number of components of the wave function has to be $M=nq$, where n is the "denominator" of e^*/e, and the minimal period of Φ is Φ_0^*/L, where L is the least common multiple of n and q.

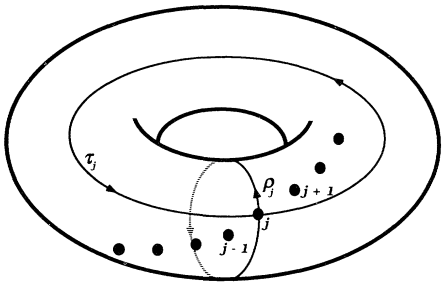

FIG. 2. Braid-group generators ρ_j and τ_j on a torus.

The above conditions have very profound physical implications. First, let us consider the cylinder case. For fractionally charged quasiparticles to exist, their wave function has to have n components with $n>1$. To specify their *many-body* states, besides the positions one needs an extra index, the index of components (or "sheets") s $(=1,\ldots,n)$. We emphasize that this index generally is *not* associated with individual quasiparticles. Equation (7) shows that the operation of moving one anyon (with others held fixed) around the hole is given by, up to some phase, the winding operator W which acts on the sheet indices. Note that $W^n=1$, so its eigenvalues are n cyclic like Z_n. Normally the Hamiltonian H, no matter how complicated it may be, with various interactions, impurities or defects, or external field all included, involves only the coordinates (or other degrees of freedom like spin) of individual particles, but not the sheet index s of the many-body wave function. So H always commutes with W, even if the tunneling process involving the creation and annihilation of a virtual pair of quasiparticle and quasihole propagating across the cylinder is included. So the eigenvalues of W, $\exp(i2\pi k/n)$ or simply $k(\mathrm{mod}n)$, give us a good quantum number, the so-called n-ality, for the many-body energy eigenstates.

Because the (k,Φ) dependence of the eigenvalues of ρ_i is through the combination $\Phi + k\Phi_0^*/n$, the energy or any

physical property satisfies

$$E(k,\Phi,\{\alpha\}) = f(\Phi + k\Phi_0^*/n, \{\alpha\}), \quad (11)$$

where $\{\alpha\}$ is a set of usual quantum numbers. Thus the energy spectrum is actually a collection of n sectors, each corresponding to a one-component system with a central flux differing from each other by Φ_0^*/n, and therefore admits a smaller period Φ_0^*/n.

Moreover, (11) implies the existence of level crossings or spectral flow, since the energy $E(k,\Phi,\{\alpha\})$ with $\{\alpha\}$ fixed is *not necessarily* a function of $\Phi + k\Phi_0^*/n$ with period Φ_0^*/n. Because the topological n-ality is a good quantum number that can never be violated by impurities, defects, or whatever, a gap can never be open at the level-crossing points unless the two levels involved have the same n-ality. This unavoidableness of level crossing violates the Von Neumann–Wigner theorem,[21] which asserts that generically levels do not cross when two parameters in a complex Hamiltonian are varied. Indeed this theorem is not applicable here, because we have a topological quantum number which does not exist generically.

A numerical result showing the pattern of level crossings is given by Fig. 3. (The details will be presented elsewhere.[20]) Note in particular that the three lowest levels flow into each other with a period $\frac{1}{3}$, but for each fixed level the period is 3 times larger. Though this kind of pattern is not typical for an anyon system, there are good reasons to believe that the ground states of a cylindrical FQH system have such a pattern of level crossings. First, we argue that the ground state of a cylindrical FQH system must also carry a topological n-ality k, since it exists for each excited state containing fractionally charged quasiparticles but is not associated with individual quasiparticles. Furthermore, at least when $\nu = 1/q$ (q odd), the energy of a given ground state

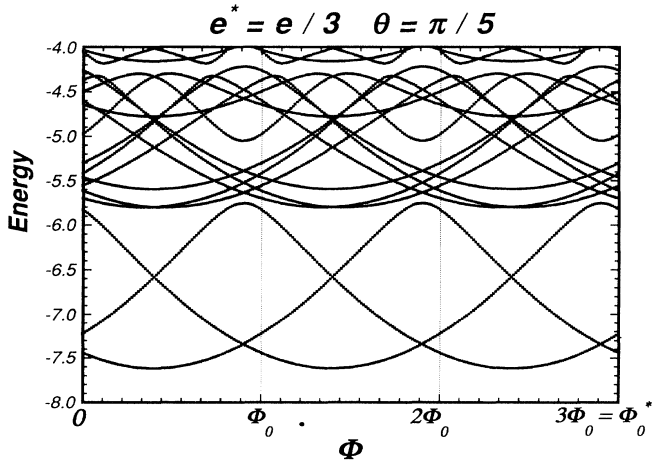

FIG. 3. Spectral flow of a three-anyon system on a 3×3 cylindrical lattice. The statistics is $\pi/5$ and the charge is $e/3$.

should be periodic in the flux with period $\Phi_0^* = qhc/e$, as Thouless recently has shown[6] for the Laughlin state on a sphere with two small holes at the poles. Then the existence of a period $\Phi_0 = \Phi_0^*/q$ implies that each time Φ changes by Φ_0, there is a ground state equivalent to the original one. Hence there must be a group of q equivalent ground states whose energies depend on flux Φ and flow into each other, in a manner like the lowest three levels in Fig. 3. Each of them has a different q-ality. Since the topological Z_q symmetry prevents transitions between the equivalent ground states, energy-level repulsion can never happen and level crossing will persist even in the presence of impurities, defects, and so on. Such a scenario is essentially what was proposed by Tao and Wu[4] several years ago and recently refined by Thouless.[6] This has been shown by Niu, Thouless, and Wu[22] to be sufficient to give the fractional quantization of the Hall conductance in the topological approach. We have a broken symmetry in the sense that originally one would expect that the winding operator W is proportional to the unit matrix, but actually it has unequal eigenvalues. The previous failure to uncover this quantum number is due to the fact that it does not exist on a disk or sphere.

A consequence of our results is that the $\nu = 1/q$ FQH *edge* states on a cylinder must, like the bulk states, carry a Z_q-like quantum number, as pointed out by Wen[23] in a different approach.

When put on a torus,[24] the Laughlin states for $\nu = 1/q$ with q odd correspond to an irreducible BGR.[13] So from gauge invariance alone we can infer that the fractional charge of quasiparticles must be an integral multiple of e/q. In the torus case, we have two noncommuting winding operators W_1 and W_2, similar to (7), respectively, corresponding to moving an anyon along different fundamental loops.[25] ($W_1 W_2 = W_2 W_1 e^{2i\theta}$, since τ_1 and ρ_1 satisfy a similar relation.[13]) They are not symmetries of a finite system: Because of tunneling of virtual quasiparticle and quasihole pairs across the torus, H should contain terms proportional to W_1 and W_2. But such terms are negligible in the thermodynamic limit,[26] so we have two noncommuting symmetries, and the spectrum has q-fold exact degeneracy for each level of anyons. As argued before, we expect the same is true for the electronic ground states that can support fractionally charged anyonic excitations. Note that our topological symmetries giving rise to the ground-state degeneracy are not the magnetic translation operators proposed in the literature:[26,27] The latter can be broken by impurities, but ours cannot.

In summary, we have shown that gauge invariance imposes significant constraints on the structure of the Hilbert space and the spectrum of a system if it supports fractionally charged (anyonic) excitations. The fascinating consequences include the necessity of multicomponent wave functions, the emergence of hidden topological Z_n symmetry with associated quantum number (the n-ality) for many-body eigenstates, and the unavoidable

occurrence of level crossings. In certain situations, gauge invariance relates the fractional charge to anyon statistics. Our topological discussion is quite general and model independent, but does not tell what underlying dynamical mechanism will give rise to the spectrum required by gauge invariance. Finally, it would be interesting to speculate on the possible relevance to quarks, which are also fractionally charged and carry a triality.

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