

The Lie Algebra of the Group of Bisections

◁A Chapter in Synthetic Differential Geometry of Groupoids▷

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Abstract

Groupoids provide a more appropriate framework for differential geometry than principal bundles. Synthetic differential geometry is the avant-garde branch of differential geometry, in which nilpotent infinitesimals are available in abundance. The principal objective in this paper is to show within our favorite framework of synthetic differential geometry that the tangent space of the group of bisections of a microlinear groupoid at its identity is naturally a Lie algebra. We give essentially distinct two proofs for its Jacobi identity.

Keywords:generalized gauge transformation, infinitesimal gauge transformation, topos-theoretic physics, synthetic differential geometry, groupoid, bisection, Lie algebra, Jacobi identity

1 Introduction and Preliminaries

1.1 Groupoids

Groupoids are, roughly speaking, categories whose morphisms are always invertible. They provide differential geometry with a more appropriate, more natural and more general framework than principal bundles. Indeed, principal bundles can be put down as (a special kind of) groupoids, though not in a canonical way. Our standard reference on groupoids is MacKenzie [7]. According to Sharpe [16], Cartan believed that differential geometry was the study of an Ehresmann connection on a principal bundle. We hold that differential geometry is the study of an infinitesimal connection on a groupoid.

Given a groupoid G over a base M with its object inclusion map $\text{id} : M \rightarrow G$ and its source and target projections $\alpha, \beta : G \rightarrow M$, we denote by $\mathfrak{S}(G)$ the totality of sections of $\alpha : G \rightarrow M$, i.e., the totality of mappings $\sigma : M \rightarrow G$ such

that $\alpha \circ \sigma$ is the identity mapping on M . Given $\sigma, \rho \in \mathfrak{S}(G)$, $\sigma * \rho \in \mathfrak{S}(G)$ is defined to be

$$(\sigma * \rho)(x) = \sigma((\beta \circ \rho)(x))\rho(x)$$

for any $x \in M$. It is well known that the set $\mathfrak{S}(G)$ is a monoid with respect to the operation $*$, where the identity is the object inclusion map $\text{id} : M \rightarrow G$. A section σ of $\alpha : G \rightarrow M$ is called a *bisection* of G if $\beta \circ \sigma$ is a bijection of M onto M . We denote by $\mathfrak{B}(G)$ the totality of bisections of G . It is well known that the set $\mathfrak{B}(G)$ is not only a submonoid of $\mathfrak{S}(G)$ but a group by itself, in which the inverse of $\sigma \in \mathfrak{B}(G)$ is given by

$$\sigma((\beta \circ \sigma)^{-1}(x))^{-1}$$

for any $x \in M$. Providing that G is a Lie groupoid over a finite-dimensional smooth manifold M , the familiar statement that the totality $\Gamma(\mathcal{A}G)$ of sections of the Lie algebroid $\mathcal{A}G$ of G is the Lie algebra of the group $\mathfrak{B}(G)$ remains metaphorical at best, unless it is courageous of you to discuss a sort of infinite-dimensional manifolds seriously or to leave the traditional set theory.

It is well known that if P is a principal bundle over M with its structure group H , and if G is its gauge groupoid $\frac{P \times P}{H}$, then $\mathfrak{B}(G)$ can be identified with the group of generalized gauge transformations of P .

1.2 Synthetic Differential Geometry

We know well that nilpotent infinitesimals were vivid and active in the realm of burgeoning analysis during the days of Newton and Leibniz and that they were rampant throughout the works of such pioneers in differential geometry as Cartan, Lie and Riemann. The 19th century witnessed that so-called sclerotic ε - δ arguments were eviscerating nilpotent infinitesimals relentlessly and leaving them in oblivion as logical anathema.

Just as various enlargements of numbers (e.g., from real numbers to complex numbers) have enriched mathematics tremendously, so does topos theory originating in sheaf theory and forcing techniques of axiomatic set theory in the middle of the preceding century. It provides mathematicians with gadgets for enlarging set theory generously. In particular, it endows differential geometers with a kind of El Dorado, in which they can do their work with a cornucopia of nilpotent infinitesimals. Such user-friendly differential geometry is called *synthetic* differential geometry, for which our standard reference is Lavendhomme [6]. To see how to build a Grothendieck topos for synthetic differential geometry, the reader is referred to Kock [3] and Moerdijk and Reyes [10]. Our discussion to follow throughout the rest of the paper is carried out internally within such a Grothendieck topos. In particular, G and M as well as \mathbb{R} (intended for the set of real numbers with a cornucopia of nilpotent infinitesimals pursuant to the general Kock-Lawvere axiom) are assumed to be entities of such a Grothendieck topos and to be microlinear. We denote by D the set $\{d \in \mathbb{R} \mid d^2 = 0\}$ as usual.

1.3 Vector Fields

In synthetic differential geometry a vector field can be looked upon in three distinct but equivalent ways, namely, a section of the tangent bundle, an infinitesimal flow and an infinitesimal transformation, for which the reader is referred to pp.69-71 of Lavendhomme [6]. The equivalence of these three viewpoints is based on the following elementary exponential laws in any topos:

$$(M^D)^M = M^{M \times D} = (M^M)^D$$

These three equivalent viewpoints can easily be uplifted to groupoids, as we will shortly see. The reader should note that since vector fields can not be identified with differential operators in synthetic differential geometry, it is by no means trivial to show that vector fields form a Lie algebra.

Given $x \in M$, we denote by $\mathcal{A}_x G$ the totality of mappings $t : D \rightarrow G$ with $t(0) = \text{id}_x$ and $(\alpha \circ t)(d) = x$ for any $d \in D$. We denote by $\mathcal{A}G$ the set-theoretic union of $\mathcal{A}_x G$'s for all $x \in M$. Proceeding set-theoretically, we can see that $X \in \Gamma(\mathcal{A}G)$ is to be reckoned to be a mapping $X : M \times D \rightarrow G$ with $X(x, 0) = \text{id}_x$ and $\alpha(X(x, d)) = x$ for any $x \in M$. Proceeding set-theoretically further, we have arrived at the most radical viewpoint, which allows us to regard $X \in \Gamma(\mathcal{A}G)$ as a mapping $X : d \in D \mapsto X_d \in \mathfrak{S}(G)$ with $X_0 = \text{id}$, namely, as a tangent vector to $\mathfrak{S}(G)$ at id . We will see in the next section that the image of X lies in $\mathfrak{B}(G) \subset \mathfrak{S}(G)$. We will adopt these equivalent three viewpoints interchangeably. The principal objective in this paper is to show that $\Gamma(\mathcal{A}G)$ or the tangent space of $\mathfrak{B}(G)$ at id forms a Lie algebra with respect to a geometrically motivated Lie bracket. Kock [2] has already established, by exploiting the correspondence between sections of $\mathcal{A}G$ and vertical right-invariant vector fields on G as orthodox differential geometers usually do, that sections of $\mathcal{A}G$ form a Lie algebra. We prefer to do so by uplifting the trinity of the three viewpoints of vector fields to the context of groupoids, which will give due status to the group $\mathfrak{B}(G)$. Since we can not identify elements of $\Gamma(\mathcal{A}G)$ with differential operators in synthetic differential geometry, it is by no means trivial to establish the Jacobi identity for $\Gamma(\mathcal{A}G)$. The discussions in this paper degenerate into classical ones in synthetic differential geometry, providing that G happens to be $M \times M$ (the pair groupoid of M).

1.4 Strong Differences

Kock and Lavendhomme [5] have introduced the notion of strong difference for microsquares into synthetic differential geometry, a good exposition of which can be seen in §3.4 of Lavendhomme [6]. Given two microsquares γ_+ and γ_- on M , their strong difference $\gamma_+ - \gamma_-$ is defined exactly when $\gamma_+|_{D(2)} = \gamma_-|_{D(2)}$, and it is a tangent vector to M . Therefore the operation $-$ is a function from $M^{D^2} \times_{M^{D(2)}} M^{D^2}$ to M^D . The following simple proposition is the prototype of the general Jacobi identity.

Proposition 1 For any $\gamma_1, \gamma_2, \gamma_3 \in M^{D^2}$ with $\gamma_1|_{D(2)} = \gamma_2|_{D(2)} = \gamma_3|_{D(2)}$, we have

$$(\gamma_1 \dot{-} \gamma_2) + (\gamma_2 \dot{-} \gamma_3) + (\gamma_3 \dot{-} \gamma_1) = 0$$

In §2 of our previous paper [12] we have relativized the notion of strong difference $\dot{-}$ for microsquares to get three strong differences $\dot{-}_1, \dot{-}_2$ and $\dot{-}_3$ for microcubes. More specifically, the set M^{D^3} can be identified with the set $(M^D)^{D^2}$ via three isomorphisms $\Psi_i : M^{D^3} \rightarrow (M^D)^{D^2}$ ($i = 1, 2, 3$) to be defined by

$$\begin{aligned}\Psi_1(\gamma)(d_1, d_2)(d) &= \gamma(d, d_1, d_2) \\ \Psi_2(\gamma)(d_1, d_2)(d) &= \gamma(d_1, d, d_2) \\ \Psi_3(\gamma)(d_1, d_2)(d) &= \gamma(d_1, d_2, d)\end{aligned}$$

for any $\gamma \in M^{D^3}$ and any $d_1, d_2, d \in D$. Let $\Phi : M^{D \times D} \rightarrow (M^D)^D$ be defined by

$$\Phi(\gamma)(d_1)(d_2) = \gamma(d_1, d_2)$$

for any $\gamma \in M^{D^2}$ and any $d_1, d_2 \in D$. By way of example, the strong difference $\dot{-} : (M^D)^{D^2} \times_{(M^D)^{D(2)}} (M^D)^{D^2} \rightarrow (M^D)^D$ for microsquares on M^D can be transferred via isomorphisms Ψ_3 and Φ to yield the strong difference $\dot{-}_3 : M^{D^3} \times_{D(2) \times D} M^{D^3} \rightarrow M^{D^2}$ for microcubes on M . I.e., given microcubes

γ_{\pm} on M , their strong difference $\gamma_+ \dot{-}_3 \gamma_-$ is defined exactly when $\gamma_+|_{D(2) \times D} = \gamma_-|_{D(2) \times D}$, and it is a microsquare on M .

We have established the following general Jacobi identity in [12]. The general Jacobi identity was revisited, and its proof was elaborated or reconsidered in [14] and [15].

Theorem 2 Let $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321}$ be six microcubes on M . As long as the following three expressions are well defined, they sum up only to vanish:

$$\begin{aligned}(\gamma_{123} \dot{-}_1 \gamma_{132}) \dot{-} (\gamma_{231} \dot{-}_1 \gamma_{321}) \\ (\gamma_{231} \dot{-}_2 \gamma_{213}) \dot{-} (\gamma_{312} \dot{-}_2 \gamma_{132}) \\ (\gamma_{312} \dot{-}_3 \gamma_{321}) \dot{-} (\gamma_{123} \dot{-}_3 \gamma_{213})\end{aligned}$$

2 The Main Result

Proposition 3 Let $X \in \Gamma(\mathcal{AG})$. For any $(d, d') \in D(2)$, we have

$$X_{d+d'} = X_d * X_{d'}$$

Proof. It suffices to note that both sides coincide in case $d = 0$ or $d' = 0$.

■

Corollary 4 *Let $X \in \Gamma(\mathcal{A}G)$. For any $d \in D$, both $\beta \circ X_d$ and $\beta \circ X_{-d}$ are the inverse to each other, so that $X_d \in \mathfrak{B}(G)$.*

Corollary 5 *Let $X \in \Gamma(\mathcal{A}G)$. For any $d \in D$, we have*

$$(X_d)^{-1} = X_{-d}$$

Proposition 6 *The group $\mathfrak{B}(G)$ of bisections of G is microlinear. The \mathbb{R} -module $\Gamma(\mathcal{A}G)$ is isomorphic to the tangent space to $\mathfrak{B}(G)$ at its identity.*

Proof. For the former statement, it suffices to note that the set $\mathfrak{B}(G)$ of bisections of G is in bijective correspondence with the set

$$\{(\varphi, \psi) \in \mathfrak{S}(G) \times \mathfrak{S}(G) \mid \varphi * \psi = \text{id} \text{ and } \psi * \varphi = \text{id}\},$$

which is undoubtedly microlinear. The latter statement follows from Corollary 4. ■

Proposition 7 *Let $X, Y \in \Gamma(\mathcal{A}G)$. The mapping*

$$\lambda : D(2) \rightarrow \mathfrak{B}(G)$$

which coincides on the axes of $D(2)$ with X and Y respectively is given by

$$\lambda(d, d') = X_d * Y_{d'} = Y_{d'} * X_d$$

Therefore we have

$$(X + Y)_d = X_d * Y_d = Y_d * X_d$$

Proof. It suffices to note that both mappings $\lambda', \lambda'' : D(2) \rightarrow \mathfrak{B}(G)$ given by

$$\lambda'(d, d') = X_d * Y_{d'}$$

$$\lambda''(d, d') = Y_{d'} * X_d$$

coincide on the axes of $D(2)$ with X and Y respectively. ■

Given $X, Y \in \Gamma(\mathcal{A}G)$, we define $\lambda : D^2 \rightarrow \mathfrak{B}(G)$ to be

$$\lambda(d_1, d_2) = Y_{-d_2} * X_{-d_1} * Y_{d_2} * X_{d_1}$$

for any $(d_1, d_2) \in D(2)$. It is easy to see that

$$\lambda(d, 0) = Y_0 * X_{-d_1} * Y_0 * X_{d_1} = X_{-d_1} * X_{d_1} = \text{id}$$

$$\lambda(0, d) = Y_{-d_2} * X_0 * Y_{d_2} * X_0 = Y_{-d_2} * Y_{d_2} = \text{id}$$

for any $d \in D$. Therefore, by Proposition 3 in §2.3 of [6], there exists a unique element of $\Gamma(\mathcal{A}G)$, called the *Lie bracket* of X and Y and denoted by $[X, Y]$, such that for any $d_1, d_2 \in D$, $[X, Y]_{d_1 d_2}$ is

$$Y_{-d_2} * X_{-d_1} * Y_{d_2} * X_{d_1}$$

Now we come to the main result of this paper.

Theorem 8 *The \mathbb{R} -module $\Gamma(\mathcal{A}G)$ is a Lie algebra with respect to $[\cdot, \cdot]$.*

Proof. It suffices to establish that for any $a \in \mathbb{R}$ and any $X, Y, Z \in \Gamma(\mathcal{A}G)$, we have

$$[aX, Y] = a[X, Y] \quad (1)$$

$$[X + Y, Z] = [X, Z] + [Y, Z] \quad (2)$$

$$[X, Y] = -[Y, X] \quad (3)$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad (4)$$

Here we deal only with the first three identities, leaving the last identity to the succeeding two sections. For (1) we have to note that

$$\begin{aligned} & [aX, Y]_{d_1 d_2} \\ &= Y_{-d_2} * (aX)_{-d_1} * Y_{d_2} * (aX)_{d_1} \\ &= Y_{-d_2} * X_{-ad_1} * Y_{d_2} * X_{ad_1} \\ &= [X, Y]_{ad_1 d_2} \end{aligned}$$

The identity (2) follows directly from (1) by dint of Proposition 10 of §1.2 of Lavendhomme [6]. For (3) It suffices to show equivalently that

$$[X, Y] + [Y, X] = 0,$$

which follows from the following computation:

$$\begin{aligned} & ([X, Y] + [Y, X])_{d_1 d_2} \\ &= [X, Y]_{d_1 d_2} * [Y, X]_{d_1 d_2} \quad (\text{by Proposition 7}) \\ &= Y_{-d_2} * X_{-d_1} * Y_{d_2} * X_{d_1} * X_{-d_1} * Y_{-d_2} * X_{d_1} * Y_{d_2} \\ &= \text{id} \end{aligned}$$

■

3 The First Approach to the Jacobi Identity

We will mimic Nishimura's [11] shortest and simplest proof of the Jacobi identity of vector fields in synthetic differential geometry. The basic idea therein is that the familiar relationship between the Lie bracket of two vector fields and the Lie derivative of a vector field with respect to the other reduces the well-known Jacobi identity of vector fields to something akin to the Leibniz rule in elementary differential calculus.

Given $X \in \Gamma(\mathcal{A}G)$ and $\sigma \in \mathfrak{B}(G)$, we denote by $\sigma_* X$ the tangent vector $d \in D \mapsto \sigma * X_d * \sigma^{-1} \in \mathfrak{B}(G)$ to $\mathfrak{B}(G)$ at id . Given $X, Y \in \Gamma(\mathcal{A}G)$, we define the *Lie derivative of Y with respect to X* , denoted by $\mathbf{L}_X Y$, to be the unique tangent vector to $\mathfrak{B}(G)$ at id such that

$$(X_{-d})_* Y - Y = d\mathbf{L}_X Y$$

for any $d \in D$. The relationship between $\mathbf{L}_X Y$ and $[X, Y]$ is as simple as follows:

Theorem 9 For any $X, Y \in \Gamma(\mathcal{A}G)$, we have $\mathbf{L}_X Y = [X, Y]$.

Proof. For any $d, d' \in D$, we have

$$\begin{aligned} & ((X_{-d})_* Y - Y)_{d'} \\ &= X_{-d} * Y_{d'} * X_d - Y_{d'} \\ &= Y_{-d'} * X_{-d} * Y_{d'} * X_d \\ & \text{[by Proposition 7]} \\ &= [X, Y]_{dd'} \\ &= (d[X, Y])_{d'} \end{aligned}$$

Therefore

$$(X_{-d})_* Y - Y = d[X, Y]$$

for any $d \in D$, which is tantamount to saying that $\mathbf{L}_X Y = [X, Y]$. ■

With the above theorem in mind, we can say that the following theorem is none other than Jacobi's identity of $\Gamma(\mathcal{A}G)$ in disguise.

Theorem 10 For all $X, Y, Z \in \Gamma(\mathcal{A}G)$, we have

$$\mathbf{L}_X [Y, Z] = [\mathbf{L}_X Y, Z] + [Y, \mathbf{L}_X Z] \quad (5)$$

Proof. It is easy to see that for any $d \in D$, we have

$$(X_{-d})_* [Y, Z] = [(X_{-d})_* Y, (X_{-d})_* Z]$$

Therefore we have

$$\begin{aligned} & (X_{-d})_* [Y, Z] - [Y, Z] \\ &= [(X_{-d})_* Y, (X_{-d})_* Z] - [Y, Z] \\ &= [d\mathbf{L}_X Y + Y, d\mathbf{L}_X Z + Z] - [Y, Z] \\ &= d([\mathbf{L}_X Y, Z] + [Y, \mathbf{L}_X Z]) \end{aligned}$$

Consequently we have

$$\mathbf{L}_X [Y, Z] = [\mathbf{L}_X Y, Z] + [Y, \mathbf{L}_X Z],$$

as was expected. ■

The reader should note that the proof of the above theorem is no more difficult than the synthetic proof of Leibniz's rule in elementary differential calculus, which is to be seen e.g., in Proposition 1 of §1.2 of Lavendhomme [6]. Now Jacobi's identity in $\Gamma(\mathcal{A}G)$ with respect to $[\cdot, \cdot]$ is an easy consequence of the preceding theorem, as we will see just below.

Proof. (The Jacobi identity) It suffices to establish the following version of the Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

for all $X, Y, Z \in \Gamma(\mathcal{A}G)$. Since we have

$$\begin{aligned} [X, [Y, Z]] &= \mathbf{L}_X[Y, Z] \\ [X, Y] &= \mathbf{L}_X Y \\ [X, Z] &= \mathbf{L}_X Z \end{aligned}$$

by dint of Theorem 9, the desired equality is no other than a reformulation of (5) of Theorem 10. ■

4 The Second Approach to the Jacobi Identity

Given $X_1, \dots, X_n \in \Gamma(\mathcal{A}G)$, we define n -microcube $X_n \otimes \dots \otimes X_1$ on $\mathfrak{B}(G)$ to be

$$(X_n \otimes \dots \otimes X_1)(d_1, \dots, d_n) = (X_n)_{d_n} * \dots * (X_1)_{d_1}$$

for any $(d_1, \dots, d_n) \in D^n$. Given an n -microcube γ on $\mathfrak{B}(G)$ and a permutation ε of the first n natural numbers, we define another n -microcube $\sum_{\varepsilon} \gamma$ on $\mathfrak{B}(G)$ to be

$$\left(\sum_{\varepsilon} \gamma \right)(d_1, \dots, d_n) = \gamma(\varepsilon(d_1), \dots, \varepsilon(d_n))$$

for any $(d_1, \dots, d_n) \in D^n$. A cycle ε of length k is usually denoted by $(j, \varepsilon(j), \varepsilon^2(j), \dots, \varepsilon^{k-1}(j))$, where j is not fixed by ε .

Proposition 11 *Let $X, Y \in \Gamma(\mathcal{A}G)$. Both $Y \otimes X$ and $\sum(X \otimes Y)$ coincide on the axes, and we have*

$$[X, Y] = Y \otimes X - \sum_{(12)}(X \otimes Y)$$

Proof. Since we have

$$\begin{aligned} (Y \otimes X)(d_1, d_2) &= Y_{d_2} * X_{d_1} \\ \sum_{(12)}(X \otimes Y)(d_1, d_2) &= X_{d_1} * Y_{d_2} \end{aligned}$$

for any $(d_1, d_2) \in D^2$, it is easy to see that

$$\begin{aligned} (Y \otimes X)(d, 0) &= X_d = \sum_{(12)}(X \otimes Y)(d, 0) \\ (Y \otimes X)(0, d) &= Y_d = \sum_{(12)}(X \otimes Y)(0, d) \end{aligned}$$

for any $d \in D$, so that the right-hand side of the desired equality is meaningful. We define a function $\lambda : D^3 \setminus \{(1, 3), (2, 3)\} \rightarrow \mathfrak{B}(G)$ to be

$$\lambda(d_1, d_2, d_3) = X_{d_1} * [X, Y]_{d_3} * Y_{d_2}$$

Then it is easy to see that

$$\begin{aligned}
\lambda(d_1, d_2, 0) &= X_{d_1} * Y_{d_2} = \sum_{(12)} (X \otimes Y)(d_1, d_2) \\
\lambda(d_1, d_2, d_1 d_2) &= X_{d_1} * [X, Y]_{d_1 d_2} * Y_{d_2} \\
&= X_{d_1} * [Y, X]_{d_1(-d_2)} * Y_{d_2} \\
&= X_{d_1} * (X_{-d_1} * Y_{d_2} * X_{d_1} * Y_{-d_2}) * Y_{d_2} \\
&= Y_{d_2} * X_{d_1} \\
&= (Y \otimes X)(d_1, d_2)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&(Y \otimes X - \sum_{(12)} (X \otimes Y))(d) \\
&= \lambda(0, 0, d) \\
&= [X, Y]_d
\end{aligned}$$

for any $d \in D$, which completes the proof. ■

Proposition 12 *Given $X, Y, Z \in \Gamma(\mathcal{AG})$, we define six microcubes on $\mathfrak{B}(G)$ as follows:*

$$\begin{aligned}
\gamma_{123} &= Z \otimes Y \otimes X \\
\gamma_{132} &= \sum_{(23)} (Y \otimes Z \otimes X) \\
\gamma_{213} &= \sum_{(12)} (Z \otimes X \otimes Y) \\
\gamma_{231} &= \sum_{(123)} (X \otimes Z \otimes Y) \\
\gamma_{312} &= \sum_{(132)} (Y \otimes X \otimes Z) \\
\gamma_{321} &= \sum_{(13)} (X \otimes Y \otimes Z)
\end{aligned}$$

Then the right-hand sides of the following three identities are meaningful, and all the three identities hold:

$$\begin{aligned}
[X, [Y, Z]] &= (\gamma_{123} \overset{\cdot}{\underset{1}{-}} \gamma_{132}) - (\gamma_{231} \overset{\cdot}{\underset{1}{-}} \gamma_{321}) \\
[Y, [Z, X]] &= (\gamma_{231} \overset{\cdot}{\underset{2}{-}} \gamma_{213}) - (\gamma_{312} \overset{\cdot}{\underset{2}{-}} \gamma_{132}) \\
[Z, [X, Y]] &= (\gamma_{312} \overset{\cdot}{\underset{3}{-}} \gamma_{321}) - (\gamma_{123} \overset{\cdot}{\underset{3}{-}} \gamma_{213})
\end{aligned}$$

Now it is easy to see that the Jacobi identity in $\Gamma(\mathcal{AG})$ follows readily from the general Jacobi identity.

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