（6）「量子力学の非局所性と統計的推測」に関する研究報告

Quantum Infodynamics: Description of Dynamics of Open Quantum Systems Based on Quantum Information Theory

Vladimír BUŽEK*, **[1]

* *Research Center for Quantum Information, Institute of Physics, Slovak Academy of Sciences,*
Dúbravská cesta 9, 845 11 Bratislava, Slovakia
** *Faculty of Informatics, Masaryk University, Botanická 68a, 602 00 Brno, Czech Republic*

Abstract

In my lecture I will analyze how information encoded in a quantum system is "diluted" in reservoirs. I will present a simple model that would allow us to understand various aspects of dynamics of open systems interacting with an environment.

Keywords: dynamics of open quantum systems, quantum information processing, master equations

I will analyze how information encoded in a quantum system is "diluted" in reservoirs. In particular, I will describe a *universal* quantum homogenizer [1, 2, 3], which is a quantum machine that takes as an input a system qubit initially in the state ρ and a set of N reservoir qubits initially prepared in the same state ξ. In the homogenizer the system qubit sequentially interacts with the reservoir qubits via the *partial swap* transformation. The homogenizer realizes, in the limit sense, the transformation such that at the output each qubit is in an arbitrarily small neighborhood of the state ξ irrespective of the initial states of the system and the reservoir qubits. This means that the system qubit undergoes an evolution that has a fixed point, which is the reservoir state ξ. The model of the homogenizer allows us to understand various aspects of the dynamics of open systems interacting with non-equilibrium environments. In particular, the reversibility *vs* or irreversibility of the dynamics of the open system is directly linked to specific (classical) information about the order in which the reservoir qubits interacted with the system qubit. I will analyze possible physical realization of quantum homogenization [4]. In addition I will discuss how entanglement is established between particles involved in homogenization process [5]. Finally I will show how a master equation governing dynamics of the system qubit during the homogenization process can be derived.

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[1]E-mail: buzek@savba.sk

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Can Quantum Non-locality Improve Quantum Estimation?

Masahito HAYASHI*[1]

*ERATO Quantum Computation and Information Project, JST, Tokyo

Abstract

In quantum system, we can obtain information only through quantum measurement. For example, we need an quantum measurement in order to estimate the density matrix of the system of interest. In this setting, the choice of quantum measurement is essential because the quantum measurement causes state demolition. Of course, in order to reliable inference, we need to prepare several numbers of the systems of interest which are equivalent with each other. Thus, we formulate this problem as statistical inference, and we call it quantum estimation when we estimate the unknown density matrix of the system of interest. This problem can be regarded as the quantum version of the estimation of probability distribution in the mathematical statistics. At least in principle, we can use quantum measurement having a quantum correlation between systems of interest for this purpose. It is an interesting problem from non-locality of quantum mechanics as well as from statistical inference whether using such a correlational quantum measurement improve the performance of the estimation. In this talk, we propose a correlational quantum measurement improving the estimation error.

Keywords: Quantum Estimation, Quantum Correlation, Quantum Gaussian State, Number Detection, Heterodyne Detection

1 Formulation of Quantum Estimation

First, we give a formulation of theoretically available quantum measurements. While the quantum state is described by density matrix (operator) ρ, i.e., a positive semi-definite matrix whose trace is 1, quantum measurement is described by a positive operator valued measure (POVM) $M = \{M_\omega\}$ on the Hilbert space $\mathcal{H}$ of the system of interest, where POVM $M = \{M_\omega\}$ consists of the partition of unity I by positive semi-definite matrixes (operators) M_ω on $\mathcal{H}$, i.e. this partition satisfies

$$\sum_\omega M_\omega = I. \tag{1}$$

When we perform the quantum measurement corresponding to POVM $M = \{M_\omega\}$ to the system whose density matrix is ρ, we obtain the data ω with the probability:

$$\mathrm{P}^M_\rho(\omega) := \operatorname{Tr} \rho M_\omega.$$

[1]E-mail: masahito@qci.jst.go.jp

When the data set $\{\omega\}$ is continuous, an integral replaces $\sum$ in the equation (1).

In quantum estimation, we assume that the unknown density matrix is included by a parameterized density family $\mathcal{S} = \{\rho_\theta \in \mathcal{S}(\mathcal{H}) | \theta \in \Theta \subset \mathbb{R}\}$. In the estimation of probability distribution in mathematical statistics, we need to assume a parameterized distribution family for reliable estimation, because it is very difficult to treat reliable estimation from finite data under an infinite dimensional family. The similar difficulty appears in the quantum case. In this assumption, any estimator is described by a pair of POVM $M = \{M_\omega\}_{\omega \in \Omega}$ and a function $\hat{\theta}$ from data set Ω to the parameter set Θ. Usually the estimation error of $(M, \hat{\theta})$ is evaluated by mean square error (MSE) $v_\theta(M, \hat{\theta})$ which is given as follows in the continuous and one-parametric case:

$$v_\theta(M, \hat{\theta}) := \int (\hat{\theta}(\omega) - \theta)^2 \operatorname{Tr} \rho_\theta M(d\omega).$$

In the multi-parametric case, the i-th MSE is given by

$$\int_{\Omega_n} (\hat{\theta}^i_n(\omega) - \theta^i)^2 \operatorname{Tr} \rho_\theta^{\otimes n} M^n(d\omega).$$

We usually assume an n-independent identically density (i.i.d.) $\rho^{\otimes n} := \underbrace{\rho \otimes \cdots \otimes \rho}_{n}$ and the i.i.d. parameterized family $\mathcal{S}_n = \{\rho_\theta^{\otimes n} \in \mathcal{S}(\mathcal{H}^{\otimes n}) | \theta \in \Theta \subset \mathbb{R}\}$. In this case, any estimator is described by a pair of POVM $M^n = \{M^n_{\omega_n}\}_{\omega_n \in \Omega_n}$ on the tensor product space $\mathcal{H}^{\otimes n}$ and the function $\hat{\theta}_n$ from Ω_n to Θ. In the one-parametric case, its MSE is given by

$$v_\theta(M^n, \hat{\theta}_n) := \int_{\Omega_n} (\hat{\theta}_n(\omega) - \theta)^2 \operatorname{Tr} \rho_\theta^{\otimes n} M^n(d\omega).$$

We often assume that the estimator $M^n, \hat{\theta}_n)$ satisfies the unbiased condition:

$$\int_{\Omega_n} \hat{\theta}_n \operatorname{Tr} \rho_\theta^{\otimes n} M^n(d\omega) = \theta, \quad \forall \theta \in \Theta. \tag{2}$$

In the asymptotic case, this condition often is replaced by the asymptotically unbiased condition:

$$\begin{aligned} \lim_{n \to \infty} \int_{\Omega_n} \hat{\theta}_n \operatorname{Tr} \rho_\theta^{\otimes n} M^n(d\omega) &= \theta, \quad \forall \theta \in \Theta \\ \lim_{n \to \infty} \frac{d}{d\theta} \int_{\Omega_n} \hat{\theta}_n \operatorname{Tr} \rho_\theta^{\otimes n} M^n(d\omega) &= 1, \quad \forall \theta \in \Theta. \end{aligned}$$

Under the asymptotically unbiased condition, the asymptotic behavior of MSE can be characterized as

$$v_\theta(M^n, \hat{\theta}_n) \lesssim \frac{1}{J_\theta n},$$

i.e., the inequality

$$\lim_{n \to \infty} n v_\theta(M^n, \hat{\theta}_n) \leq \frac{1}{J_\theta}$$

holds, where J_θ is called the SLD Fisher information and defined by[1, 2]

$$J_\theta := \operatorname{Tr} L_\theta^2 \rho_\theta, \quad \frac{d\rho_\theta}{d\theta} = \frac{1}{2}(L_\theta \rho_\theta + \rho_\theta L_\theta)$$

The lower bound given in (3) is attained by adaptive measurement M^n defined by

- $M^n = \{M^{(1)}_{\omega_1} \otimes \cdots \otimes M^{(n)}_{\omega_1,\dots,\omega_{n-1},\omega_n}\}_{(\omega_1,\dots,\omega_n)\in\Omega^n}$,
- $M^{(j)}_{\omega_1,\dots,\omega_{j-1},\omega_j} = (M^{(j)}_{\omega_1,\dots,\omega_{j-1},\omega_j})^\dagger \geq 0$,
- $\int_\Omega M^{(j)}_{\omega_1,\dots,\omega_{j-1},\omega_j} \, d\omega_j = I \quad (k = 1, \dots, n)$.

Since any quantum measurement having quantum correlation does not satisfy the above condition, such a correlational measurement does not improve MSE in the asymptotic setting in the one-parametric case.

2 Quantum Gaussian States Family

Thus, we need to discuss a multi-parametric state family to research the quantum correlational effect in state estimation. In this paper, we treat the quantum Gaussian states family as the typical example having such an effect. Let's consider a Hilbert space $L^2(\mathbb{R})$, whose normal orthogonal system is given by Hermite functions $\{|n\rangle\}$. In this system, the coherent vector $|\alpha\rangle_a := \sum_{k=0}^\infty e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^k}{\sqrt{k!}} |k\rangle (\alpha \in \mathbb{C})$ is an eigen vector of annihilation operator a which defined as $a|n\rangle = \sqrt{n}|n-1\rangle$. The state $\rho_{\alpha,0} := |\alpha\rangle_a {}_a\langle\alpha|$ is called a coherent state and its Gaussian mixture $\rho_{\zeta,N} := \frac{1}{\pi N} \int e^{-\frac{|\alpha-\zeta|^2}{N}} |\alpha\rangle_a {}_a\langle\alpha| \, d\alpha$ is called a quantum Gaussian state. In the physical system of photon whose frequency is ν, the quantum Gaussian state $\rho_{\zeta_0 e^{-\frac{ct}{2}-i\nu t}, \overline{n}(1-e^{-ct})}$ is the final state of the following Master equation when the initial state is the coherent state $|\zeta_0\rangle_a {}_a\langle\zeta_0|$:

$$\frac{d\rho}{dt} = -i[\nu a^\dagger a, \rho] - \frac{c\overline{n}}{2}(a a^\dagger \rho - 2a^\dagger \rho a + \rho a a^\dagger) - \frac{c(\overline{n}+1)}{2}(a a^\dagger \rho - 2a\rho a^\dagger + \rho a^\dagger a),$$

where c is coupling constant, and $\overline{n}$ is the average photon number.

In the quantum Gaussian states family $\{\rho_{\zeta,N} | \zeta \in \mathbb{C}, N > 0\}$, the following POVM is very useful: One is the number detection $\mathbf{N}$, the other is the heterodyne detection $\mathbf{H}$:

$$\begin{aligned} \mathbf{N} &: \quad k \mapsto |k\rangle\langle k| \\ \mathbf{H} &: \quad \alpha \mapsto |\alpha\rangle_a {}_a\langle\alpha|. \end{aligned}$$

Concerning the subfamily $\{\rho_{0,N} | \zeta \in \mathbb{C}, N > 0\}$, it is sufficient to estimate only the parameter N. In this n-i.i.d. case, the following estimator is the optimal[3, 4]: First, we perform the number detection $\mathbf{N}$ individually and obtain n data $k_1, \dots, k_n$. The final estimate is decided as $\hat{k} := \sum_{i=1}^n \frac{k_i}{n}$. In this case, the unbiased condition (2) is satisfied, and the MSE is calculated as

$$\frac{1}{n}\sum_{k=0}^\infty (k-N)^2 \operatorname{Tr} \rho_{0,N} |k\rangle\langle k| = \frac{N(N+1)}{n}.$$

$N(N+1)$ coincides with the inverse of the SLD Fisher information of this family.

Next, we consider the subfamily $\{\rho_{\zeta,N}|\zeta \in \mathbb{C},\}$ which depends on N, i.e., we estimate only the complex parameter ζ. In this case, the average data by the heterodyne detection is optimal estimator. In other word, we perform the heterodyne detection $\mathbf{H}$ individually, and obtain the n data $\alpha_1, \dots \alpha_n$, and decide the final estimate as the average $\hat{\zeta}_n := \sum_{i=1}^n \frac{\alpha_i}{n}$. It also satisfies the unbiased condition, and its MSE is calculated as

$$\frac{1}{n}\int_{\mathbb{C}} |\zeta-\alpha|^2 \operatorname{Tr} \rho_{\zeta,N}|\alpha\rangle_a\, {}_a\langle\alpha|\, d\alpha = \frac{1}{n}\int_{\mathbb{C}} |\zeta|^2\, {}_a\langle\alpha|\rho_{0,N}|\alpha\rangle_a\, d\alpha = \frac{N+1}{n}.$$

This estimator is optimal.

However, in the estimation in the quantum Gaussian states family $\{\rho_{\zeta,N}|\zeta \in \mathbb{C}, N > 0\}$, we have to discuss the simultaneous estimation of N and ζ. If we perform number detection, the obtained information about ζ is very small. If we perform the heterodyne detection, the estimation of N is boiled down to the estimation of variance of normal distribution $\frac{1}{\pi(N+1)}e^{-\frac{|\alpha-\zeta|^2}{N+1}}$. In this case, $\hat{N}_{1,n} := \frac{1}{n-1}\sum_{i=1}^n |\alpha_i - \hat{\zeta}|^2 - 1$ is suitable for the estimator of N . Its MSE is calculated as

$$\begin{aligned}
&\mathrm{E}_{\zeta,N}\left(\frac{1}{n-1}\sum_{i=1}^n |\alpha_i-\hat{\zeta}|^2-(N+1)\right)^2 = \mathrm{E}_{0,N}\left(\frac{1}{n-1}\sum_{i=1}^n |\alpha_i-\hat{\zeta}|^2-(N+1)\right)^2\\
=&\mathrm{E}_{0,N}\left(\frac{1}{n-1}\sum_{i=1}^n |\alpha_i-\hat{\zeta}|^2\right)^2-(N+1)^2 = \frac{n^2}{(n-1)^2}\mathrm{E}_{0,N}\left(\frac{1}{n}\sum_{i=1}^n |\alpha_i-\hat{\zeta}|^2\right)^2-(N+1)^2\\
=&\frac{n^2}{(n-1)^2}\mathrm{E}_{0,N}\left(\frac{1}{n}\sum_{i=1}^n |\alpha_i|^2-|\hat{\zeta}|^2\right)^2-(N+1)^2\\
=&\frac{n^2}{(n-1)^2}\mathrm{E}_{0,N}\left(\left(\frac{1}{n}\sum_{i=1}^n |\alpha_i|^2\right)^2-2\frac{1}{n}\sum_{i=1}^n |\alpha_i|^2|\hat{\zeta}|^2+|\hat{\zeta}|^4\right)-(N+1)^2 = \frac{(N+1)^2}{n-1}.
\end{aligned}$$

The MSE $\frac{(N+1)^2}{n-1}$ is larger than the one by the number detection. This trade-off is solved as follows[5]: First, we perform the time evolution as:

$$\rho_{\zeta,N}^{\otimes n} \mapsto \rho_{\sqrt{n}\zeta,N} \otimes \rho_{0,N}^{\otimes n-1}.$$

This time evolution can be performed by using beam splitter. Next, we perform the heterodyne detection in the first system whose state is $\rho_{\sqrt{n}\zeta,N}$, and perform the number detection other systems whose state is $\rho_{0,N}$. Letting the these data be $\hat{\zeta}$ and $k_1 \dots, k_{n-1}$, we can use data $\hat{\zeta}$ as the estimate of ζ and the average $\hat{N}_n := \sum_{i=1}^{n-1} \frac{k_i}{n-1}$ as the estimate of N. The former MSE is equivalent with the optimal one, and the later MSE equals $\frac{N(N+1)}{n-1}$, which is asymptotically equivalent with the optimal one.

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Restricted Linear Statistical Estimation

Yoshiyuki TSUDA*[1]

*Imai Quantum Computation and Information Project, ERATO, JST, 5-28-3-201, Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan

Abstract

We consider statistical estimation problems using only linear maps between sample spaces and parameter spaces. Several kinds of restrictions of the linear maps will be introduced. As one example, this formulation includes the estimation problem of the location parameter of complex amplitudes of multimode coherent state with the thermal Gaussian noise.

Keywords: Covariance matrix, Lie groups, Unbiased estimation

1 Introduction

Let $X = \mathbb{R}^{n_1}$ and $\Theta = \mathbb{R}^{n_2}$ are real vector spaces ($n_1 \geq n_2$) and let $\Phi : \Theta \to X$ be a real linear injective map. Let $\mathcal{T}$ be a set of real linear maps. Let V_1 and V_2 are respectively a $n_1 \times n_1$ real matrix and a $n_2 \times n_2$ real matrix. For $\Psi \in \mathcal{T}$, let

$$R(\Psi) = \mathbf{tr}(\Psi V_1 {}^t\Psi V_2). \tag{1}$$

We consider a problem to minimize $R(\Psi)$ for $\Psi \in \mathcal{T}$. In this paper, we do not distinguish transposition and adjoint.

If V_1 and V_2 are positive symmetric and the restriction for $\Psi \in \mathcal{T}$ is

$$\Psi \in \mathcal{T} \Longleftrightarrow \Psi \circ \Phi = \text{identity on } \Theta, \tag{2}$$

then this problem is equivalent to a classical statistical estimation problem considering only linear unbiased estimators. In that case, X is a sample space, Θ is a parameter space, $\Phi : \Theta \to X$ determines the mean value of the data with respect to the true value $\theta \in \Theta$ of the parameter of a family of probability distributions on X having a constant variance V_1. V_2 is a weight for the risk function. We can consider V_1^{-1} as a positive metric on X, and Φ pull-backs it and we obtain a positive metric $F = {}^t\Phi V_1 \Phi$ on V_2. If $\Psi_0 \in \mathcal{T}$ is optimal in the sense that Ψ_0 minimizes R, if and only if

$$\Psi_0 V_1 {}^t\Psi_0 = F^{-1} \tag{3}$$

holds. Such Ψ_0 is called the best linear linear estimator (BLUE). If the family of distribution is normal (or Gaussian), then F is equal to the Fisher information matrix, and Eq. (3) says that Ψ_0 attains the Cram er-Rao bound.

[1]E-mail: tsuda@qci.jst.go.jp

Making the condition of $\Psi \in \mathcal{T}$ more strict, we obtain various kinds of problems minimizing $R(\Psi)$ for $\Psi \in \mathcal{T}$. One of these types of problems is equivalent to the estimation of the complex amplitude of multimode Gaussian state which is one of the most fundamental problems of quantum estimation [1, 2, 3, 4], which is closely related to asymptotic theory of quantum estimation [5, 6, 7, 8].

Through this paper, we identify linear maps and bilinear forms with matrices by naturally fixed basis of linear spaces.

2 Restriction

Define Y as a vector subspace of X that satisfies $X = \mathrm{Im}\Phi \oplus Y$. Let W and W_Y be the restrictions of V_1 to $\mathrm{Im}\Phi$ and Y, respectively, and suppose that V_1 can be written as $V_1 = W \oplus W_Y$. This condition for V_1 corresponds to the case that the main system $\mathrm{Im}\Phi$ and the ancillary system Y are independent. As a restriction of $\Psi \in \mathcal{T}$, we require that $\Phi \in \mathcal{T}$ satisfies the condition (2). Then, it is sufficient to consider the cases $\Psi \in \mathcal{T}$ is of the form

$$\Psi = \Phi^{-1} \oplus \Psi_Y \tag{4}$$

where each element of the direct sum is just the restriction of Ψ to the corresponding subspaces $\mathrm{Im}\Phi$ and Y, so our choice is limit to the freedom of Ψ_Y. Since $R(\Psi)$ can be written as

$$R(\Psi) = \mathbf{tr}(\Phi^{-1}W\,{}^t\Phi^{-1}V_2) + \mathbf{tr}(\Psi_Y W_Y\,{}^t\Phi^{-1}V_2), \tag{5}$$

the problem is reduced to minimize

$$R_Y(\Psi_Y) = \mathbf{tr}(\Psi_Y W_Y\,{}^t\Phi^{-1}V_2). \tag{6}$$

Let $\mathcal{T}_Y$ be a class of linear maps $\Psi : Y \to \Theta$ satisfying some conditions. In this paper, the restrictions for $\mathcal{T}_Y$ is that $\mathcal{T}_Y$ is a Lie group.

3 Non compact cases

First, we consider the case that V_2 is positive symmetric and W_Y is the identity. In this case, if $\mathcal{T}_Y$ is a compact Lie group, $R_Y(\Psi_Y)$ is constant and the problem is obvious, so $\mathcal{T}_Y$ must be a non compact Lie group. Since $R_Y(\Psi_Y)$ depends only on $\Psi_Y\,{}^t\Psi_Y$, the problem is to minimize a function on $\mathcal{T}_Y/H$ where H is a maximal compact subgroup of $\mathcal{T}_Y$.

3.1 $Sp(n, \mathbb{R})$

The most important case is that $\mathcal{T}_Y$ is a symplectic group $Sp(n, \mathbb{R})$ (n_2 is even and $n = n_2/2$). This problem is equivalent to choose the direction of homodyne detections for a complex n dimensional mode.

The optimal solution is

$$\Psi_Y = \sqrt{\sqrt{V_2}^{-1}\sqrt{\sqrt{V_2}\,{}^tAV_2A\sqrt{V_2}}\sqrt{V_2}^{-1}} \tag{7}$$

where A is an antisymmetric matrix

$$A = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad (I_n : n \text{ dimensional identity}). \tag{8}$$

3.2 $SO(p,q)$

The previous example is related to the antisymmetric matrix A. In contrast, the case of $\mathcal{T}_Y = SO(p,q)$ $(p+q=n_2)$ is related to a symmetric matrix

$$S = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}. \tag{9}$$

Similarly to (7) we can see that

$$\Psi_Y = \sqrt{\sqrt{V_2}^{-1}\sqrt{\sqrt{V_2}SV_2S\sqrt{V_2}}\sqrt{V_2}^{-1}}., \tag{10}$$

In proofs for the optimality of (7) and (10), we just use the Lagrange's multiplier method. That method itself can only guarantee a necessary condition for the local optimality. In our cases, however, we can see that the optimal solution can be seen as a fixed point for a involution of the Lie groups. Since such a fixed point is unique if the Lie group is non compact ([9]), we can see that the local condition is sufficient for the global optimality.

3.3 $SL(n,\mathbb{R})$

If $\mathcal{T}_Y = SL(n,\mathbb{R})$, the optimal solution Ψ_Y is a linear transformation

$$V_2 \mapsto {}^t\Psi_Y V_2 \Psi_Y = (\det V_2) I_n. \tag{11}$$

3.4 Other non compact cases

The above examples are all irreducible real Riemannian symmetric spaces. We can consider others like $SL(n,\mathbb{H})$, $SU(p,q;\mathbb{C})$, $U(P,q;\mathbb{H})$, $SO(n,\mathbb{H})$, and complex forms. Let a Lie group G be one of them. Then, there is an extensions of the group G $G \subset GL(*,K) = F$ which is minimal among $K \in \{\mathbb{C},\mathbb{H}\}$. (The inclusion relation is, for example, $GL(n,\mathbb{H}) \subset GL(2n,\mathbb{C})$.) If $V_2 \in F$, then we can obtain the optimal solutions in the same way as (7), (10), (11).

4 Compact cases

Let $\mathcal{T}_Y$ is a compact Lie group in $F = GL(n,K)$ $(K \in \{\mathbb{R},\mathbb{C},\mathbb{H}\})$. Suppose that $V_2, W_Y \in F$ and they are not the identity. We consider $\mathbb{H} \subset \mathbb{C}^2 \subset \mathbb{R}^4$ and any element in $GL(*,K)$ is faithfully represented by a real matrix by the natural way. So the adjoint is just the transposition of the matrix. We assume that V_2 and W_Y are both symmetric or both antisymmetric according to the transposition for adjoint.

A necessary condition for Ψ_Y to be optimal is $V_2 W_Y = W_Y V_2$. Since $V_2, W_Y \in F$, there is $\Psi' \in \mathcal{T}$ such that

$$\Psi W_Y {}^t\Psi V_2 = V_2 \Psi W_Y {}^t\Psi. \tag{12}$$

Let $\mathcal{W}$ be the set of $\Psi W_Y {}^t\Psi$ satisfying (12). There is a discrete subgroup G of $\mathcal{T}$ isomorphic to the symmetric group $\mathfrak{S}_n$. Since $\Psi W_Y {}^t\Psi$ and V_2 are both symmetric or both antisymmetric, $\mathrm{tr}(\Psi W_Y {}^t\Psi V_2)$ can be written as

$$\mathrm{tr}(\Psi W_Y {}^t\Psi V_2) = \begin{cases} \sum_{i=1}^n \lambda_i \mu_i & V_2 \text{ and } W_Y \text{ are symmetric} \\ -\sum_{i=1}^n \lambda_i \mu_i & V_2 \text{ and } W_Y \text{ are antisymmetric} \end{cases} \tag{13}$$

where λ_i and μ_i are the absolute values of eigenvalues of V_2 and W_Y, respectively, and

$$\lambda_1 \leq \cdots \leq \lambda_n. \tag{14}$$

Therefore, if V_2 and W_Y are symmetric, then the optimal solution is one that permutes $\mu_1, ..., \mu_n$ to $\mu'_1, ..., \mu'_n$ being

$$\mu'_1 \geq \cdots \geq \mu'_n, \tag{15}$$

and, if V_2 and W_Y are symmetric, then

$$\mu'_1 \leq \cdots \leq \mu'_n. \tag{16}$$

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Entanglement as Interaction in Advance: Dense Coding and Teleportation

David MERMIN*[1]

* *Cornell University, Ithaca, New York*

Abstract

Dense coding and teleportation are quantum protocols that exploit shared entanglement to facilitate communication between two distant parties. The same communication can be achieved classically, but only by introducing additional interaction between the parties. I shall derive these exotic quantum protocols from their straightforward classical versions, showing that the interaction needed classically is still present in the quantum case, hidden away as the interaction necessary to prepare the shared entanglement. What remains remarkable in the quantum case is that this part of the interaction can take place *before* the information to be transmitted has been chosen.

Keywords: Entanglement, Dense Coding, Teleportation, Quantum circuit

The circuit diagrams developed to represent the sequence of gates used in a quantum computation can also provide economical and insightful derivations of certain quantum mechanical procedures, without the need for any analytical manipulations of quantum states.

I shall illustrate this by deriving two quantum protocols, dense coding [1,2] and teleportation [3,4], from simple classical circuit diagrams that transparently achieve the same results as the quantum protocols by exploiting direct dynamical interactions between the two parties. (By a classical circuit diagram I mean one whose initial and final states are classically meaningful [i.e. states of the "computational basis"], every gate of which takes classically meaningful states into other classically meaningful states.)

Circuits realizing the quantum protocols are extracted from their parent classical circuits by expanding each classical gate into products of gates, some of which are quantum gates that do not preserve classically meaningful states. By then allowing rearrangements in the order in which commuting gates act one arrives at circuit-theoretic versions of the quantum protocols.

This procedure makes clear the classical parentage of the quantum protocols. It shows that the quantum protocols have just as much dynamical interaction as the classical ones. But a crucial part of that interaction is absorbed into the preparation of the initial shared entangled state. This part of the interaction can be executed before anybody has even specified the two bits to be encoded in a single qubit or the unknown state to be teleported. The role of entanglement in the protocols is to store interaction, unrealized, until the time arrives when it is needed.

All the gates used in these constructions are simple enough for somebody unfamiliar with quantum circuit diagrams (and unacquainted with dense coding or teleportation) to be able to follow the derivations.

[1]E-mail: ndm4@cornell.edu

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Entanglement of identical particles and reference phase uncertainty

J.A. VACCARO*, H.M. WISEMAN**, and F. ANSELMI*

*Quantum Physics Group, STRC, University of Hertfordshire, College Lane, Hatfield AL10 9AB, U.K.

**Centre for Quantum Computer Technology, Centre for Quantum Dynamics, School of Science, Griffith University, Brisbane, Queensland 4111 Australia.

Abstract

We have recently introduced a measure of the entanglement of identical particles, E_{P}, based on the principle that entanglement should be *accessible* for use as a resource in quantum information processing. We show here the strong connection between local particle number conservation and reference phase uncertainty and we demonstrate the complementarity between local particle number conservation and the accessible entanglement.

Keywords: entanglement, identical particles, quantum information, superselection rules

Introduction. Entanglement is an essential resource for quantum information processing. The non separability of the wavefunction of two distinct systems is the usual hallmark of an entangled state. However, the symmetric or antisymmetric wavefunctions of collections of identical particles is *inherently* non separable. A crucial question then is how to quantify the entanglement of identical particles. The approach of Zanardi and others [1] is to calculate the entanglement of the quantum field modes occupied by the particles, E_{M}. But E_{M} can be non-zero even for the case of a single particle. An alternate approach [2] is to examine the non separability of the wavefunction beyond that required by symmetrization or antisymmetrization. The difficulty here, however, is that there is no fixed partition into distinct systems.

The approach we take [3] is to insist that the entanglement be *accessible* in the sense that it could be used as a resource for quantum information processing. This requires strict *partite separation* and the entanglement to be accessible using *local operations* only. These restrictions are equivalent to a local particle number superselection rule. Entanglement in the presence general superselection rules have been explored further in [4].

In this paper, we briefly review our measure of bipartite entanglement of identical particles [3]. We then explore the relationship between local particle number conservation and reference phase uncertainty, and discuss the complementarity between the accessible entanglement and the conservation of local particle number.

Entanglement of identical particles. We imagine two well-separated parties, Alice and Bob, sharing a collection of N identical particles, such as atoms or electrons etc., which are in the pure state $|\Psi\rangle_{\mathrm{AB}}$. Many particles of interest in quantum information processing are composite (e.g. atoms) and so their number is not conserved in general. However, there are underlying quantities that are strictly conserved such as electric charge and lepton number. These underlying conservation laws imply that the number $N\eta$ is conserved, where η is a conserved-quantity number (e.g. lepton number) associated with each composite particle. For brevity, we refer in the following to N being conserved, although the arguments apply to the strict conservation of $N\eta$.

The entanglement shared between two parties can not increase under the action of local (separable) operations performed independently at Alice's and Bob's sites. Local operations include unitary transformations as well as non unitary measurements. In contrast, the shared entanglement can increase under the action of nonlocal operations, such as transporting a quantum system from one site to the other. Entanglement measures must, therefore, exclude the possibility of nonlocal manipulations and so particle exchange between the parties is forbidden. Hence, the number of particles at each site is unchanged by local operations:

$$[\hat{N}_i, \hat{L}_i] = 0 \tag{1}$$

where $\hat{N}_i$ is the particle number operator and $\hat{L}_i$ is an arbitrary Hamiltonian or observable local to site i. For the shared entanglement to be accessible for use in quantum information processing, we require that it be able to be transferred (e.g. locally teleported) to a set of local quantum registers, such as regular (distinguishable) qubits at each site. The local transfer involves local operations only. After the transfer the entanglement in the local quantum registers can be measured by conventional means. The *accessible entanglement* in the shared particle system is the maximum entanglement that can be transferred to the local quantum registers. The actual number of particles at each site can be measured after the transfer operation without disturbing the accessible entanglement. But since $\hat{N}_i$ commutes with all $\hat{L}_i$, we could equally measure the local particle number first, and then transfer the entanglement to the local quantum registers, without altering the accessible entanglement. The measurement of local particle number $\hat{N}_i$ projects the state $|\Psi\rangle_{\rm AB}$ into the state $|\Psi_n\rangle_{\rm AB}$ with probability P_n, where

$$|\Psi_n\rangle_{\rm AB} = \frac{\hat{\Pi}_n|\Psi\rangle_{\rm AB}}{\sqrt{P_n}},\ P_n = {}_{\rm AB}\langle\Psi|\hat{\Pi}_n|\Psi\rangle_{\rm AB}\ , \tag{2}$$

and $\hat{\Pi}_n$ is the projector onto states with n particles at Alice's site and $N-n$ at Bob's. Here $|\Psi_n\rangle_{\rm AB}$ represents field modes occupied by a fixed number of particles at each site; the entanglement of these modes can be transferred to the local registers and is therefore accessible. This tells us that the accessible entanglement in the state $|\Psi\rangle_{\rm AB}$ is quantified by [3]:

$$E_{\rm P}(|\Psi\rangle_{\rm AB}) \equiv \sum\nolimits_n P_n E(|\Psi_n\rangle_{\rm AB}) \tag{3}$$

where $E(|\Psi_n\rangle_{\rm AB}) = S(\hat{\rho}_{\rm A}^{(n)})$, $S(\hat{\rho})$ is the binary von Neumann entropy $-{\rm Tr}(\hat{\rho}\log_2\hat{\rho})$, and $\hat{\rho}_{\rm A}^{(n)}$ is the reduced density matrix $\hat{\rho}_{\rm A}^{(n)} = {\rm Tr}_B[|\Psi_n\rangle_{\rm AB}\langle\Psi_n|]$. In essence, Eq. (3) results from a *local* particle number superselection rule in that, due to Eq. (1), the coherences between subspaces of differing local particle number are not observable by local means.

We now give some examples. We use the second quantized notation where $|n,m\rangle_{\rm AB}$ represents n particles in a field mode at Alice's site and m particles in a field mode at Bob's site. Sharing a single particle between Alice and Bob, $\frac{1}{\sqrt{2}}(|0,1\rangle_{\rm AB} + |1,0\rangle_{\rm AB})$, would yield the projected states $|0,1\rangle_{\rm AB}$ or $|1,0\rangle_{\rm AB}$ with equal probability after a measurement of local particle number at each site, and so the accessible particle entanglement is $E_{\rm P} = 0$. Independently sharing 2 particles, $\frac{1}{2}(|0,1\rangle_{\rm AB} + |1,0\rangle_{\rm AB}) \otimes (|0,1\rangle_{\rm AB} + |1,0\rangle_{\rm AB})$, gives the projected states $|0,1\rangle_{\rm AB} \otimes |0,1\rangle_{\rm AB}$, $|1,0\rangle_{\rm AB} \otimes |1,0\rangle_{\rm AB}$ and $\frac{1}{\sqrt{2}}(|0,1\rangle_{\rm AB} \otimes |1,0\rangle_{\rm AB} + |1,0\rangle_{\rm AB} \otimes |0,1\rangle_{\rm AB})$ with probabilities 1/4, 1/4 and 1/2, respectively. The first two states have no entanglement whereas the last has 1 ebit and so the average entanglement is $E_{\rm P} = 1/2$. This illustrates a striking feature of $E_{\rm P}$ in that it is super additive: $E_{\rm P}(|\Psi\rangle \otimes |\Phi\rangle) \geq E_{\rm P}(|\Psi\rangle) + E_{\rm P}(|\Phi\rangle)$. The super additivity is a direct consequence of the inherent indistinguishability of the particles.

Reference phase uncertainty. The absence of the entanglement in the shared single-particle state, $\frac{1}{\sqrt{2}}(|0,1\rangle_{\rm AB} + |1,0\rangle_{\rm AB})$, can be traced to uncertainty in the difference between local reference phases at the two sites. Reference phase uncertainty has recently been discussed in relation to continuous variable teleportation [5]. It occurs here when we transfer the state of the shared particle to a pair of regular (distinguishable) qubits, as follows.

Let Alice have a very large number $M \gg 1$ of ancillary particles in a particular field mode. An operation is then performed which shares the particles with another mode at Alice's site to produce the state $\sum_{n=0}^{M} \sqrt{\mathcal{P}_n}|M-n,n\rangle_{\rm A}$ where $\mathcal{P}_n$ is approximately Poissonian $\mathcal{P}_n \propto \frac{\mu^{2n}e^{-\mu^2}}{n!}$ and μ^2 is approximately the average number of particles in the second field mode. We can rewrite this state as being proportional to

$$\int_{2\pi} |\psi(\theta)\rangle_{\rm A}\hat{\Pi}_M|\mu e^{i\theta}\rangle_{\rm A}\frac{d\theta}{2\pi} \tag{4}$$

where $|\mu e^{i\theta}\rangle = e^{-\mu^2/2}\sum_{n=0}^{\infty}\mu^n e^{in\theta}|n\rangle/\sqrt{n!}$ is a coherent state, $\hat{\Pi}_M = \sum_{n=0}^{M}|n\rangle\langle n|$ projects onto the subspace of states with up to M particles and $|\psi(\theta)\rangle = \sum_{n=0}^{M} e^{-in\theta}|M-n\rangle/\sqrt{M+1}$ is a phase state [6]. We now imagine that the particles in the second mode are shared equally with a further $m-1$ modes at Alice's site via linear "beam splitter" interactions. Each component $\hat{\Pi}_M|\mu e^{i\theta}\rangle_{\rm A}$ in the superposition in Eq. (4) is replaced with $(\hat{\Pi}_M|\mu' e^{i\theta}\rangle_{\rm A})^{\otimes m}$, approximately, where $\mu' = \mu/\sqrt{m}$. A corresponding process is performed at Bob's site with his local ancillary system, resulting in the combined ancillary state

$$\int_{2\pi} |\psi(\theta)\rangle_{\rm A}(\hat{\Pi}_M|\mu' e^{i\theta}\rangle_{\rm A})^{\otimes m}\frac{d\theta}{2\pi} \otimes \int_{2\pi} |\psi(\phi)\rangle_{\rm B}(\hat{\Pi}_M|\mu' e^{i\phi}\rangle_{\rm B})^{\otimes m}\frac{d\phi}{2\pi}\ . \tag{5}$$

The transfer of the state of the shared particle $\frac{1}{\sqrt{2}}(|0,1\rangle_{\rm AB} + |1,0\rangle_{\rm AB})$ to a pair of regular qubits $|\underline{0,0}\rangle_{\rm AB}$ can now be performed using the method of Mayers [7]. We use an underline to distinguish the states of a regular qubit, $|\underline{0}\rangle$, $|\underline{1}\rangle$, (such as two orthogonal electronic states of an atom) from the Fock states of the field modes $|0\rangle$, $|1\rangle$, $\cdots$. Alice performs a local CNOT operation using her local shared-particle mode as the control and her local regular qubit as the target. She then performs a controlled operation with her regular qubit as the control and the state $|\psi(\theta)\rangle_{\rm A}$ of the first ancillary mode and her single-particle mode as the target, where the latter particle is transferred to the first ancillary mode if her qubit is in the state $|\underline{1}\rangle$ and no change otherwise. Bob performs the corresponding operations at his site. Tracing over the rest of the system gives the end result that, with probability $1 - 2/(M+1)$, the qubit registers are in the mixed state:

$$\frac{1}{2}\int_{2\pi}\frac{d\theta}{2\pi}\int_{2\pi}\frac{d\phi}{2\pi}\left(e^{-i\theta}|\underline{1,0}\rangle_{\rm AB} + e^{-i\phi}|\underline{0,1}\rangle_{\rm AB}\right)\left(e^{i\theta}{}_{\rm AB}\langle\underline{1,0}| + e^{i\phi}{}_{\rm AB}\langle\underline{0,1}|\right)$$
$$= \frac{1}{2}\left(|\underline{0,1}\rangle_{\rm AB}\langle\underline{0,1}| + |\underline{1,0}\rangle_{\rm AB}\langle\underline{1,0}|\right)\ . \tag{6}$$

As predicted [3], there is no entanglement here.

Entanglement and particle number complementarity. It is interesting to see how the entanglement can be increased by exchanging particles and violating both the restriction to local operations and the local particle number superselection rule. The entanglement increases if we fix the relative phase difference between the two terms in the superposition state on the right-hand side of Eq. (6). One way of doing this is to measure the phase difference $\theta - \phi$ between the Alice's and Bob's coherent states in Eq. (5) and apply a corresponding phase shift to one of the qubits. The measurement will require an interference experiment that will collapse the superpositions in Eq. (5) to a specific value of $\theta - \phi$. Any resolution of the phase difference $\Delta(\theta - \phi)$ requires a minimum uncertainty ΔN in the number of particles transferred from one site to the other. The uncertainty in particle number is given by $\Delta N = \sqrt{\overline{N}}$ where $\overline{N}$ is the mean particle number of the coherent state transferred between sites. The phase difference is approximately Gaussian distributed with a variance of $\Delta^2(\theta - \phi) \approx 1/(2\overline{N})$ for large-amplitude coherent states [6]. After an ideal phase difference measurement and the local phase shifting operation, the state of the qubits becomes

$$\frac{1}{2}\left[|\underline{0,1}\rangle_{\mathrm{AB}}\langle\underline{0,1}| + C\left(|\underline{1,0}\rangle_{\mathrm{AB}}\langle\underline{0,1}| + |\underline{0,1}\rangle_{\mathrm{AB}}\langle\underline{1,0}|\right) + |\underline{1,0}\rangle_{\mathrm{AB}}\langle\underline{1,0}|\right] \tag{7}$$

where $C = e^{-1/(4\overline{N})}$ is the average of $e^{i(\theta-\phi)}$. The entanglement of formation [8] of this state is given by $E_F \approx \frac{1}{2}(1 + |C|^2)$ for $\overline{N} \gg 1$. Clearly E_F approaches 1 ebit as $\overline{N}$, the mean number transferred, increases. The Heisenberg uncertainty relation for phase and number differences gives $|C|^2 = |\langle e^{i(\hat{\phi}_{\mathrm{A}} - \hat{\phi}_{\mathrm{B}})}\rangle|^2 \le \langle \Delta^2(\hat{N}_{\mathrm{A}} - \hat{N}_{\mathrm{B}})\rangle / [1 + \langle \Delta^2(\hat{N}_{\mathrm{A}} - \hat{N}_{\mathrm{B}})\rangle]$ for physical states, where $\hat{\phi}_i$ and $\hat{N}_i$ are the Pegg-Barnett phase operator [6] and particle number operator, respectively, for site i, and $\langle \Delta^2 \hat{Q}\rangle$ is the variance in $\hat{Q}$. In particular, $\langle \Delta^2(\hat{N}_{\mathrm{A}} - \hat{N}_{\mathrm{B}})\rangle = 2\overline{N}$ for the coherent states. The entanglement E_F can be increased for a fixed mean number $\overline{N}$ of particles transferred, by preparing particular ancillary states with a larger value of $\langle \Delta^2(\hat{N}_{\mathrm{A}} - \hat{N}_{\mathrm{B}})\rangle$.

Another method for increasing the accessible entanglement is to share N particles between Alice and Bob in the state $\sum_{n=0}^{N} c_n |N - n, n\rangle_{\mathrm{AB}}$. For example, $E_{\mathrm{P}} = 1 - 1/(2\overline{N} + 1)$ for the combined state $\frac{1}{\sqrt{2}}(|0,1\rangle_{\mathrm{AB}} + |1,0\rangle_{\mathrm{AB}}) \otimes \sum_{n=0}^{N} \frac{1}{\sqrt{N+1}} |N - n, n\rangle_{\mathrm{AB}}$ where, here, $\overline{N} = N/2$ is the mean number of ancillary particles at each site. Other methods are currently being explored.

Discussion. Only manipulations by local operations are permissible when quantifying the accessible entanglement in a system. Operations that change local particle number are therefore forbidden and this gives rise to a local particle-number superselection rule. This concept underlies the definition of E_P, the entanglement of identical particles. Transferring particles from one site to another violates the restriction to local operations and the local superselection rule, and in doing so increases the accessible entanglement. This demonstrates the complementarity between local particle number conservation and accessible entanglement.

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Bell inequalities detect efficient entanglement

Valerio SCARANI[1], Antonio ACÌN[2], Nicolas GISIN[3], Michael M. WOLF[†4]

Group of Applied Physics, University of Geneva (Switzerland)
† Technische Universität Braunschweig (Germany)

Abstract

Bell inequalities have been introduced in physics as a tool to discriminate Quantum Physics from alternative models inspired from the classical world. But recent results show that Bell's inequalities have a legitimacy also *within* Quantum Physics: a state that violate an inequality is entangled enough to be "efficient" in quantum information protocols. In this talk, I will focus on one of these results: Bell inequalities are linked to the degree of distillability of N-qubit entangled states.

Keywords: Bell inequalities, multi-particle entanglement, distillability.

Quantum correlations were noticed to be astonishing by Einstein-Podolski-Rosen and by Schrödinger back in 1935. In particular, the EPR paper stressed that the predicted correlations could not be explained by exchange of a signal, since the entangled particles could be at an arbitrary distance from one another. If signal exchange is excluded, in the classical world we know only another mechanism to establish correlations: common preparation at the source. This was ruled out by John Bell in 1964: the predicted quantum correlations violate a condition ("Bell's inequality", BI) that should hold if the correlations were established at the preparation. All the experiments performed since the Aspect experiment in 1982 confirm quantum physics.

Nowadays, although one should not forget the detection loophole until its direct experimental test, for most physicists the debate on quantum correlations is closed: *entanglement does exist*, and moreover it has been recognized as a *ressource* needed to perform tasks that would classically impossible [1]. The question arises then: is it still worth investigating BI actively? Of course, the interpretational content of the BI should shape any physicist's view of the world, but in the research, shouldn't we rather concentrate on quantum information processing? The main message of the present work is good news: there is no need for a disjunctive answer! We should indeed pay great attention to quantum information processing, *and* Bell inequalities are a powerful tool for this investigation. Specifically, it has been proved that:

1. Consider the quantum key distribution protocols with entangled states, in the presence of Eve's optimal individual eavesdropping: if the quantum state shared by Alice and Bob violates a Bell inequality, then the error rate is low enough to allow one-way error correction and privacy amplification — that is, efficient secret key extraction from the raw key [2].

[1]E-mail: valerio.scarani@physics.unige.ch
[2]Presently at: Institute of Photonic Sciences, Barcelona, Spain; e-mail: antonio.acin@upc.es
[3]E-mail: nicolas.gisin@physics.unige.ch
[4]Presently at: Max-Planck-Institut, Garching, Germany; e-mail: michael.wolf@mpq.mpg.de

2. If a N-qubit state violates a Mermin-Klyshko inequality, a particular inequality in the Werner-Wolf-Zukowski-Brukner (WWZB) family [3], it can be used for multi-partite secret key sharing protocols [4].

3. If a N-qubit state violates a WWZB inequality there is some distillable entanglement in the state; moreover, the amount of the violation is associated to the degree of distillability. A similar result holds for the Uffink inequality [5].

4. To some two qu-dits Bell inequality one can associate a communication complexity protocol that works more efficiently with quantum than with classical information [6].

In all these cases, the Bell inequality acts as a *detector of efficient entanglement*. It is an interesting open problem, to determine to which extent this holds in general.

Here I focus on item 3 of the previous list. In this extended abstract I will just give the basic ideas. The references for all the statements below can be found in the published papers [5].

The entanglement of a quantum state ρ is distillable if, out of many copies of it, one can extract maximally entangled states (two-qubit singlets) using only local operations and classical communication (LOCC). Operationally, this means the following: if a source S produces a state which is weakly entangled but distillable, then one can build a new source S', that is less efficient but produces strongly entangled states, by simply appending local devices to the ports of S and allowing the partners to communicate. In other words, if we have S, then we can build S' and run any quantum information protocol like teleportation. The notion of distillability is not trivial because, in all quantum composed systems but $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$, there exist so-called bound-entangled states, that are entangled but not distillable.

Here we study quantum systems composed of $N > 2$ qubits, described by the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes N}$. In such a case, when the system is composed of more than two sub-systems, the notion of distillability is not even univoque. The strongest requirement is "full distillability": any two partners can distill a singlet by LOCC. The weakest requirement is "bipartite distillability": the N partners split into two groups of n_A and $n_B = N - n_A$ partners, and the state is distillable with respect to this partition n_A/n_B. Specifically: within each group, the most general transformations are allowed; but only classical communication is allowed between one group and the other.

Obviously, if N is large, all kind of intermediate situations between "full distillability" and "bipartite distillability" can be found. We have demonstrated an interesting, quantitative link between this hierarchy or *degree of distillability* and the amount of violation of the WWZB Bell inequalities [3]. This family of inequalities is characterized by the following two facts: (i) They are correlation inequalities; more specifically, they are linear functions of correlation coefficients. In the quantum formalism, the corresponding Bell operator B_N is a linear combinations of $\sigma_{n_1} \otimes \sigma_{n_2} \otimes ... \sigma_{n_N}$. (ii) Two and only two settings per site are considered: that is, the inequality uses only two settings $\hat{n}_j$ and $\hat{n}'_j$ for each qubit j.

This family of inequalities has an interesting property. Let us set the local boundary at $\mathrm{Tr}(\rho B_N) = 1$. Then the maximal amount of violation is $\mathrm{Tr}(\rho B_N) = 2^{(N-1)/2}$, obtained only for some inequalities in the family and for the N-qubit GHZ state. In-between, the amount of violation defines some boundaries. For instance, if M qubits among the N are entangled,

$\mathrm{Tr}(\rho B_N)$ cannot be larger than $2^{(M-1)/2}$; in other words, if the violation exceeds $2^{(M-1)/2}$, one can be sure that at least $M+1$ qubits are entangled.

The Uffink inequalities are a generalization of the WWZB family: they are also correlation inequalities using two settings per qubit, but the condition is quadratic, or alternatively, is linear but an additional parameter is free for optimization apart from the $2N$ settings.

We can now explain our results, that are summarized in two theorems.

Theorem 1. *Any N-qubit state ρ that violates one of the WWZB inequalities is at least bipartite distillable (Fig. 1).*

In clear: if one can find a Bell operator B_N in the WWZB family such that $\mathrm{Tr}(\rho B_N) > 1$, then there exists at least one partition of the N parties into two groups n_A/n_B such that the two groups can distill a singlet. Loosely speaking, Theorem 1 ensures that in any state that violates a WWZB Bell's inequality there is some distillable entanglement, although in order to extract it some parties may need to join.

Theorem 2. *Suppose that the N-qubit state ρ violates one of the WWZB inequalities by an amount of*

$$Tr(\rho B_N) > 2^{\frac{N-p}{2}} \tag{1}$$

for a given integer p such that $2 \leq p \leq N$. Then any *ensemble of p qubits can be divided into two subgroups, and a singlet can be distilled between these subgroups by means of operations which are local with respect to the $N-p+2$ parties (Fig. 2).*

In particular, if $\mathrm{Tr}(\rho B_N) > 2^{\frac{N-2}{2}}$, then the state is fully distillable. In a similar way, we could also prove that the violation of the Uffink inequality implies full distillability.

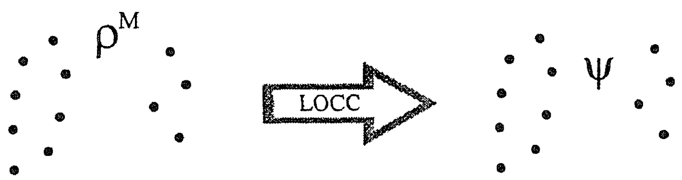

Figure 1: Theorem 1 illustrated for $N = 12$ qubits. The state $|\Psi\rangle$ is a two-qubit maximally entangled state.

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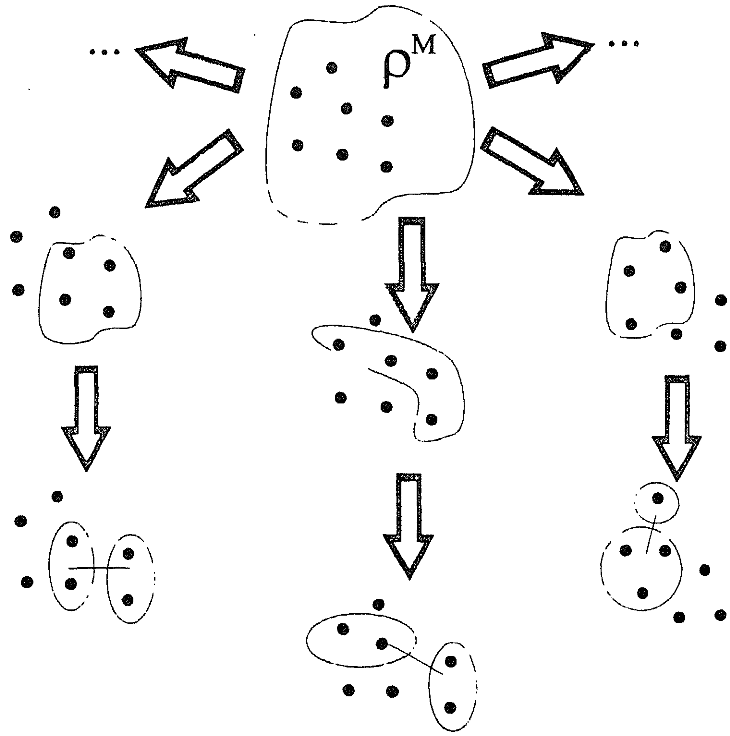

Figure 2: Theorem 2 illustrated for $N = 7$ qubits and a violation measured by $p = 4$. As in Fig. 1, the hollow arrows represent LOCC operations, and the thick links represent the singlet state. Any three $N - p = 3$ qubits can perform a suitable measurement and communicate its result to the others. The four remaining qubits end up with a state which is bipartite distillable. Note that only three out of the $C_3^7 = 7!/(3!\,4!) = 35$ possibilities are shown.

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Anomalous quantum states in finite macroscopic systems

Akira SHIMIZU*[1], Takayuki MIYADERA**[2], and Akihisa UKENA*[3]

*Department of Basic Science, University of Tokyo
3-8-1 Komaba, Tokyo 153-8902, Japan
** Department of Information Sciences, Science University of Tokyo
Chiba 278-8510, Japan.

Abstract

We consider finite macroscopic systems, i.e., systems of large but finite sizes, which we believe are poorly understood as compared with small systems and infinite systems. We focus on pure states that do not have the 'cluster property,' which means that they have long-range correlations. Such a pure state is entangled macroscopically, and is quite anomalous in view of many-body physics because it does not approach any pure state in the infinite-size limit. However, we often encounter such anomalous states when studying finite macroscopic systems, such as quantum computers with many qubits and finite systems that will exhibit symmetry breaking in the infinite-size limit. We study stabilities of such anomalous states in general systems. In contrast to the previous works, we obtain general and universal results, by making full use of the locality of the theory. Using the general results, we discuss the appearance of a classical world, the origin of symmetry-breaking in finite systems, and a key to realizing quantum computers with many qubits.

Keywords: macroscopic system, cluster property, entanglement, decoherence

A quantum system composed of N qubits can handle data of size 2^N. In quantum information theory, one is usually interested in the asymptotic behaviors of quantities of interest as the size of the data is increased. To study the asymptotic behavior, one must take the data size large but finite. Therefore, quantum information theory treats quantum systems with large but *finite* degrees of freedom, $1 \ll N < +\infty$. We call such a system a 'finite macroscopic system.'

Finite macroscopic systems are poorly understood as compared with small systems and infinite systems. In particular, there exist anomalous pure states that do not exist in infinite systems. For example, consider two Néel states of an antiferromagnet consisting of N spins,

$$|\mathrm{AF}_+\rangle \equiv |1010\cdots\rangle, \ |\mathrm{AF}_-\rangle \equiv |0101\cdots\rangle, \tag{1}$$

where $|1\rangle$ and $|0\rangle$ denote the spin-up and -down states, respectively. In the antiferromagnetic Ising model, these two states are the ground states of a finite system, having the same energy,

[1]E-mail: shmz@ASone.c.u-tokyo.ac.jp
[2]E-mail: miyadera@is.noda.tus.ac.jp
[3]E-mail: ukena@ASone.c.u-tokyo.ac.jp

each approaching a 'vacuum state' of an infinite system as $N \to \infty$. It is obvious that a superposition of them,

$$|\mathrm{C}\rangle \equiv \frac{1}{\sqrt{2}}|\mathrm{AF}_+\rangle + \frac{1}{\sqrt{2}}|\mathrm{AF}_-\rangle \tag{2}$$

is also a pure state having the same energy as $|\mathrm{AF}_\pm\rangle$. However, it can be rigorously proved that this state does not approach any pure state as $N \to \infty$, but approaches a mixed state. A convenient way of judging this is to examine the 'cluster property,' which is defined in infinite systems as follows [1]: A quantum state is said to have the cluster property if $\langle \delta\hat{a}(x)\delta\hat{b}(y)\rangle \to 0$ as $|x-y| \to \infty$ for any local operators $\hat{a}(x)$ and $\hat{b}(y)$ at x and y, respectively, where $\delta\hat{a}(x) \equiv \hat{a}(x) - \langle\hat{a}(x)\rangle$ and $\delta\hat{b}(y) \equiv \hat{b}(y) - \langle\hat{b}(y)\rangle$ [2]. Here, by a *local operator at x* we mean a finite-order polynomial of field operators and their finite-order derivatives at position x [3]. For *infinite* quantum systems, there is a fundamental theorem [1]: *Any pure state has the cluster property.* Since $\langle \mathrm{C}|\delta\hat{\sigma}_z(x)\delta\hat{\sigma}_z(y)|\mathrm{C}\rangle \not\to 0$, $|\mathrm{C}\rangle$ with $N \to \infty$ does not have the cluster property (whereas $|\mathrm{AF}_\pm\rangle$ do), hence $|\mathrm{C}\rangle$ does not approach any pure state as $N \to \infty$. Therefore, $|\mathrm{C}\rangle$ with finite N is a quite anomalous state. Although it may be possible to judge simply by intuition that $|\mathrm{C}\rangle$ is anomalous, because it has a very simple form, we can investigate using the cluster property whether more complicated pure states are anomalous or not.

Since we are interested in a finite macroscopic system, for which $1 \ll N < +\infty$, we generalize the concept of the cluster property to the case of finite systems. It reads roughly as follows: (The precise definition is described in Ref. [4].) Let $\Omega(\epsilon)$ be the size of the region outside which correlations of fluctuations of any local operators become smaller than ϵ. We consider a sequence of systems with various values of N and associated states, where the states with different values of N are similar to each other [5]. We say that the states (for large N) of the sequence have the cluster property if for any $\epsilon > 0$ one can make $\Omega(\epsilon) \ll N$ by taking N large enough.

The lack of the cluster property means the existence of a long-range correlation(s). Therefore, for a pure state, it means an entanglement. Since a small number of Bell pairs do not destroy the cluster property, the lack of the cluster property means a *macroscopic* entanglement. Among many definitions of entanglement for systems with large N, the entanglement in the above sense is related most directly to many-body physics. Since many-body physics has been developed along with many experiments, the entanglement in the above sense is also related directly to properties of real systems. For example, the following inseparable state has the cluster property, hence is *not* entangled macroscopically;

$$|\mathrm{W}\rangle \equiv \frac{1}{\sqrt{N}}\left[|100\cdots 0\rangle + |010\cdots 0\rangle + |001\cdots 0\rangle + \cdots + |000\cdots 1\rangle\right]. \tag{3}$$

Although some other definitions of entanglement classify this state as an entangled state, it is a quite normal state in many-body physics and experiments: It represents, e.g., a low-lying excited state of an insulating solid, in which a single Frenkel exciton is excited on the ground state. Such a state can be easily generated experimentally. It is therefore more reasonable to classify $|\mathrm{W}\rangle$ as a state with infinitesimal entanglement. Furthermore, some other definitions of entanglement lead to an unphysical result that a Slater determinant of identical fermions is entangled. In contrast, according to our definition, such a state is not entangled at all. Moreover, it is guaranteed by the fundamental theorem mentioned above that a macroscopically entangled state by our definition is quite an anomalous state in the sense that it does not approach any pure state as $N \to \infty$.

It is convenient to consider additive operators, which is defined as the sum of local operators;

$$\hat{A} = \sum_{x} \hat{a}(x). \tag{4}$$

For example, if we take $\hat{a}(x) = (-1)^x \hat{\sigma}_z(x)$ for a one-dimensional spin systems, $\hat{A}$ becomes the z-component of the staggered magnetization. If a quantum state satisfies $\langle(\delta\hat{A})^2\rangle \leq O(N)$ for *every* additive operator, we call it a 'normally-fluctuating state' (NFS). If the correlation $|\langle\delta\hat{a}(x)\delta\hat{a}(y)\rangle|$ of any local operator $\hat{a}(\cdot)$ decays quickly for large $|x-y|$, then the state has the cluster property and is an NFS. If, on the other hand, a pure state has anomalously-large fluctuations as $\langle(\delta\hat{A})^2\rangle = O(N^2)$ for *some* of additive operators, we call it an 'anomalously-fluctuating state' (AFS). It is easy to show that an AFS does not have the cluster property, hence is entangled macroscopically. Therefore, an AFS is quite an anomalous state.

When studying finite macroscopic systems, we often encounter AFSs. For example, consider a finite system which will exhibit symmetry breaking as $N \to \infty$. When the order parameter does not commute with the Hamiltonian of the system, it is known that the *exact* ground state for finite N is not a symmetry-breaking state but the *symmetric* ground state that possesses all the symmetries of the Hamiltonian [6, 7, 8]. The well-known diagram, in which symmetry-breaking states have the lowest energy, is a result of a mean-field approximation. Anomalous states, such as AFSs, cannot be obtained by the mean-field approximation. Another example is states of quantum computers. We show that AFSs also appear during quantum computation in a quantum computer performing Shor's factoring algorithm [9]. Therefore, it is important to explore properties of anomalous states, such as AFSs, of finite macroscopic systems.

For this purpose, we study the stability of quantum states of general macroscopic systems of finite sizes, against weak classical noises, weak perturbations from environments, and local measurements [4]. In contrast to the previous works, we obtain general and universal results, by making full use of the locality of the theory.

We say that a pure state is 'fragile' if its decoherence rate Γ is anomalously great in the sense that

$$\Gamma \sim K N^{1+\delta}, \tag{5}$$

where K is a function of microscopic parameters, and δ is a positive constant. This is an anomalous situation in which the decoherence rate *per qubit*, Γ/N, grows with increasing N as $\sim KN^{\delta}$. In contrast, $\delta = 0$ is a normal situation. We show that an NFS never becomes fragile in weak perturbations from *any* noises or environments. Hence, the stability of NFSs, which are typical normal states [10], against decoherence is shown most generally, independently of details of models, for the first time.

Regarding fragility of AFSs, on the other hand, we find that an AFS becomes *either fragile or non-fragile* depending on the spectral densities of the noises or correlation functions in the environments. Although one might think that anomalous states such as AFSs would always be fragile, our general result is against such naive expectations. This means that we must go beyond the decoherence rate in order to study the stability of anomalous states.

For this purpose, we propose the new criterion of stability: We say a quantum state is 'stable against local measurements' if the result of a local measurement is not affected by another local measurement at a distant point. (The precise definition is described in Ref. [4].) According to experiences, macroscopic states seem to have this stability, i.e., they do not change significantly by measurement of only a tiny part of the macroscopic system. Indeed, we show the following theorem: *If a quantum state has the cluster property, then it is stable against local measurements, and vice versa.* Namely, state changes induced by any local measurements are small for normal states that have the cluster property, whereas anomalous states that do not have the cluster property, such as AFSs, are changed drastically by measurements of some relevant local observables.

Therefore, we have successfully clarified what stabilities are related to what anomalies of quantum states of general systems. Using these general results, we discuss the origin of symmetry-breaking in finite systems, roles of anomalous states in quantum computers with many qubits, and the emergence of a classical macroscopic world from quantum theory.

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[2] The cluster property should not be confused with the absence of the long-range order. For example, symmetry-breaking vacua have *both* the long-range order and the cluster property [1, 4].

[3] There is another way of defining local operators, hence another way of defining the cluster property: A local operator *around* x is a finite-order polynomial of field operators and their finite-order derivatives at positions in any finite region around x. A quantum state is said to have the cluster property if $\langle \delta\hat{a}(x)\delta\hat{b}(y)\rangle \to 0$ as $|x-y| \to \infty$ for any local operators $\hat{a}(x)$ and $\hat{b}(y)$ *around* x and y, respectively. Ref. [1] employed this definition. If a state has the cluster property in this sense, then it also has the cluster property defined in the text.

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[5] For example, consider quantum computers of various sizes which all perform the same algorithm with different sizes of inputs. Then, as has been confirmed explicitly in Ref. [9], a state in a computer of some N is similar to a state(s) in a computer of another N, because they are generated by the same algorithm. Another example is the ground-state wavefunctions of many particles in a sphere for the same particle density, for various sizes of spheres.

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[10] Here, the 'normal state' does not mean the normal state in the physics of superconductors: The BCS state is a normal state in the sense of the present paper.

Einstein-Podolsky-Rosen Correlation in General Relativity

Hiroaki TERASHIMA[1] and Masahito UEDA[2]

Tokyo Institute of Technology, Tokyo 152-8551, Japan

Abstract

We study the Einstein-Podolsky-Rosen (EPR) correlation in the regime of general relativity. We show that both acceleration and gravity deteriorate the perfect anti-correlation of an EPR pair of spins in the *same* direction. To maintain the perfect EPR correlation, observers must measure the spins in the appropriately chosen different directions that depend on the velocity of the particles, the curvature of the spacetime, and the positions of the observers.

Keywords: Quantum communication, EPR correlation, General relativity

1 Introduction

The Einstein-Podolsky-Rosen (EPR) correlation [1] is the non-local quantum correlation that originates in the entanglement. It is not just a strange feature of quantum theory but also offers a vital resource in quantum communication such as in quantum teleportation [2]. Therefore, for precise quantum communication, it is important to understand properties of the EPR correlation in various physical situations. Recently, the entanglement have been discussed in the regime of special relativity [3, 4, 5, 6, 7] using the Lorentz transformation of spin.

In this paper, we consider the EPR correlation in the regime of general relativity by introducing the gravitational field. Our motivation is that in the context of general relativity the spin of a particle is defined only locally with respect to the local inertial frame. Note that the gravitational field in general relativity is described by the curved spacetime which entails a breakdown of the global rotational symmetry. We clarify how to extract the *non-local* correlation for quantum communication from the *locally* defined spins. As an example, we consider circularly moving particles near the Schwarzschild black hole.

2 Local Inertial Frame

Consider a massive spin-1/2 particle in the curved spacetime with metric $g_{\mu\nu}(x)$. To discuss the spin state of the particle, we must introduce the local inertial frame at each point. The local inertial frame is represented by the vierbein $\{e_a{}^{\mu}(x)\}$ defined by

$$e_a{}^{\mu}(x)\, e_b{}^{\nu}(x)\, g_{\mu\nu}(x) = \eta_{ab}, \tag{1}$$

[1] E-mail: terasima@stat.phys.titech.ac.jp
[2] E-mail: ueda@ap.titech.ac.jp

where η_{ab} denotes the Minkowski metric diag$(-1, 1, 1, 1)$ and repeated indices are to be summed. Here and henceforth, it is tacitly assumed that Greek letters run over the four general coordinate labels and Latin letters run over the four local inertial coordinate labels $0, 1, 2, 3$. The vierbein specifies the general coordinate transformation from the general coordinate system x^μ to the local inertial frame x^a at each point and transforms a tensor in the general coordinate system into that in the local inertial frame. The inverse of the vierbein $\{e^a{}_\mu(x)\}$ is defined by

$$e^a{}_\mu(x)\, e_a{}^\nu(x) = \delta_\mu{}^\nu, \qquad e^a{}_\mu(x)\, e_b{}^\mu(x) = \delta^a{}_b. \tag{2}$$

The choice of the local inertial frame is not unique since the inertial frame remains inertial under the Lorentz transformation. Thus, the choice of the vierbein has the degree of freedom of the local Lorentz transformation. The spin-1/2 particle in the curved spacetime is defined as a particle whose one-particle states furnish the spin-1/2 representation of this local Lorentz transformation rather than the general coordinate transformation.

3 Spin Precession

With the local definition of the spin, the motion of the particle gives rise to a spin precession due to a change in the local inertial frame and due to the acceleration by an external force. Suppose that the particle is located at a spacetime point x^μ and is moving with four-velocity $u^\mu(x)$ normalized as $u^\mu(x)\, u_\mu(x) = -c^2$. This motion is not geodesic in the presence of an external force. The four-momentum of the particle with mass m becomes $p^\mu(x) = mu^\mu(x)$ in the general coordinate system, and it becomes $p^a(x) = e^a{}_\mu(x)\, p^\mu(x)$ in the local inertial frame at that point x^μ. After an infinitesimal proper time $d\tau$, the particle moves to a new point $x'^\mu = x^\mu + u^\mu(x)d\tau$. In the local inertial frame at this new point x'^μ, the four-momentum becomes $p^a(x') = p^a(x) + \delta p^a(x)$ due to a change in the local inertial frame and due to acceleration caused by the external force. The resultant change $p^a(x) \to p^a(x) + \delta p^a(x)$ is an infinitesimal Lorentz transformation $\delta^a{}_b + \lambda^a{}_b(x)d\tau$, where

$$\lambda^a{}_b(x) = -\frac{1}{mc^2}\left[\, a^a(x)\, p_b(x) - p^a(x)\, a_b(x)\,\right] + \chi^a{}_b(x) \tag{3}$$

with the four-acceleration $a^a(x)$ and the change in the local inertial frame $\chi^a{}_b(x)$ defined by

$$a^a(x) \equiv e^a{}_\mu(x)\ \left[\, u^\nu(x) \nabla_\nu u^\mu(x)\,\right], \qquad \chi^a{}_b(x) \equiv u^\mu(x)\ \left[\, e_b{}^\nu(x) \nabla_\mu e^a{}_\nu(x)\,\right]. \tag{4}$$

Note that the Lorentz transformation rotates the spin according to the Wigner rotation [8]. When the four-momentum of the particle is p^a, the Wigner rotation caused by the Lorentz transformation $\Lambda^a{}_b$ is given by

$$W^a_b(\Lambda, p) = \left[L^{-1}(\Lambda p)\, \Lambda\, L(p)\right]^a{}_b, \tag{5}$$

where $L^a{}_b(p)$ is the standard Lorentz transformation defined by

$$L^0{}_0(p) = \gamma, \quad L^0{}_i(p) = L^i{}_0(p) = p^i/mc, \quad L^i{}_k(p) = \delta_{ik} + (\gamma - 1)\, p^i\, p^k/|\vec{p}|^2, \tag{6}$$

with $\gamma = \sqrt{|\vec{p}|^2 + m^2c^2}/mc$ and $i, k = 1, 2, 3$. Substituting Eq. (3) into Eq. (5), we find the spin precession during the motion of the particle.

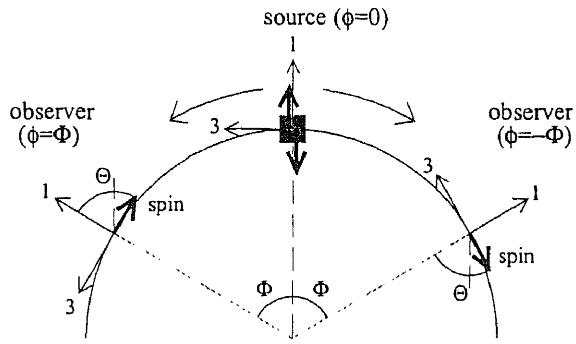

Figure 3: An EPR gedankenexperiment in the Schwarzschild spacetime.

4 EPR Correlation

We now consider the EPR correlation in the gravitational field. To be specific, we consider circularly moving particles at radius r on the the equatorial plane in the Schwarzschild spacetime. In the spherical coordinate system (t, r, θ, ϕ), the metric of the Schwarzschild spacetime reads

$$g_{\mu\nu}dx^\mu dx^\nu = -f(r)c^2dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7}$$

where $f(r) = 1 - (r_s/r)$ with r_s being the Schwarzschild radius, and the four-velocity of the circularly moving particles are given by

$$u^t(x) = \frac{\cosh\xi}{\sqrt{f(r)}}, \qquad u^\phi = \pm\frac{c\sinh\xi}{r}, \tag{8}$$

where ξ is the rapidity in the local inertial frame. Since this motion is not geodesic, we assume that an external force is applied to the particles in order to sustain the circular motion. Two observers and an EPR source are assumed to be static, whose azimuthal angles ϕ are given by $\pm\Phi$ and 0, respectively (see Fig. 3). The static local inertial frame on which the observers measure the spins is specified by

$$e_0{}^t(x) = \frac{1}{c\sqrt{f(r)}}, \qquad e_1{}^r(x) = \sqrt{f(r)}, \qquad e_2{}^\theta(x) = \frac{1}{r}, \qquad e_3{}^\phi(x) = \frac{1}{r\sin\theta}. \tag{9}$$

First, the EPR source emits a pair of entangled particles to opposite directions with four-momenta $p^a_\pm(x) = (mc\cosh\xi, 0, 0, \pm mc\sinh\xi)$ in the spin-singlet state,

$$\frac{1}{\sqrt{2}}\Big[\,|p^a_+,\uparrow\rangle|p^a_-,\downarrow\rangle - |p^a_+,\downarrow\rangle|p^a_-,\uparrow\rangle\Big] \tag{10}$$

which is defined in terms of the local inertial frame at $\phi = 0$. The particles move with the spin precession and reach the observers after the proper time $r\Phi/c\sinh\xi$. Then, in terms of the local inertial frames at $\phi = \Phi$ and $-\Phi$, the state is described as

$$\frac{1}{\sqrt{2}}\left[\cos\Theta\Big(|p^a_+,\uparrow\rangle|p^a_-,\downarrow\rangle - |p^a_+,\downarrow\rangle|p^a_-,\uparrow\rangle\Big) + \sin\Theta\Big(|p^a_+,\uparrow\rangle|p^a_-,\uparrow\rangle + |p^a_+,\downarrow\rangle|p^a_-,\downarrow\rangle\Big)\right], \tag{11}$$

where

$$\Theta = \Phi \cosh \xi \left[1 - \frac{r_s}{2rf(r)} \right] \sqrt{f(r)}. \tag{12}$$

Since the spins are rotated in opposite directions on the equatorial plane through the angle Θ, the spin-singlet state is mixed with the spin-triplet state. Therefore, if the spins are measured in the same direction with respect to the local inertial frames at $\phi = \pm\Phi$, the perfect anti-correlation is deteriorated. To extract the perfect EPR correlation, the observers must rotate the directions of measurement in opposite directions through the angle Θ. We emphasize that this angle Θ is not equal to the trivial rotation angle of the local inertial frames Φ. In the non-relativistic limit of $c \to \infty$ and $r_s \to 0$, the angle Θ reduces to Φ. However, in general relativity, the angle Θ is different from Φ due to the curvature of the spacetime and due to the acceleration by the external force. Even if the spins are measured in the same direction (expected in non-relativistic theory), the perfect anti-correlation is deteriorated.

If there is an uncertainty in identifying the position of the observers Φ, it is difficult to extract the EPR correlation near the event horizon $r \sim r_s$ since the ratio Θ/Φ is very large. This is because the spin precession is extremely rapid near the event horizon due to the strong gravity. Actually, right on the event horizon $r \to r_s$, the ratio Θ/Φ becomes infinite. This divergence originates from the singularity of the static vierbein (9) and that of the circularly moving four-velocity (8). To circumvent these singularities, the observers should use a different local inertial frame and different particles to extract the EPR correlation near the event horizon.

5 Conclusions

In conclusion, we have shown that both acceleration and gravity deteriorate the perfect EPR correlation if the spins are measured in the *same* direction. To maintain the perfect EPR correlation, the spins must be measured in the directions that depend on the velocity of the particles, the curvature of the spacetime, and the positions of the observers. Near the event horizon of a black hole, these appropriate directions depend so sensitively on the positions of the observers that even a very small uncertainty in the identification of the observer's position leads to a fatal error in quantum communication. The choice of the vierbein and four-velocity is important to communicate non-locally in the curved spacetime using the EPR pair of spins.

Acknowledgments

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Multi-Photon Entanglement and Bell-type Experiments

M. BOURENNANE*,**, S. GAERTNER*,**, M. EIBL*,**, C. KURTSIEFER*,
K. SAUCKE*, M. WEBER*, J. VOLZ*, M. ŻUKOWSKI*** and
H. WEINFURTER*,**[1]

* *Sektion Physik, Ludwig-Maximilians-Universität, D-80799 Munich, Germany*
***Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany*
****Instytut Fizyki Teoretycznej i Astrofizyki Uniwersytet Gdański, PL-80-952 Gdansk, Poland*

Abstract

Polarization entangled multi-photon states obtained from parametric down-conversion are used for the first four observer Bell-experiment. While the violation of a Bell-inequality is a possible tool to analyze entanglement, the detection loophole prevents one to use such experiments as tests of local realism. We present a scheme to circumvent the problem in an atom optics experiment.

Keywords: Entanglement, Bell-type experiments

The process of spontaneous parametric down-conversion (SPDC) offers currently the best way to generate entangled photon pairs. There, photons from an intense light beam are converted to pairs of daughter photons in an optically nonlinear material. In this conversion process conservation laws cause strong correlations between various properties of the generated photons. Particularly, type-II SPDC offers a method to directly generate pairs of polarization-entangled photons [1]. Recently, pulsed down-conversion enabled the simultaneous observation of more than just two photons. This formed the basis for first experiments with 3- and 4-photon GHZ states [2], where interferometric setups were used to generate the desired multiphoton entanglement out of two pairs of photons.

Four-photon entangled states can be also obtained directly from parametric down conversion without overlapping single photons [3, 4]. Here we analyze the properties of this four-photon state. The high stability of the source enabled the first experimental violation of the local realistic condition of a generalized Bell-inequality for four observers. However, the detection loophole prevents one from using such measurements as tests of local realistic theories. We present a scheme which should allow to close this loophole by using the entanglement between a pair of atoms, which can be detected with high efficiency.

Analyzing the process of SPDC, one observes that not only pairs of entangled photons can be emitted. The emission of four photons becomes possible in a second order process, when two photons of the pump light are simultaneously downconverted into two spatial modes a_0 and b_0 (Fig. 1). The remarkable feature of this four-photon state is that it is not simply the product of two entangled pairs. Due to their bosonic nature, the emission of two photons with identical

[1]E-mail: harald.weinfurter@physik.uni-muenchen.de

polarization into the same direction is twice as probable as the emission of two photons with orthogonal polarization. This very fundamental interference causes entanglement between the four photons emitted by type-II SPDC.

To make the four-photon entanglement accessible, we split each of the two outputs of the type-II SPDC by non-polarising beam splitters. Furthermore, we select events such that one photon is detected in each of the resulting four outputs (a, b, c, and d) of the beam splitters. The state of the four detected photons is then given by [3]:

$$\begin{aligned}|\Psi^{(4)}\rangle_{abcd} = \sqrt{\frac{1}{3}}[\quad & |HHVV\rangle \quad +|VVHH\rangle \\ -\frac{1}{2}(\quad & |HVHV\rangle \quad +|HVVH\rangle + \\ & |VHHV\rangle \quad +|VHVH\rangle)]_{abcd}\end{aligned} \tag{1}$$

The four entries in the kets describe the polarisation (horizontal-H and vertical-V) of the photons in the arms a, b, c, and d. In order to obtain the particular form of the above state it is necessary to compensate birefringence in the SPDC source and of the beam splitters with compensation crystals right behind the SPDC crystal and additional quartz plates in the reflected output arms of the beam splitters (not shown in Fig. 1) such that the two-photon state $|\Psi^-\rangle$ can be observed between arms a and c, a and d, etc.

One can write down a Bell inequality which summarizes all possible local realistic constraints on the correlation function for the case of each local observer measuring the polarizations along two alternative directions [3, 5]. For this purpose the observers in the four modes ($x = a, b, c, d$) perform measurements corresponding to a polarization observable with eigenvectors $|l_x, \phi_x\rangle = \sqrt{1/2}(|R\rangle_x + l_x e^{i\phi_x}|L\rangle_x)$, and eigenvalues $l_x = +1, -1$ and thereby obtain the value of the correlation function defined as the expectation value of the product of the four local polarization observables.

Let us introduce the shorthand notation $E(\phi_a^k, \phi_b^l, \phi_c^m, \phi_d^n)$ for the correlation functions deduced from the observed count rates for the full set of 2^4 local directions, with $k, l, m, n = 1, 2$ denoting which of the two alternative phase settings was chosen by the local observer measuring in arm x ($x = a, b, c, d$). The generalized Bell inequality gives an upper bound for the observed correlations in a local realistic description and reads [5]

$$\begin{aligned}S^{(4)} \quad = \quad & \frac{1}{16} \sum_{s_a, s_b, s_c, s_d = \pm 1} \\ & \left| \sum_{k,l,m,n=1,2} s_a^k s_b^l s_c^m s_d^n \, E(\phi_a^k, \phi_b^l, \phi_c^m, \phi_d^n) \right| \le 1.\end{aligned} \tag{2}$$

The maximal violation of this inequality for state (2) is obtained when three observers, ($x = a, c, d$) perform polarization analysis along $\pm\pi/4$ and the observer in mode b chooses between $\phi_b^1 = 0$ or $\phi_b^2 = \pi/2$. Then the quantum prediction is as high as $S_{QM}^{(4)} = 1.886$ [3] and results in a violation of the inequality (9), whenever the correlation function has visibility greater 53%. For a four-photon GHZ state one obtains $S_{QM}^{(4)} = 2\sqrt{2}$ and a critical visibility of $1/\sqrt{8} \approx 35\%$.

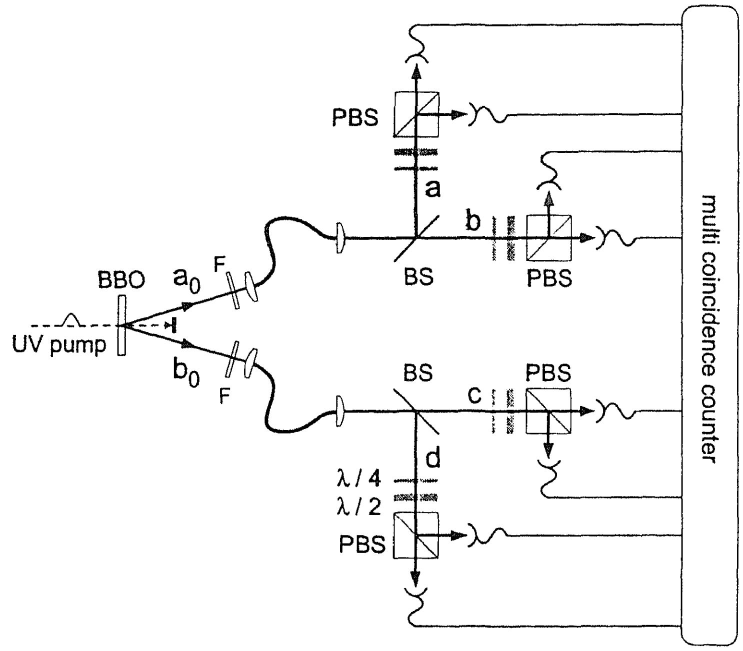

Figure 4: Experimental setup to observe four-photon entanglement obtained directly from type-II down conversion. The four photons are emitted from the BBO crystal into two spatial modes a_0 and b_0, passed through 3 nm interference filters (F), and distributed into the four modes a, b, c, d by 50% – 50% beam splitters (BS). To characterise the polarisation-entangled four-photon state $|\Psi^{(4)}\rangle$ (2), a polarisation analysis in various bases is performed in each mode using $\lambda/4$ and $\lambda/2$ plates in front of polarising beam splitters (PBS) and single photon avalanche detectors. Joint photodetection events in the four arms are recorded in a multi-coincidence unit.

Fig. 2 shows all 256 fourfold coincidence probabilities necessary for the analysis. They were recorded in blocks of 16 coincidence rates corresponding to the 16 phase settings, with an average measurement time of 1.5 hours per frame. For evaluating the generalized Bell inequality we used the raw data without any correction for background, collection or detection efficiency. The resulting value $S^{(4)} = 1.664 \pm 0.028$ clearly violates the boundary for local realistic theories and thus proves the entanglement of $|\Psi^{(4)}\rangle$.

This value is also higher than the bound for bipartite entanglement ($S^{(4)}_{bipartite} \leq \sqrt{2}$) [6] and thus confirms that the observed state has at least tripartite entanglement. Yet, in order to unambiguously test four-particle entanglement, the Bell inequality is not suited, as there are tripartite entangled states giving values up to $S^{(4)}_{tripartite} \leq 2$. Although the possible tripartite entangled states do not exhibit the observed correlations and are thus ruled out by our measurements, the recently developed entanglement witness would be the proper tool [7].

In spite of this clear violation of the Bell-inequality the detection loophole rules out any contradiction with local realistic theories. Even if the required detection efficiency becomes lower in multi-party experiments, the experimental difficulties in generating entanglement between many separated observers will prevent conclusive results from such multi-photon experiments.

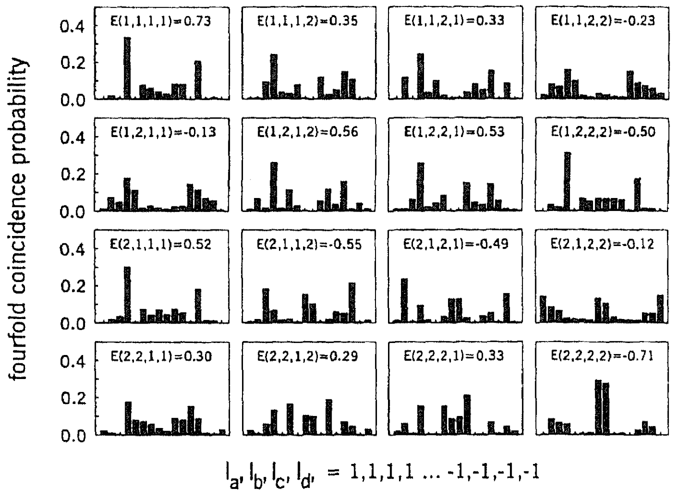

Figure 5: Fourfold coincidence probabilities for the evaluation of a four-particle Bell inequality. For the sixteen settings of the analyzer phases $\phi_a, \phi_b, \phi_c, \phi_d$, the normalized count rates $p_{k,l,m,n}$ obtained are used to evaluate a generalized Bell inequality (9), leading to $S^{(4)} = 1.664 \pm 0.028$. This clearly exceeds the bound of 1 given for local realistic theories and proves the entanglement of the observed state. For this measurement the acquisition time for each individual frame was 1.5 hours, with about 670 fourfold coincidence events per hour.

We propose to use the entanglement between the internal state of a single trapped atom with the polarization of an emitted photon as the basis for a loophole free experiment [8]. Entanglement swapping of two such photons will entangle the internal, long lived states of two remote atoms which in turn can be detected with high efficiency. We will present the scheme of this Bell-type experiment and the first tests of a trap for single Rubidium atoms.

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Quantum Information with Macroscopic Cesium Gas Samples

Brian JULSGAARD*[1]

* *Department of Physics and Astronomy, University of Aarhus, Ny Munkegade, Building 520, OK-8000, Aarhus C*

Abstract

In the rapidly advancing science of quantum information our experimental contribution has concentrated on the interface between strong pulses of light with macroscopic samples of cesium atoms. In the talk we review the interaction between polarized light and the atomic ground state spin, and we explain how this interaction enable us to create an entangled state between two samples of cesium atoms. We also give an outlook for future experimental progress.

[1]E-mail: brianj@phys.au.dk

Violation of Local Realism as a Resource

Marek ZUKOWSKI*[1]

* *Institute of Theoretical Physics and Astrophysics, University of Gdansk, ul. Wita Stwosza 57 PL-80-952 Gdansk, Poland*

Abstract

New Bell inequalities will be presented, and it will be shown that their violations can be utilized to beat some classical limits in communication complexity problems. All this will be shown for systems involving entanglement of sub-systems which are qubits as well as for ones of a higher dimensionality.

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[1]E-mail: zukowski@ap.univie.ac.at

Physics of Points and Walls in Quantum Mechanics

Taksu CHEON*, Tamás FÜLÖP** and Izumi TSUTSUI**[1]

* *Kochi University of Technology, Tosa Yamada, Kochi 782-8502, Japan*
** *Institute of Particle and Nuclear Studies, KEK, Tsukuba 305-0801, Japan*

Abstract

Points and walls are basically trivial objects in classical mechanics, but they are far from trivial in quantum mechanics due to their arbitrariness admitted theoretically. We discuss briefly how the arbitrariness arises when quantized, and mention several intriguing quantum phenomena which become available if the arbitrariness can be controlled. These include duality, supersymmetry, Berry phase, quantum anholonomy, copy by tunneling/caustics, and quantum force as a statistical effect of a wall.

Keywords: Quantum singularity, Duality, Berry phase, Quantum tunneling

1. Family of singular points and walls in quantum mechanics

In classical mechanics, a point (or a wall without thickness) will have no characteristics and is basically trivial. In contrast, quantum mechanics allows for a $U(2)$ family of them on a line, for instance, because the singularity caused by the point or wall can be dealt with boundary conditions that preserve the unitarity [1] but such conditions are not unique: there exist a $U(2)$ family of conditions and, accordingly, the $U(2)$ family for points and walls. Thus, quantum points or walls are significant physically and can exhibit a variety of phenomena which have no classical counterparts. In view of the recent progress of nanotechnology, we may soon be able to manufacture objects that can be regarded as points and walls — possibly in the form of quantum dots or junctions of semiconductors — so that such exotic phenomena can be observed in laboratories. Here we mention some of these.

2. Duality, supersymmetry, Berry phase and quantum anholonomy

The $U(2)$ family of quantum points are described by four parameters, which are characterized by their distinctive physical roles: two of them determine the energy spectra of the system under the points while the other two specify the scattering (reflection and transmission) properties [2]. Roughly speaking, the parameter space $U(2)$ is a product of the spectral space given by a torus T^2 (more precisely, it is T^2/Z_2 which is a Möbius strip [3]) and the isospectral space given by a sphere S^2 (Figure 6). The structure of the family then allows us to play around with the quantum points by observing what happens if we change the parameters properly. The simplest of them occurs when we compare two 'antipodal' points in their spectra and the strengths of interactions. Here we find that their spectra are exactly the same even though their strengths of interactions are the reciprocal of each other, which is *Duality* in the spectrum

[1] E-mail: izumi.tsutsui@kek.jp; http://research.kek.jp/people/itsutsui/index_e.html

under the strong-weak coupling exchange [4]. One can then find a subfamily forming a circle S^1 on the torus where the above phenomenon takes place on their own. At these self-dual points, all energy levels are doubly degenerate and the system possesses a certain *supersymmetry* [5].

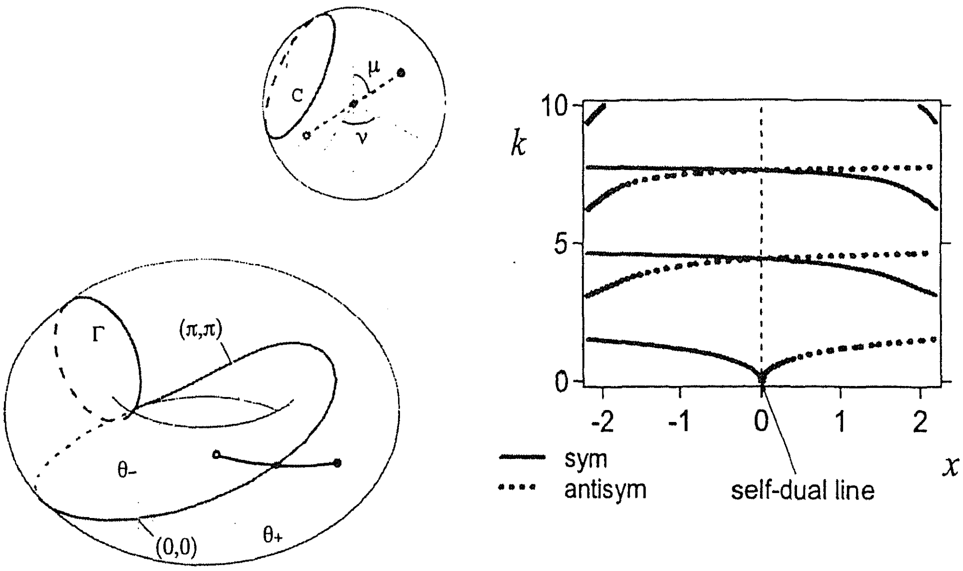

Figure 6: (Left) $U(2)$ parameter space as a product of a torus and a sphere. Changes along the loop on these spaces produce the Berry phase and the quantum anholonomy. (Right) Energy levels at 'antipodal' points on the torus. At the centre is the self-dual point where the system acquires supersymmetry.

A less trivial phenomenon arises if we control the quantum points continuously along a loop on the isospectral sphere S^2 while fixing the point on the spectral torus T^2. After completing the loop, one finds that the state acquires an extra phase proportional to the area encircled by the loop. This is nothing but the *Berry phase* associated with the loop, where the phase is given by the magnetic flux of a fictitious Dirac monopole at the centre of the sphere S^2. Moreover, if we instead take a nontrivial loop on the torus T^2 while fixing the point on the sphere S^2, then after the loop is completed we find that the energy levels are shifted without changing the spectrum. This is an example of phenomena called *quantum anholonomy* which may be of use in pumping energy out of the system indirectly by manipulating the parameters [2].

3. Quantum tunneling and copy

In the presence of a wall represented by a potential $V(x)$ which diverges at a point (like, *e.g.*, the Coulomb potential), we may pick up the singular point and treat it similarly to the aforementioned quantum points. This means that such potential singularities can also be characterized by the $U(2)$ group [6], and hence the phenomena mentioned above can occur here as well. Because of the potential, however, we have more interesting possibilities, one of which occurs when we combine quantum tunneling with 'caustics' which are originally found in geometrical optics [7].

Classically, caustics arise when the final position becomes insensitive to the initial velocity, and this leads to recurrence of states in quantum mechanics. The simplest example of caustics is found in the harmonic oscillator, but it is known that caustics remain even if the oscillator couples with the square inverse potential as $V(x) = \frac{m\omega^2}{2}x^2 + g\frac{1}{x^2}$. When the coupling constant g is large, the right and left half lines, $x > 0$ and $x < 0$, are separated physically and no information can be transmitted between them. However, if g is sufficiently small and satisfies $0 < g < \frac{3\hbar^2}{8m}$, then quantum tunneling is allowed even though the potential at $x = 0$ is infinitely high. Remarkably, one finds in this case that the recurrence occurs in duplicate, that is, the original profile of a state on the right half $x > 0$, say, is reproduced on the left $x < 0$ as well as on the right $x > 0$ at later times. This implies that one may 'copy' a profile prepared on one side to the other between subsystems which are completely separated classically [8]. (This copying process in not in conflict with the no-go theorem [9] of quantum cloning, because the process takes place in one Hilbert space rather than two as presumed in the theorem.)

4. Quantum force on a partition wall

Consider a quantum well with a partition wall at the centre of the well (Figure 7). If we put identical particles into the well so that the same number of particles are distributed on the right and the left of the partition wall, then the physical property of the wall will show up in various statistical quantities. Take, for simplicity, the Dirichlet boundary conditions for the left and right sides of the well, and suppose that the partition wall enforces the Dirichlet boundary condition for its left and the Neumann for its right (which is specified by a particular element of the $U(2)$). If we measure the net force $\Delta F(T)$ acting on the partition under various temperatures T, we may expect that it has a nonvanishing value $\Delta F(0)$ in the low temperature limit $T \to 0$ due to the difference in the spectrum of the subsystems on the right and left. This can be confirmed easily by direct computations.

On the other hand, we also naïvely expect that $\Delta F(T)$ will vanish in the high temperature limit $T \to \infty$ because the difference of the spectrum due to the boundary conditions will become irrelevant in the classical regime. This, however, is not the case: in fact, we observe that $\Delta F(T)$ decreases up to a certain temperature to achieve a minimum and then increases indefinitely, *e.g.*, for the bosonic particles, we find asymptotically $\Delta F(T) \sim \sqrt{T}$ for $T \to \infty$ [10]. Interestingly, the force $\Delta F(T)$ is generated purely by the boundary effect which is available only at the quantum level, but nonetheless it does not vanish in the high temperature limit where the statistics becomes classical (the Bose-Einstein is replaced in effect by the Maxwell distribution).

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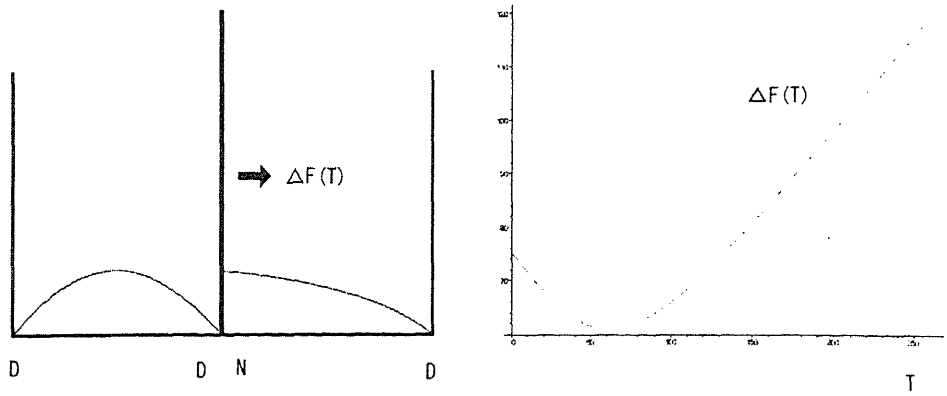

Figure 7: (Left) Quantum well with a partition wall at the centre. A net force $\Delta F(T)$ arises due to the distinct boundary conditions at the partition ('D' = Dirichlet and 'N' = Neumann boundary condition) to which the energy eigenstates (the lowest is shown in each subsystem) must obey. (Right) The force $\Delta F(T)$ exhibits a nontrivial behaviour as a function of temperature T, achieving a minimum before it diverges as $T \to \infty$.

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Non-commutative Geometry, the Bohm approach and non-locality

Basil J. HILEY*[1]

* *Theoretical Physics Research Unit, Birkbeck College, University of London, Malet Street, London, WC1E 7HX*

Abstract

The appearance of non-locality in quantum entangled states that is clearly demonstrated in the Bohm approach to quantum mechanics led Bell to ask whether all theories that attributed properties to local entities would exhibit non-locality or whether this feature was only a particular property of the Bohm approach. As Bell subsequently showed this feature is quite general and not confined to the Bohm approach. In spite of the interest generated by entangled states over the last two decades, there has been little interest in the Bohm approach and it is perceived as being in some way flawed. Yet all the predictions of the standard approach are *exactly* reproduced. Indeed Bohm was only re-expressing the Schrödinger equation in terms of its real and imaginary parts and then exploring what the resulting equations imply for the physics. We find that one of the equations has a form very similar to the Hamilton-Jacobi equation of classical mechanics, which makes a comparison between the two approaches straightforward. The difference arises from an extra term that looks like a new form of potential, the quantum potential. It is this term that can completely account for the difference in the behaviour of particles in both cases. Furthermore this quantum potential is a mathematical expression of the non-locality that appears in the quantum domain. The quantum potential vanishes in the classical domain leaving our familiar local classical world.

With all this going for it why then is there a general dismissive attitude to the Bohm interpretation? It cannot be the weird properties that have to be attributed to the quantum potential because the quantum phenomena are themselves very strange and 'counter intuitive'. May be the basic problem is the belief that the uncertainty principle implies that because we cannot measure simultaneously position and momentum, the motion of a quantum particle cannot be described in an $r^{3N} \times p^{3N}$ phase space, the natural state space of classical mechanics. But that is precisely what the Bohm approach does and incidentally exactly what the Wigner-Moyal approach does as well. The latter also reproduces *exactly* the standard quantum results. How is this possible?

The reason why such a description is possible in both cases is because the variables used are not eigenvalues of the usual x and p operators. In the case of the Wigner-Moyal approach, the x and p operators are constructed from a pair of operators corresponding to a pair of points in configuration space. In this case x and p are the mean operators $x = (x_1+x_2)/2$ and $p = (p_1+p_2)/2$. These operators commute and therefore the corresponding phase space is constructed from the simultaneous eigenvalues of these operators. In the

[1]E-mail: b.hiley@bbk.ac.uk

Bohm approach the x is the eigenvalue of the x operator, while the p turns out to be a mean momentum $\bar{\bar{p}}$ already defined in Moyal's seminal 1949 paper. This variable is not an eigenvalue of the p operator. Moyal even shows that the equation that transports $\bar{\bar{p}}$ is the real part of Schrodinger equation used by Bohm. Thus the Wigner-Moyal approach already implicitly contains the quantum potential although Moyal does not draw attention to it. Thus there is an intimate relation between the Wigner-Moyal approach and that of Bohm. They are each constructing x-p phase spaces but these phase spaces are different.

How then are we to understand the appearance of *different* phase spaces? To understand why this is possible let us assume that primary relevance can be given to the algebraic structure of the operator formalism and proceed in the spirit of non-commutative geometry. To explain what I mean by this consider the traditional approach in which one builds up a field theory by first defining a topological space, giving it a local differential structure and then imposing a metric before building a structure of functions on that space. Gel'fand shows that we can also proceed in the opposite direction. If we start with the algebra of functions first then, provided this algebra is commutative, we can abstract from the algebra all the properties of the underlying space. The points of the space are the maximal ideals defined by the functions. However if the algebra is non-commutative then there is no unique underlying space and we have to be content with what are known as 'shadow manifolds'. The phase spaces are then a direct consequence of this fact. Thus the Bohm approach can be considered as just one of these shadow manifolds. Here the position representation has been singled out. There are others depending on other representations and indeed Bohm-type interpretations can be constructed for all such representations thus restoring the full symplectic symmetry to the Bohm approach. The apparent lack of this symmetry has been the source of criticisms of the approach.

The algebraic approach requires the minimal left and right ideals to play the role of the wave function and its complex conjugate. Then for the single particle, the algebraic equations equivalent to Schrödinger's equation are

$$i\frac{d\rho}{dt} + [\rho, H]_- = 0 \text{and} \quad , \rho\frac{dS}{dt} + \frac{1}{2}[\rho, H]_+ = 0$$

where ρ is the density *operator* and S is a phase *operator*. No quantum potential appears in either of these two equations which, it must be stressed once again, are operator equations. However once we project these equations into a representation then the first equation gives the conservation of probability while the second becomes the quantum Hamilton-Jacobi equation used by Bohm which contains the quantum potential explicitly. Thus we see that the quantum potential is essentially an *apparent* potential appearing because we are projecting the quantum process into an inappropriate space for its description. It is reminiscent of the gravitational force in general relativity where the geodesic in curved space becomes a curve when it is projected into a flat Euclidean space. Unlike the gravitational force the apparent quantum force is non-local and only acts between particles involved in the entangled state.

The particular representation chosen is determined by the experimental set up and therefore the apparent force is determined by the details of experimental arrangement.

This is in exact accordance with Bohr's position when he insists the "word phenomenon should refer exclusively to the observations obtained under specified circumstances, including an account of the whole experimental arrangement ". Our proposal is that this can also be understood if we regarded the quantum potential as an *information potential* which is active in the dynamics of the evolution of a system modifying its behaviour in response to the significant environmental conditions in which the system finds itself. It is in terms of this notion of active information that the evolution of quantum processes can be given meaning. Indeed it then accounts for the non-locality in terms of the systems responding to a common pool of information and as Maroney and Hiley have shown, it also can account for the recently reported teleportation experiments.
[More detailed background to this work can be found at www.bbk.ac.uk/tpru]

Entanglement and Violations of Heisenberg's Noise-Disturbance Uncertainty Relation

Masanao OZAWA[1]

Tôhoku University, Sendai

Abstract

Heisenberg's reciprocal relation between position measurement noise and momentum disturbance holds, if those noise and disturbance are statistically independent of the state of the measured object, but otherwise can be violated. We discuss two distinct types of possible violations of Heisenberg's relation, and explore the role of entanglement, in the light of the universally valid reformulation of the Heisenberg's relation recently obtained by the present author.

Keywords: Measurement, Uncertainty, Noise, Disturbance, Entanglement

Heisenberg's uncertainty relation for position measurement noise and momentum disturbance can be formulated as follows: *For every measurement of the position Q of a mass with root-mean-square (rms) noise $\epsilon(Q)$, the rms disturbance $\eta(P)$ of the momentum P of the mass caused by the interaction of this measurement always satisfies the relation*

$$\epsilon(Q)\eta(P) \geq \frac{\hbar}{2}. \tag{1}$$

Heisenberg [1] explained the physical intuition underlying the above relation by discussing the γ ray microscope thought experiment, and claimed that this relation is a straightforward consequence of the canonical commutation relation $QP - PQ = i\hbar$. Shortly after, Heisenberg's argument is refined by Kennard [2], who introduced the notion of standard deviation and proved the relation

$$\sigma(Q)\sigma(P) \geq \frac{\hbar}{2}, \tag{2}$$

where $\sigma(X)$ stands for the standard deviation of an observable $X = Q, P$ in a given state.

Many text books have associated the formal expression of "Heisenberg's uncertainty relation" to Eq. (2), but also associated the physical meaning of "Heisenberg's uncertainty relation" to Eq. (1). However, the universal validity of Eq. (1) has been criticized in many ways. In fact, the "resolution power" of the γ ray microscope cannot be identified in any interpretation with the standard deviation of the mass position in the state to be measured. Moreover, no one has succeeded in proving Eq. (1) for general measurements even by applying Eq. (2) not only to the mass state but also to the apparatus state. Undoubtedly, this has caused serious confusions among physicists on the status of this leading principle of quantum mechanics. Recently, the present author [3, 4, 5] gave rigorous and general formulation for the notions of noise and

[1]E-mail: ozawa@mailaps.org

disturbance and proved the new relations for Heisenberg's uncertainty principle that includes Eq. (1) and Eq. (2) as special cases.

It is well-known that every measuring instrument $\mathcal{I}$ is associated with a probabiltiy operator valued measure (POVM) Π and a trace-preserving completely positive map (TPCPM) T. If the input state is ρ, the probability of obtaining the outcome in a Borel set Δ is given by $\mathbf{tr}[\Pi(\Delta)\rho]$ and the output state is given by $T\rho$. Then, the *first and the second moment operators of* Π, denoted by $O(\Pi)$ and $O^{(2)}(\Pi)$, are defined by

$$O(\Pi) = \int x \, d\Pi(x), \tag{3}$$

$$O^{(2)}(\Pi) = \int_{\mathbf{R}} x^2 \, d\Pi(x). \tag{4}$$

We define the *distance* $d_\rho(\Pi, A)$ of POVM Π from observable A in ρ by

$$d_\rho(\Pi, A) \;=\; \mathbf{tr}[O^{(2)}(\Pi) - O(\Pi)A - AO(\Pi) + A^2)\rho]^{1/2}. \tag{5}$$

Then, the rms noise of this instrument for measuring observable A in the input state ρ is given by

$$\epsilon(A, \rho) \;=\; d_\rho(\Pi, A), \tag{6}$$

and the rms disturbance of an observable B caused by this instrument in the input state ρ is given by

$$\eta(B, \rho) \;=\; d_\rho(T^* E^B, B), \tag{7}$$

where T^*E^B stands for the POVM defined by $(T^*E^B)(\Delta) = T^*[E^B(\Delta)]$ for any Borel set Δ.

We now introduce the *mean noise operator* $n(A)$ for observable A and the *mean disturbance operator* $d(B)$ for observable B defined by

$$n(A) \;=\; O(\Pi) - A, \tag{8}$$

$$d(B) \;=\; O(T^*E^B) - B. \tag{9}$$

Then, we obtain the model-independent universally valid noise-disturbance uncertainty relation: *Every instrument satisfies the relation*

$$\epsilon(A, \rho)\eta(B, \rho) + \frac{1}{2}|\,\mathbf{tr}\{[n(A), B]\rho\} + \mathbf{tr}\{[A, d(B)]\rho\}| \geq \frac{1}{2}\,|\mathbf{tr}([A, B]\rho)| \tag{10}$$

for any state ρ *for which all the relevant terms are finite.*

We say that an instrument $\mathcal{I}$ makes an *unbiased measurement* of A, if the mean output is equal to the mean of the observable A in the input state, i.e., $O(\Pi) = A$. We say that an instrument $\mathcal{I}$ makes an *unbiased disturbance* of B, if $\mathcal{I}$ does not change the mean of B, i.e., $O(T^*E^B) = B$. We say that $\mathcal{I}$ has *statistically independent noise* for A, if the mean noise does not depend on the input state ρ, or equivalently, if the mean noise operator $n(A)$ is a constant operator, i.e., $n(A) = rI$ for some $r \in \mathbf{R}$. We say that $\mathcal{I}$ has *statistically independent disturbance* for B, if the mean disturbance does not depend on the input state ρ, i.e., $d(B) = rI$ for some $r \in \mathbf{R}$.

The model-independent universally valid noise-disturbance uncertainty relation leads to rigorous conditions on what instrument satisfies Heisenberg's noise-disturbance uncertainty relation: *An instrument $\mathcal{I}$ satisfies Heisenberg's noise-disturbance uncertainty relation, i.e.,*

$$\epsilon(A,\rho)\eta(B,\rho) \geq \frac{1}{2}|\,\mathrm{tr}([A,B]\rho)|$$

for any state ρ for which all the relevant terms are finite, if one of the following conditions holds:

(i) The mean noise operator commutes with B and the mean disturbance operator commutes with A, i.e.,

$$[n(A),B] = 0, \tag{11}$$
$$[d(B),A] = 0. \tag{12}$$

(ii) The instrument $\mathcal{I}$ has both statistically independent noise for A and statistically independent disturbance for B.

(iii) The instrument $\mathcal{I}$ makes both unbiased measurement of A and unbiased disturbance of B.

By applying the Schwarz inequality to all terms in the left-hand-side of Eq. (10), we now obtain the generalized noise-disturbance uncertainty relation: *For any instrument $\mathcal{I}$, we have the relation*

$$\epsilon(A,\rho)\eta(B,\rho)+\epsilon(A,\rho)\sigma(B,\rho)+\sigma(A,\rho)\eta(B,\rho) \geq \frac{1}{2}|\,\mathrm{tr}([A,B]\rho)| \tag{13}$$

for any state ρ for which all the relevant terms are finite.

From the above, we have the following trade-off relations for precise A measurements or B-non-disturbing measurements: *If the instrument $\mathcal{I}$ does not disturb B, we have*

$$\epsilon(A,\rho)\sigma(B,\rho) \geq \frac{1}{2}|\,\mathrm{tr}([A,B]\rho)|. \tag{14}$$

If $\mathcal{I}$ precisely measures A, we have

$$\sigma(A,\rho)\eta(B,\rho) \geq \frac{1}{2}|\,\mathrm{tr}([A,B]\rho)|. \tag{15}$$

For finitely accessible input states, i.e., $\sigma(Q),\sigma(P) < \infty$, Eq. (13) excludes the possibility of having both $\epsilon(Q)=0$ and $\eta(P)=0$ simultaneously. However, it is possible to have $\eta(P)=0$ uniformly over every input state or alternatively to have $\epsilon(Q)=0$ uniformly. In both cases, Eq. (1) is violated uniformly with $\epsilon(Q)\eta(P)=0$. Thus, we have two types of uniform violations of Eq. (1); we shall refer to the former as *type I* and the latter as *type II.*

In type I violations, by substituting $\eta(P)=0$ in Eq. (13), we have

$$\epsilon(Q)\sigma(P) \geq \frac{\hbar}{2}. \tag{16}$$

The above relation even allows to have $\epsilon(Q)\to 0$ with $\sigma(P)\to\infty$, and, as shown below, we have a model realizing relations $\eta(P)=0$ and $\epsilon(Q)\to 0$ with $\sigma(P)\to\infty$. In this case, the small noise is compensated not by the large momentum disturbance but by the large initial momentum uncertainty. From Eq. (2), this means that without disturbing the momentum the position can be measured as precisely as our initial knowledge on the object position. This rather natural possibility has been excluded from Eq. (1).

Similarly, in type II violations, $\sigma(Q)$ and $\eta(P)$ are constrained as

$$\sigma(Q)\eta(P) \geq \frac{\hbar}{2}, \tag{17}$$

so that the small momentum disturbance is compensated by the large initial position uncertainty, and actually a model in Ref. [6] realizes relations $\epsilon(Q) = 0$ and $\eta(P) \to 0$ with $\sigma(Q) \to \infty$. From Eq. (2), this implies the possibility of the precise position measurement with only disturbing the momentum as much as the initial momentum uncertainty. Since Eq. (1) has prohibited the precise position measurement without infinite momentum transfer, this opens a new possibility of precision measurements of the mass position and similar physical quantities.

A Type I violation can be obtained by reformulating the Einstein-Podolsky-Rosen thought experiment [7]. Let the measured system be a two-particle system comprising one-dimensional particles 1 and 2 with positions Q_1, Q_2 and momenta P_1, P_2, respectively, and consider the following process of measuring Q_1; our measuring apparatus is assumed to couple only to particle 2 and to precisely measure Q_2, but then to output this measured value of Q_2 as the outcome of the indirect Q_1 measurement. This is generally not a good measurement of Q_1; however, the interaction for this measurement does not disturb P_1, so that $\eta(P_1) = 0$ uniformly. In this case, we can show that $\epsilon(Q_1)^2 = \langle (Q_1 - Q_2)^2 \rangle$ for any input state. On the other hand, for any small $\alpha > 0$, we can choose a two-particle state ψ such that $\langle (Q_1 - Q_2)^2 \rangle < \alpha^2$. Thus, in all such states, we can measure Q_1 with $\epsilon(Q_1) < \alpha$ without disturbing P_1.

The importance of the above example is the abundance of such state ψ. Let $\mathcal{H}_i$ be the state space of particle i for $i = 1, 2$. There exist a unitary operator U on the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and a state $\eta' \in \mathcal{H}_2$ such that for every η the state $\psi = U(\eta \otimes \eta')$ satisfies the condition $\langle (Q_1 - Q_2)^2 \rangle < \alpha^2$.

A measurement is of type II violation if and only if for any Δ the operator $\Pi(\Delta)$ is the spectral projection of Q corresponding to Δ. All the possible state changes associated with those measurements of type II violation were described in Ref. [10]

We have shown that Heisenberg's reciprocal relation between position-measurement noise and momentum disturbance holds for every measurement with statisitcally independent noise and disturbance, and also clarifies the limitation of Heisenberg's relation. We proposed a new universally valid relation among measurement error, disturbance, and initial uncertainties. This relation reveals possibilities of measurements beyond Heisenberg's relation and clarifies the new constraints for measurements beyond Heisenberg's relation.

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Violation of local uncertainty relations by entangled N-level systems

Holger F. HOFMANN[1] and Shigeki TAKEUCHI[2]

JST-PRESTO, Research Institute for Electronic Science, Hokkaido University, Sappporo, Japan

Abstract

Entangled states can violate the statistical limitations imposed by a local application of the uncertainty principle. In this presentation, we derive some local uncertainty limits for N-level systems and discuss their application to the detection and characterization of NxM mixed state entanglement.

Keywords: Entanglement verification, N-level uncertainties, Local uncertainty relations

Originally, entanglement has been defined as a situation where the correlations between measurement results in two spatially separated systems are so strong that the uncertainty principle appears to be violated [1]. However, the conventional uncertainty relation for position and momentum can only be applied in continuous variable systems. A more general definition of the uncertainty criterion for entanglement requires a formulation of N-level uncertainty limits for arbitrary sets of properties. This generalization cannot be based on commutation relations, since the expectation value of the commutator is always zero for an eigenstate of one of the two properties concerned. We have therefore recently proposed a reformulation of the uncertainty principle based on uncertainty sums. The sum uincertainties for two level systems then allow a derivation of local uncertainty limits for all separable states, providing sufficient criteria for experimental entanglement verification [2].

Sum uncertainty relations can be defined for any set of observables $\{\hat{A}_i\}$ of a physical system. The uncertainty of each observable $\hat{A}_i$ for a given quantum state $\hat{\rho}$ is then defined as the variance of measurement results obtained in von Neumann measurements of $\hat{A}_i$,

$$\delta A_i^2 = \mathrm{Tr}\{\hat{\rho}\hat{A}_i^2\} - (\mathrm{Tr}\{\hat{\rho}\hat{A}_i\})^2. \quad (1)$$

This variance can only be zero if the system is in an eigenstate of $\hat{A}_i$. Therefore, the sum of all uncertainties δA_i^2 can only be zero if there exists a common eigenstate of all $\hat{A}_i$. Otherwise, there exists a non-trivial uncertainty limit U_A, such that

$$\sum_i \delta A_i^2 \geq U \quad (2)$$

for all possible states of the system. This formulation of uncertainty defines a global limit of the simultaneous predictability of observables in a quantum system. However, as pointed

[1]E-mail: h.hofmann@osa.org

[2]E-mail: takeuchi@es.hokudai.ac.jp

out in [2], maximally entangled states allow the precise prediction of measurement outcomes for each property $\hat{A}_i$ in A from the measurement of a corresponding property $-\hat{B}_i$ in B. This is possible because the maximally entangled state is a simultaneous eigenstate of the joint properties $\hat{A}_i + \hat{B}_i$ with an eigenvalue of zero. On the other hand, separable states are limited by the local uncertainty limits of U,

$$\sum_i \delta(A_i + B_i)^2 \geq 2U. \tag{3}$$

Any violation of such a local uncertainty limit therefore proofs that A and B are entangled.

Starting from this general observation, it is possible to construct various sum uncertainty relations and their corresponding local uncertainty limits. The most simple example is the local uncertainty relation for the singlett entanglement of two spin-l systems ($N = 2l + 1$ levels),

$$\sum_{i=x,y,z} \delta(L_i(A) + L_i(B))^2 \geq 2l, \tag{4}$$

where the operators $\hat{L}_i$ are the three components of angular momentum. However, this selection of operators may not be optimal for some cases. For example, it has been shown that the detection of bound 3x3 entanglement can be achieved by defining an asymmetric local uncertainty relation based on the eight generators of the SU(3) operator algebra [3]. This result indicates that local uncertainty relations can even be violated by states that cannot be distilled to singlett form. It may therefore be an interesting question whether every entangled state violates some local uncertainty relation, or if there exist another type of entanglement that does not violate any local uncertainty.

In general, local uncertainty relations provide directly observable criteria for entanglement. They thus allow an identification of entanglement in the context of measurement outcomes, providing deeper insights into the practical aspects of quantum non-locality.

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Simultaneous measurements of spin, signal locality, and uncertainty

Erika ANDERSSON*[1], Stephen M. BARNETT*[2] and Alain ASPECT**

*Department of Physics and Applied Physics, University of Strathclyde, Glasgow G4 0NG, UK

**Laboratoire Charles Fabry de l'Institut d'Optique, UMR 8501 du CNRS, 91403 Orsay Cedex, France

Abstract

We present a connection between the locality principle and the bound on the accuracy of simultaneous measurements of spin of a S=1/2 particle, along any two different directions. The measurement bound is shown to be directly related to the uncertainty relation for the product of the variances. The resulting uncertainty relation is tighter than the Arthurs-Goodman form of the uncertainty relation for simultaneous measurements.

Keywords: Simultaneous measurements, Locality, Uncertainty, Non-commuting observables

It has long been appreciated that simultaneous measurements of non-commuting observables are possible, at the expense of an increase in the variances of the measured observables [1, 2]. In such a measurement, we have to accept some additional noise over the intrinsic quantum uncertainty. This noise appears as an increase in the variances of the jointly measured observables, over and above the Heisenberg limit.

Let us suppose that we measure the spin of a $S = 1/2$ particle simultaneously along two directions, $\mathbf{a}$ and $\mathbf{a}'$. At the moment, we do not need to think about how to achieve this joint measurement. If measured separately, the observables are given by $\hat{\mathbf{A}} = \mathbf{a}\cdot\hat{\sigma}$ and $\hat{\mathbf{A}}' = \mathbf{a}'\cdot\hat{\sigma}$. We denote the values obtained in the simultaneous measurement by A_S and A'_S. For any quantum state, we ask that the expectation values of the simultaneously measured observables must be proportional, with real factors α and α', to the expectation values of the separately measured observables. The variances of the jointly measured observables may then be written

$$\begin{aligned}(\Delta A_S)^2 &= \overline{A_S^2} - \overline{A_S}^2 = 1 - \alpha^2\langle\hat{\mathbf{A}}\rangle^2 \\ (\Delta A'_S)^2 &= \overline{A'^2_S} - \overline{A'_S}^2 = 1 - \alpha'^2\langle\hat{\mathbf{A}}'\rangle^2, \end{aligned} \tag{1}$$

where we have also noted that the measurement result, ± 1, always equals $+1$ when squared. In general, the joint measurement of $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$ results in an increase in their variances as compared to separate measurements, and this forces $|\alpha|$ and $|\alpha'|$ to be smaller than 1. The precise upper bound on $|\alpha|$ and $|\alpha'|$ stems from the fact that a joint probability distribution must exist for A_S and A'_S, and this bound will depend on the directions of $\mathbf{a}$ and $\mathbf{a}'$. Such a bound has previously been derived by considering all possible generalised measurement operators describing the joint measurement [6]. In the following, we present a derivation using the locality principle and the

[1] E-mail: erika@phys.strath.ac.uk
[2] E-mail: steve@phys.strath.ac.uk

existence of entangled states. This requires no further assumptions about the joint measurement itself, other than the definitions made above.

The locality principle, as it is defined here, ensures that no operation performed on one of a pair of entangled states can be detected by observation of its entangled partner [3]. Consider two quantum systems prepared in the singlet state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right). \tag{2}$$

Two observers have access to one quantum system each. On quantum system 2, observer 2 will make a measurement of spin *either* along $\mathbf{b}$ or along $\mathbf{b}'$. This yields the results ± 1 with equal probability. On quantum system 1, observer 1 will make a simultaneous measurement of spin along two directions, $\mathbf{a}$ and $\mathbf{a}'$. Because of locality, the probabilities for the results observer 1 obtains cannot depend on any action taken by observer 2. Observer 1 cannot tell whether observer 2 measured $\mathbf{b}\cdot\hat{\sigma}_2$ or $\mathbf{b}'\cdot\hat{\sigma}_2$. This will provide a bound on how accurately observer 1 can perform the simultaneous measurement.

Let us denote the measurement results by A_S, A'_S, B and B'; these are all ± 1. Suppose that observer 2 has measured spin along $\mathbf{b}$. The probability that observer 1 obtains $A_S = A'_S$ can then be written

$$p(A_S = A'_S) = p(A_S = A'_S = B) + p(A_S = A'_S = -B). \tag{3}$$

The probabilities on the right hand side exist, because joint probability distributions must exist for the triples A_S, A'_S, B (and also for A_S, A'_S, B'); in each run of the experiment, three observables are measured. Because probabilities are positive,

$$p(A_S = A'_S = B) + p(A_S = A'_S = -B) \geq |p(A_S = A'_S = B) - p(A_S = A'_S = -B)|. \tag{4}$$

We can use the correlation functions $E(A,B) = p(A = B) - p(A = -B) = \overline{AB}$ to write

$$p(A_S = A'_S = B) - p(A_S = A'_S = -B) = \frac{1}{2}\left[E(A_S, B) + E(A'_S, B)\right], \tag{5}$$

finally giving us

$$p(A_S = A'_S) \geq \frac{1}{2}|E(A_S, B) + E(A'_S, B)|. \tag{6}$$

In a similar way, if we assume that observer 2 has measured spin along $\mathbf{b}'$, we can derive

$$p(A_S = -A'_S) \geq \frac{1}{2}\left[E(A_S, B') - E(A'_S, B')\right]. \tag{7}$$

Since the probabilities on the left hand sides of these two inequalities are independent of whether observer 2 measured spin along $\mathbf{b}$ or $\mathbf{b}'$, adding the two inequalities, and noting that $p(A_S = A'_S) + p(A_S = -A'_S) = 1$, we obtain

$$|E(A_S, B) + E(A'_S, B)| + |E(A_S, B') - E(A'_S, B')| \leq 2. \tag{8}$$

This inequality bears great resemblance to Bell inequalities [4, 5]. This is not surprising, since the existence of joint probability distributions, or "hidden variables", is the assumption underlying both inequality (8) and Bell inequalities. The difference is, that violations of Bell inequalities mean that either hidden variables cannot exist, or quantum mechanics has to be nonlocal, whereas in the present case, since joint probability distributions necessarily exist for jointly measured observables, inequality (8) *must be satisfied.*

Inequality (8) places restrictions on the correlations between observables of the two quantum systems. We would like obtain a bound on the joint measurement of $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$, involving only observer 1 and quantum system 1. If spin is measured only along $\mathbf{a}$ on quantum system 1, and along $\mathbf{b}$ on quantum system 2, the correlation function for the singlet state is given by

$$E(A,B) = \langle\psi^-|\mathbf{a}\cdot\hat{\sigma}_1\mathbf{b}\cdot\hat{\sigma}_2|\psi^-\rangle = -\mathbf{a}\cdot\mathbf{b}. \tag{9}$$

Since simultaneous measurements reduce expectation values by factors α and α' for any state, we must have

$$E(A_S,B) = \alpha\langle\psi^-|\mathbf{a}\cdot\hat{\sigma}_1\mathbf{b}\cdot\hat{\sigma}_2|\psi^-\rangle = -\alpha\mathbf{a}\cdot\mathbf{b}, \tag{10}$$

and similarly for $E(A'_S,B), E(A_S,B')$ and $E(A'_S,B')$. Using this in (8) gives

$$|(\alpha\mathbf{a}+\alpha'\mathbf{a}')\cdot\mathbf{b}| + |(\alpha\mathbf{a}-\alpha'\mathbf{a}')\cdot\mathbf{b}'| \leq 2. \tag{11}$$

This must be valid for any choice of $\mathbf{b}$ and $\mathbf{b}'$. The left hand side is maximised when $\mathbf{b}$ is parallel to $\alpha\mathbf{a}+\alpha'\mathbf{a}'$ and $\mathbf{b}'$ parallel to $\alpha\mathbf{a}-\alpha'\mathbf{a}'$, giving

$$|\alpha\mathbf{a}+\alpha'\mathbf{a}'| + |\alpha\mathbf{a}-\alpha'\mathbf{a}'| \leq 2. \tag{12}$$

This condition, linking $\alpha, \alpha', \mathbf{a}$ and $\mathbf{a}'$, is the same as obtained in [6]. Geometrically this means that the sum of the diagonals in a parallellogram, with $\alpha\mathbf{a}+\alpha'\mathbf{a}'$ and $\alpha\mathbf{a}-\alpha'\mathbf{a}'$ as its sides, must be less than 2. Unless $\mathbf{a}$ and $\mathbf{a}'$ are parallel, this forces both $|\alpha|$ and $|\alpha'|$ to be strictly less than 1 ($\mathbf{a}$ and $\mathbf{a}'$ are unit vectors). The smaller $|\alpha|$ and $|\alpha'|$ are, the more smeared the jointly measured observables are, since this increases their variances according to equations (1).

The derivation in [6] was made by considering the possible generalised measurement operators describing the joint measurement, whereas the present derivation is based on the locality principle and the existence of entangled states. The locality principle is here used much in the same way as the energy conservation principle may be used to solve physical problems. The derivation is not tied to any particular model of joint quantum measurements.

In the following, we show how the bound (12) and the uncertainty relation for the jointly measured observables are related. Squaring the expression (12), and noting that $|\alpha\mathbf{a}\pm\alpha'\mathbf{a}'|^2 = \alpha^2+\alpha'^2\pm 2\alpha\alpha'\mathbf{a}\cdot\mathbf{a}'$, we obtain

$$|\alpha\mathbf{a}+\alpha'\mathbf{a}'||\alpha\mathbf{a}-\alpha'\mathbf{a}'| \leq 2-\alpha^2-\alpha'^2. \tag{13}$$

Squaring this expression once more, and cancelling terms on both sides, we obtain

$$\alpha^2+\alpha'^2-\alpha^2\alpha'^2(\mathbf{a}\cdot\mathbf{a}')^2 \leq 1. \tag{14}$$

Denoting $\mathbf{a}\cdot\mathbf{a}'$ by $\cos\theta$, we may also write this as

$$\frac{(1-\alpha^2)(1-\alpha'^2)}{\alpha^2\alpha'^2} \geq \sin^2\theta. \tag{15}$$

The uncertainty in the joint measurement arises from two sources: The intrinsic uncertainty in the quantum observables, and the fact that they are measured jointly. The product of the jointly measured square variances can be broken up in four terms as

$$\begin{aligned}(\Delta A_S)^2(\Delta A'_S)^2 &= (1-\alpha^2\langle\hat{\mathbf{A}}\rangle^2)(1-\alpha'^2\langle\hat{\mathbf{A}}'\rangle^2) \\ &= \alpha^2\alpha'^2(1-\langle\hat{\mathbf{A}}\rangle^2)(1-\langle\hat{\mathbf{A}}'\rangle^2)+(1-\alpha^2)(1-\alpha'^2) \\ &\quad +(1-\alpha^2)\alpha'^2(1-\langle\hat{\mathbf{A}}'\rangle^2)+(1-\alpha'^2)\alpha^2(1-\langle\hat{\mathbf{A}}\rangle^2). \end{aligned} \tag{16}$$

In this expression, $1-\langle\hat{\mathbf{A}}\rangle^2$ and $1-\langle\hat{\mathbf{A}}'\rangle^2$ are the "bare" variances of $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$, when measured separately. The quantities $(1-\alpha^2)/\alpha^2$ and $(1-\alpha'^2)/\alpha'^2$ are contributions coming from the fact that the measurement is a joint measurement. A lower bound on their product is given by (15), which is now seen to be an uncertainty relation giving a lower bound on the uncertainty associated purely with the fact that $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}'$ are quantum observables which are measured *jointly*. This bound can be shown to be tight, meaning that there is always a measurement such that equality can be reached, and it does not depend on the measured state at all, only on the measured quantum observables.

An upper bound on the bare variances is obtained from the Heisenberg uncertainty relation,

$$(\Delta\hat{\mathbf{A}})^2(\Delta\hat{\mathbf{A}}')^2=(1-\langle\hat{\mathbf{A}}\rangle^2)(1-\langle\hat{\mathbf{A}}'\rangle^2)\geq\frac{1}{4}|\langle[\hat{\mathbf{A}},\hat{\mathbf{A}}']\rangle|^2=\sin^2\theta|\langle\mathbf{a}_\perp\cdot\hat{\sigma}\rangle|^2, \tag{17}$$

where $\mathbf{a}_\perp$ is perpendicular to both $\mathbf{a}$ and $\mathbf{a}'$. Using (15) and (17), we obtain

$$\frac{(\Delta A_S)^2(\Delta A'_S)^2}{\alpha^2\alpha'^2}\geq\sin^2\theta(1+|\langle\mathbf{a}_\perp\cdot\hat{\sigma}\rangle|)^2 \tag{18}$$

as a bound on the total uncertainty for the simultaneous measurement. This inequality is tighter than the uncertainty relation for simultaneous measurements given by Arthurs and Goodman [2],

$$\frac{(\Delta A_S)^2(\Delta A'_S)^2}{\alpha^2\alpha'^2}\geq|\langle[\hat{\mathbf{A}},\hat{\mathbf{A}}']\rangle|^2=4\sin^2\theta|\langle\mathbf{a}_\perp\cdot\hat{\sigma}\rangle|^2. \tag{19}$$

The Arthurs-Goodman uncertainty relation is not restricted to measurements of spin. The reason why relation (18), which applies only to simultaneous measurements of spin, is tighter, is that we have used a tight bound on the external mesurement uncertainty (15) in its derivation.

As a final remark, in contrast to the relation (15), the Heisenberg uncertainty relation (17) will not always be tight. Schrödinger showed early on that a tighter uncertainty relation can be obtained [7],

$$(\Delta\hat{\mathbf{A}})^2(\Delta\hat{\mathbf{A}}')^2=(1-\langle\hat{\mathbf{A}}\rangle^2)(1-\langle\hat{\mathbf{A}}'\rangle^2)\geq\frac{1}{4}|\langle[\hat{\mathbf{A}},\hat{\mathbf{A}}']\rangle|^2+\frac{1}{4}(\langle\hat{\mathbf{A}}\hat{\mathbf{A}}'+\hat{\mathbf{A}}\hat{\mathbf{A}}'\rangle-2\langle\hat{\mathbf{A}}\rangle\langle\hat{\mathbf{A}}'\rangle)^2, \tag{20}$$

where the second term is the correlation between $\hat{A}$ and $\hat{A}'$. If, in the derivation of the bound for jointly measured observables, this relation is used instead of the ordinary Heisenberg uncertainty relation, the result will be a tighter bound on the joint measurements as well.

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Predicting Quantum Measurement Outcomes Using Local Hidden Variables

Hsu Li-Yi*[1]

* *Physics Division, National Center of Theoretical Sciences, Hsinchu, Taiwan, Republic of China*

Abstract

It is well-known that local hidden variable theories always lead to parity contradictions in predicting some specific outcomes of Greenberger-Horne-Zeilinger (GHZ) state. On the other hand, the local realism hypothesis and quantum theory can make the same predictions in the other outcomes, which containing no knowledge in distinguishing local realism hypothesis and quantum theory. In our study, we focus on the ability of the local realism hypothesis in predicting the GHZ-state outcomes. We consider the prediction ability quantitatively and how to optimize the prediction ability. Moreover, we can represent the prediction ability via the recursive relation.

Keywords: Quantum entanglement, Local hidden variable,

In the celebrated EPR paper, Einstein, Podolsky, and Rosen argued that quantum theory is incomplete [1]. Their premises are quite plausible propositions about locality, reality, and completeness. According to the local realism hypothesis, measurement outcomes are intrinsically deterministic, but appear to be probabilistic because some degrees of freedom are not accurately known. Early and ongoing experiments showed the violation of Bell inequality [2]. These experiments not only lend support to quantum mechanics, but also demonstrate the presence of the entanglement. Recently, Mermin demonstrated the parity contradiction between the local hypothesis and quantum theory in predicting the measurement outcomes of a three-particle Greenberger-Horne-Zeilinger (GHZ) state [3]. Several studies focused on the GHZ-like parity contradictions in the multiparticle multilevel cases, in which, as Mermin pointed out, the local realism hypothesis and quantum theory predict different outcomes of at least *one* experiment [4][5].

Our study explores GHZ-like parity contradictions quantitatively. We do not answer the following question. How many experiments do the local realism hypothesis and quantum theory predict different outcome parities. Instead, we consider how many experiments the local realism hypothesis and quantum theory predict the *same* outcome parities. Logically, if the local realism hypothesis and quantum theory predict the same experiment outcomes, such outcomes provides no knowledge of distinguishing the local realism hypothesis and quantum theory. Suppose that he local realism hypothesis and quantum theory can predict at most $S(N)$ experiment outcome parities in the N-particle two-level case. Consequently, we need at most $S(N)+1$ experiment outcome parities to judge that the particles are of quantum-mechanics or of local realism.

We consider Mermin's gedanken demonstration in the N-particle case. At first, N outcome parities are previously known. From quantum-mechanical view, information of these known

[1]E-mail: cdhu6@phys.ntu.edu.tw

outcome parities is sufficient to infer the maximally-entangled state vector. On the other hand, from local realism viewpoint, information of these known outcome parities are sufficient for finding out all possible instruction sets. It is barely possible to find $S(N)$ by enumerating all combinations of N outcome parities. Hence, we consider the lower bound of $S(N)$, denoted by $S_L(N)$. To find out $S_L(N)$, we can assume that these N known outcomes have the same parities. In this way, it is easily verified that

$$\begin{aligned} S_L(N) &= \max\{\sum_{j=0}^{k} C^N_{4j+i},\ i = 0,1,2,3\} - N \\ &= \max\{\sum_{j=0}^{k} C^N_{4j+i},\ i = 0,2 \text{ or } i = 1,3\} - N. \end{aligned} \tag{1}$$

Furthermore, we can express $S_L(N)$ via the recursive relation. The basic idea of deriving the recursive relation is to consider the experimental settings of the known parities in the N-particle case from those in $(N-1)$-particle case. We can show that

$$S_L(N) = 2^{N-3} + 2S_L(N-2) + N - 4. \tag{2}$$

with initial conditions $S_L(4) = 2$ and $S_L(5) = 5$. It is believed that the lower bound $S_L(N)$ is compact.

In fact, in Mermin's gedanken demonstration, the outcomes parities are random for half of the experimental settings. Many studies and the above discussion only consider those experiments with constant outcome parities. Now we just consider whether the outcome parity under some experimental setting is constant or not. In this case, we can argue that both the local realism hypothesis and quantum theory lead to exactly the same predictions. As a result, in the N-particle case, the ratio of the same predictions is

$$\frac{2^{N-1} + S(N)}{2^N - N} \tag{3}$$

Moreover, the same conclusion can be drawn to the multiparticle multilevel case [5].

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On Thermal Spin States through Beam Splitters

Damian MARKHAM*[1], Mio MURAO** and Vlatko VEDRAL*

*Imperial College, London, UK
**University of Tokyo, Tokyo, Japan

Abstract

We investigate the entanglement generated by passing a thermal spin state through a beam splitter. In the infinite temperature case this can be seen as creating distillable entanglement from a maximally mixed state through unitary operations. It is the truncation of the state that allows for entanglement generation. The output entanglement is investigated for different temperatures and it is found that more randomness - in the form of higher temperature - is better for this set up.

Keywords: Thermal, Spin, Beam Splitter, Entanglement

1 Background

We know that in the infinite dimensional case of the harmonic oscillator, if a state can be described as a statistical mixture of Glauber states, this state cannot be used to generate entanglement using a beam splitter [1, 2].

Here we discuss the finite dimensional case and look at the transition to the infinite dimensional case for thermal spin states. For a finite dimensional system these represent, in some sense, our "most classical" states.

We must be careful here to define what we mean by finite space, since the beam splitter can only be truly defined on an infinite dimensional space. Here when we refer to a state of spin S, we mean that the maximum number state that can be occupied in the input beams is equal to $2S$.

For these thermal states to be physical, such a truncation must be physically imposed - this may be appropriate for example for a finite number of photons in a mode, or atoms in a Bose-Einstein Condensate (BEC).

We find that for any finite dimensional thermal state entanglement is generated when passed through a beam splitter. Even more, we devise an explicit protocol illustrating the distillability of the entanglement (which is not necessarily implied by the state being non-seperable)

[1]E-mail: d.markham@imperial.ac.uk

2 Beam Splitter

As a two mode Fock state, $|m,n\rangle$ enters a beam splitter, in general we ge the output state

$$U_{bs}|m,n\rangle = \sum_{M=0}^{(m+n)} f(m,n,M)|M,m+n-M\rangle, \tag{1}$$

where

$$\begin{aligned} f(m,n,M) &= \sum_{p=max(0,M-n)}^{min(m,M)} \frac{n!m!\sqrt{M!(m+n-M)!}}{p!(p-M+n)!(m-p)!(M-p)!} \\ &\quad \times T^p \bar{T}^{(p-M+n)} R^{(M-p)} R^{(m-p)} (-1)^{(M-p)}, \end{aligned} \tag{2}$$

and where T and R are the complex transition and reflection coefficients, with normalisation $|T|^2+|R|^2=1$. We use this as our definition of the beam splitter on the general number state (i.e. not restricted to optics). We notice that the phases of R and T can be considered as a local phase change on the basis states of the first mode. Thus we assume, without loss of generality that R and T are real. For a general input state the beam splitter operation causes entanglement between two output beams. In fact, the only pure state this is not true of is the Glauber state.

As a measure of entanglement for mixed output states we take the logarithmic negativity $E_{\mathcal{N}}$, defined for a given state ρ as,[3],

$$\begin{aligned} E_{\mathcal{N}}(\rho) &= \log_2 ||\rho^{T_A}||_1 \\ &= \log_2\{\sum_i |\mu_i|\}, \end{aligned} \tag{3}$$

where μ_i are the eigenvalues of ρ^{T_A}, the partial transpose of ρ in subspace of particle A. The logarithmic negativity is a widely used measure of entanglement, indicating inseparability for mixed states. We note that a positive value of this measure does not necessarily imply the existence of useful, i.e. distillable, entanglement (however, for our cases we prove distillability by independent means).

We can also consider optimality of entanglement generation in terms of the beam splitter reflectivity R (since we are deling only with real values, T is set by R through $|T|^2 = 1-|R|^2$). Numerics shows us that on the most part optimality is found when $R=T=1/\sqrt{2}$. Where this isn't the case, the difference is small and the qualitative trends remain the same. Since in this paper we deal with mostly qualitiative results we always use $R=T=1/\sqrt{2}$.

3 Thermal States Through a Beam Splitter

We now consider two thermal states incident on a beam splitter different temperatures,

$$\rho_{in} = \sigma_{T_1} \otimes \sigma_{T_2}, \tag{4}$$

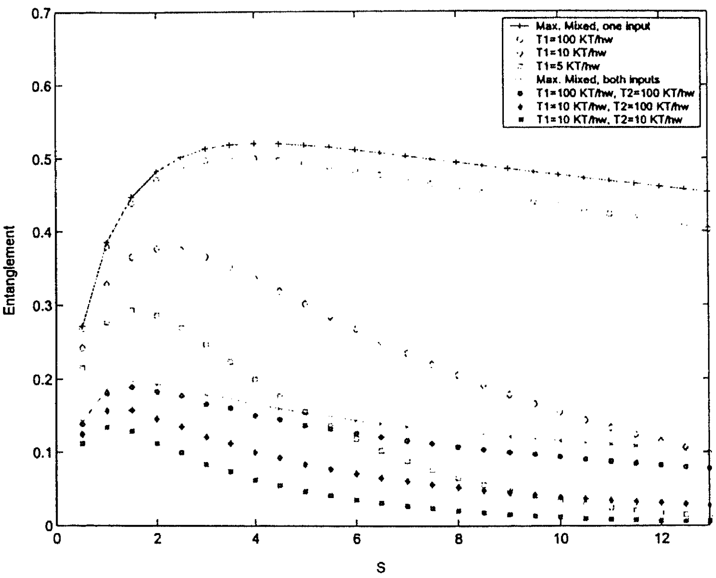
where

$$\sigma_{T_i} = \frac{\sum_{n=0}^{2S} e^{-n\hbar\omega/KT_i}|n\rangle\langle n|}{Z_i}, \tag{5}$$

and $Z_i = \sum_{n=0}^{2S} e^{-n\hbar\omega/KT_i}$ is the partition function. The output state after the beam splitter transformation is then,

$$\begin{aligned} \rho_{out} &= U_{bs}\rho_{in}U_{bs}^{\dagger} \\ &= \frac{1}{Z_1 Z_2}\sum_{m=0}^{2S}\sum_{n=0}^{2S} e^{(-\frac{m}{KT_1}-\frac{n}{KT_2})\hbar\omega} \\ &\quad \sum_{M=0}^{(m+n)}\sum_{M'=0}^{(m+n)} f(m,n,M)\overline{f(m,n,M')} \\ &\quad \times|M,m+n-M\rangle\langle M',m+n-M'|. \end{aligned} \tag{6}$$

Entanglement of various thermal states against S of the input states for $R = T = 1/\sqrt{2}$. The top four, blue, data sets correspond to when one port has a thermal state entering it and the other only the vacuum . The bottom four, red, data sets are when both ports have thermal states entering them. We see the general trend of an initial peak followed by a slow decline (this is more obvious for lower temperatures).



In general the case where the vacuum enters one port gives more entanglement - this can be explained as resulting from the fact the two beams do not destructively interfere with one another.

Starting from the maximally mixed state, which represents an infinite temperature thermal state, the height of the peak is less for lower temps and the peak is occurs earlier. We might expect the entanglement to be greater for lower temperatures, and that the maximally mixed state gives the lowest entanglement. The observed trend can be explained by noticing that for lower temperatures the higher dimensional states are not as populated, restricting the possible entanglement. For the zero temperature we have the ground state which offers no entanglement.

For all the output states mentioned here we can devise a protocol to distill a minimal amount of entanglement. From equation (6) we can see that if local projections are made on either arm, onto the subspace spanned by the states $|0\rangle\langle 0|$ and $|4S\rangle\langle 4S|$ the remaining state is an entangled state of the form $\rho = F|0,0\rangle\langle 0,0| + (1-F)(|0,4S\rangle + |4S,0\rangle)(\langle 0,4S| + \langle 4S,0|)$, which has entanglement for any finite F. For finite S, F is also finite (hence we have entanglement). This can be considered as a two dimensional state and it can be easily shown that it has negative partial transpose, which, for a two dimensional, bipartite state implies distillable entanglement [4]. As S goes to infinity, F goes to one and no entanglement can be found in this way as we expect. All measurements where the projection falls onto the remaining space are discarded. This scheme shows that for any finite S we indeed have distillable entanglement for all the thermal states, though it is very inefficient and it destroys most entanglement.

4 Discussion

- The general trend of generated entanglement is similar for the pure spin coherent states (SCS) and the thermal spin states, that is an initial rise followed by a slow decline.
- We can also discuss the transition in S as an approach to classicality [5]. With respect to this, we see that the approach is slow and appears to be only reached in the infinite limit.
- We may be surprised to get distillable entanglement at all from maximally mixed states using a unitary operation. The answer is of course that either we cannot consider the beam splitter a unitary on the spin space, or that it is the truncation of the thermal state that allows entanglement. This is interesting in terms of how much purity a system needs to create entanglement and can be related to other works, e.g. [6], [7].

A natural extension of this work would be to investigate the significance of the SCS decomposition of finite dimensional systems. Although they are clearly different, it can be hoped that an SCS decomposition may tell us things as interesting as the infinite Glauber state version, which allows us to call states classical and has meaningful constraints with respect to physical operations, for example linear operations on gaussian states, no entanglement can be generated.

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Observation of Antinormally-ordered Intensity Correlation of Electromagnetic Field via Stimulated Parametric Down-conversion

Koji USAMI[*,**,1], Yoshihiro NAMBU[**,***], Bao-Sen SHI[****]
Akihisa TOMITA[***,****] and Kazuo NAKAMURA[*,**,***]

[*] *Tokyo Institute of Technology, Yokohama, Japan*
[**] *CREST, JST, Tokyo, Japan*
[***] *Fundamental Reseach Laboratories, NEC corporation, Tsukuba, Japan*
[****] *ERATO, JST, Tokyo, Japan*

Abstract

When photons are detected by stimulated emission, rather than by absorption, antinormally-ordered photodetection can be realized. Since the detection responds not only to actual photons but also to zero-point fluctuations in the concered modes via spontaneous emission, photon-counting statistics of the detection is distinct from those of the standard normally-ordered photodetection in the certain regime, where even coherent states exhibit the bunching effect and the super-Poissonian fluctuation. We will report on our latest experimental results measuering antinormally-ordered intensity correlations of vacuum and coherent states of electromagnetic field by making use of stimulated parametric down-conversion.

Keywords: Antinormally-ordered correlation, Zero-point fluctuation, Stimulated parametric down conversion

Introduction. Since Planck's quantization hypothesis of electromagnetic field and Einatien's photoelectric theory were appered, quantum nature of electromagnetic field has been intensively explored [1, 2, 3]. The clear evidence of quantized electromagnetic field, that is, *photon*, was finally provided via the photon antibunching effect by measuring normally-ordered intesity correlation of resonance fluorescence from a sodium atom [4]. The normally-ordered photodetection (NOPD) theory due to Glauber [5], which stems from the fact that electromagnetic field is detected by an absorption process, has been playing the central role in the exploration of nonclasical states of electromagnetic field.

Insesitivity to zero-point fluctuations of the NOPD is, on the one hand, the very reason why Planck's spectrum of black-bady radiation is convergent regardless of the divergent term due to them [3, 6], and, on the other hand, why some information is lost during the detection process and thus the initial density matrix cannot be *logically reversible*, that is, cannot be calculated from the post-detection density matrix and the readout of the detecting apparatus [7].

When photons are detected by stimulated emission, however, antinormally-ordered photodetection (ANPD) can be realized. The detection responds not only to actual photons but also to zero-point fluctuations in the concerned modes via spontaneous emission. Then the photon-counting statistics of the detection is distinct from those of the standard NOPD in the regime where the average photon-occupation number in the modes is small [8]. In this regime, even

[1] E-mail: usami@frl.cl.nec.co.jp

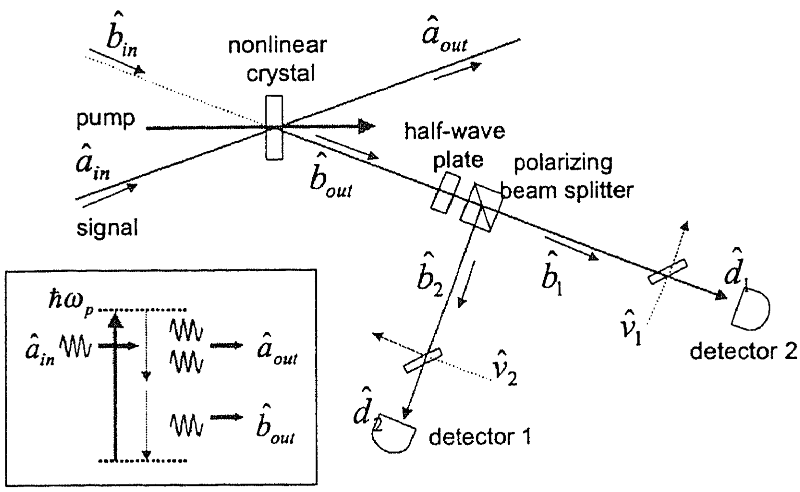

Figure 8: Schematic illustration of the antinormally-ordered photodetection based on stimulated parametric down-conversion.

coherent states exhibit the photon bunching effect and the super-Poissonian photon-number fluctuation due to the zero-point fluctuations. Because of the sensitivity to zero-point fluctuations of the ANPD, it is possible to coserve the system's information during a photodetection process [7]. Thus, the ANPD may serve, for example, an alternative way of monitering quantum system for its feedback control [9].

We will report on our latest experimental results measuering antinormally-ordered intensity correlations of vacuum and coherent states of electromagnetic field by making use of stimulated parametric down-conversion (STDC).

Theory. The process of stimulated emission in a pulsed paramtric down-converter [10, 11, 12] is well suited to realize the ANPD and to observe the antinormally-ordered intensity correlation, because of its large nonlinear response even in the single-pass configuration and its ability to overcom the slow response time of the detectors [11, 12]. The schematic illustration of the STDC-based ANPD is shown in Fig. 8. Since we are deeling with traveling-wave field, the annihilation operator of our interest is the time-dependent and continuous-mode one, which can be written as $\hat{a_{in}}(t) = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} d\omega \hat{a_{in}}(\omega)\exp(-i\omega t)$ with the narrow-bandwidth approximation [1]. Here, $\hat{a_{in}}(\omega)$ represents an annihilation operator for a mode of frequency ω, and satisfies the commutation relation $[\hat{a_{in}}(\omega), \hat{a_{in}}^{\dagger}(\omega')] = \delta(\omega - \omega')$.

The operator $\hat{a_{in}}(\omega)$ is coupled with the operator $\hat{b_{in}}(\omega_p - \omega)$ via parametric interaction with the pump field of frequency ω_p, and evolved into $\hat{a_{out}}(\omega) = \hat{a_{in}}(\omega)\cosh[s(\omega)] - \hat{b_{in}}^{\dagger}(\omega_p - \omega)\exp[-i\vartheta(\omega)]\sinh[s(\omega)]$, while the operator $\hat{b_{in}}(\omega_p - \omega)$ becomes $\hat{b_{out}}(\omega_p - \omega) = \hat{b_{in}}(\omega_p - \omega)\cosh[s(\omega)] - \hat{a_{in}}^{\dagger}(\omega)\exp[i\vartheta(\omega)]\sinh[s(\omega)]$, where, several paramters in the interaction, such as the complex second-order nonlinear susceptibility and length of the crystal, are included in the parameters $s(\omega)$ and $\vartheta(\omega)$ [1, 13]. When the modes which are relevant to the operators $\hat{b_{in}}(\omega_p - \omega)$ are vacua and the parameters, $s(\omega)$ and $\vartheta(\omega)$, are assumed to be constant with respect to ω, the NOPD of the field represented by $\hat{b_{out}}(t) = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} d\omega \hat{b_{out}}(\omega)\exp(-i\omega t)$ coincides with the ANPD of the field $\hat{a_{in}}(t)$ up to the constant factor, that is, $\langle \int_t^{t+T} dt' \hat{b_{out}}^{\dagger}(t')\hat{b_{out}}(t')\rangle =$

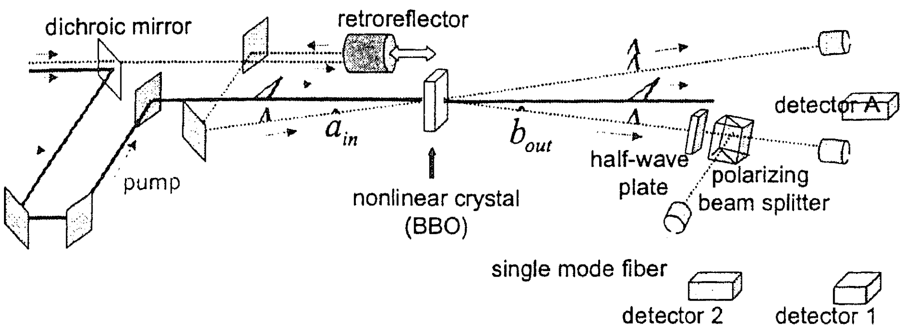

Figure 9: Experimental setup.

$\sinh^2[s]$ $\langle \int_t^{t+T} dt' \hat{a_{in}}(t') \hat{a_{in}}^\dagger(t') \rangle$. Here we use the commutation relation for the operators $\hat{b_{out}}(\omega)$ and $\hat{a_{in}}(\omega)$, t is the initial time of the detection and T is its duration. The angle brackets indicate quantum-mechanical expectation values.

The antinormally-ordered intensity correlation [8] of the field $\hat{a_{in}}(t)$ can be straightforwardly observed as follows. By splitting the field $\hat{b_{out}}(t)$ into $\hat{b_1}(t)$ and $\hat{b_2}(t)$, the standard Hanbury Brown-Twiss interferometer [1] is formed as shown in Fig. 8. Here, the detector's imperfect quantum efficiencies and several optical losses during propagation are modeled by introducing auxiliary vacuum fields $\hat{v_1}(t)$ and $\hat{v_2}(t)$ [1]. Then the normalized normally-ordered intensity correlation of the two fields, $\hat{d_1}(t)$ and $\hat{d_2}(t)$ of Fig. 8, turns out to be the antinormally-ordered counterpart of the field $\hat{a_{in}}(t)$, that is,

$$\frac{\langle \int_{t_1}^{t_1+T} dt' \int_{t_2}^{t_2+T} dt'' \ \hat{d_1}^\dagger(t') \hat{d_1}(t') \hat{d_2}^\dagger(t'') \hat{d_2}(t'') \rangle}{\langle \int_{t_1}^{t_1+T} dt' \ \hat{d_1}^\dagger(t') \hat{d_1}(t') \rangle \langle \int_{t_2}^{t_2+T} dt' \ \hat{d_2}^\dagger(t') \hat{d_2}(t') \rangle} = \frac{\langle \int_{t_1}^{t_1+T} dt' \ \int_{t_2}^{t_2+T} dt'' \ \hat{a_{in}}(t') \hat{a_{in}}(t'') \hat{a_{in}}^\dagger(t'') \hat{a_{in}}^\dagger(t') \rangle}{\langle \int_{t_1}^{t_1+T} dt' \ \hat{a_{in}}(t') \hat{a_{in}}^\dagger(t') \rangle \langle \int_{t_2}^{t_2+T} dt' \ \hat{a_{in}}(t') \hat{a_{in}}^\dagger(t') \rangle}, \tag{1}$$

regardless of the splitting ratio at the polarizing beam splitter, the quantum efficiencies of the detectors, or optical losses.

Comparing with the normally-ordered intensity correlation, it can be easily shown that the antinormally-ordered intesity correlation involves excess contributions, which are due to the stimulated emissions from the zero-point fluctuations in the concerned modes, through the bosonic commutation relation [8]. Since the stimulated emissions from the zero-point fluctuations, that is, the spontaneous emissions, are inherently thermal [14, 12], even coherent states exhibit the photon bunching effect and the super-Poissonian photon-number fluctuation.

Experimental setup and Current status. The rough sketch of our experimental setup is shown in Fig. 9. With a pulsed strong pump field (wave length: 400nm, average power: 200mW, pulse duration: 100fs, and repetition rate: 82MHz) from the second harmonics of the mode-locked Ti:Sapphire laser, the type-I, nondegenarate paramtric down-converter (2mm-thick BBO crystal) is formed. A vacuum field, or a heavily attenuated coherent field from the fundamental of the same laser is the signal field represented by the operator $\hat{a_{in}}$. In the latter case, the single photon-counting rate for the field $\hat{b_{out}}$ should be enhanced owing to the stimulated emissions. Figure 10 shows the enhancement as a function of optical-pass-length difference between the pump field and the signal field.

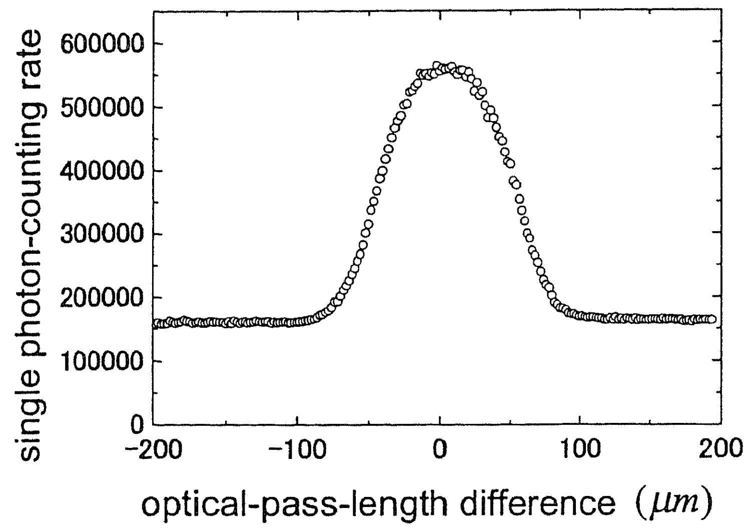

Figure 10: Measured enhancement of the photon-counting rate for the field $\hat{b}_{out}$ due to the stimulated parametric down-conversion.

The antinormally-ordered intensity correlations will be measured via the standard Hanbury Brown-Twiss interferometer consisted of the half-wave plate, the polarizing beam splitter, and the detectors 1 and 2 (Si-avalanche photodiodes: Perkin Elmer SPCM-AQR14). It is known that the ultra-broadband nature of the spontaneous parametric down-conversion (SPDC) makes it difficult to observe the photon bunching effect, namely, the antinormally-ordered intensity correlation of the vacuum field [12]. Currently, we are trying to improve the signal-to-noise ratio for observing the clearer photon bunching effect (up to now, the value of the normalized intensity correlation is 1.3 with 10nm-FWHM interference filter). We will present our latest experimental results at the workshop.

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