Multisheet configuration space and fractional quantum statistics

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We show that if one uses a multisheet configuration space for a system of identical particles, then fractional quantum statistics in two dimensions can be geometrically characterized as a topological quantum number for going from one sheet to the next. Furthermore, the spectrum of such a q-sheet system is shown to consist of the spectra of all species of anyons obeying different fractional statistics $\theta=2\pi p/q$ $(p=0,1,\ldots,q-1)$ with the same denominator q. A lattice computation has been done to verify this. Various aspects of the multisheet formalism are discussed.

Geometrization of physics (together with unification) has been one of the main trends in contemporary physics since Einstein's general relativity[1] and the Kaluza-Klein theory.[2] The recent success of Yang-Mills gauge theory for unifying all nongravitational interactions represents a strategic triumph of the mainstream.[3] And this trend becomes even more intensive in recent attempts of unified string theories.[4] However, quantum statistics, which is known to play a crucial role in quantum physics, is traditionally considered as something nongeometrical. In recent years it has become clear that quantum statistics is closely related to the topology of the many-body configuration space[5,6]—namely, statistics can be characterized by a one-dimensional unitary representation of the first homotopy group of the many-body configuration space. In this paper, as an attempt towards more direct geometrization of statistics, we describe a simple geometric characterization for *fractional* statistics in two-dimensional quantum mechanics as a topological quantum number associated with a *multisheet* configuration space.

It is well known that in two dimensions exotic statistics other than normal bosons and fermions are allowed.[7–10] They are described by an angular parameter θ. We will show that when $\theta/2\pi$ is fractional, a simple geometric characterization can be given to θ with the use of a multisheet configuration space for a system of identical particles. More concretely, if one considers a system with a q-sheet configuration space, then a fractional statistics parameter $\theta=2\pi p/q$ can be viewed as a topological quantum number associated with going from one sheet to the next. In particular, one may use the wave function of such a q-sheet system to give a unifying treatment for all species of anyons obeying different statistics with the same denominator q, in the sense that the spectrum of the system consists of spectra of anyons with different $\theta=2\pi p/q$ $(p=0,1,\ldots,q-1)$.

Let us start with the braid-group formalism for fractional statistics.[9] Consider anyons in a planar system. The many-body configuration space for the identical particles is

$$C=(R^2\times R^2\times\cdots\times R^2-D)/P_N$$

where D consists of configurations with two or more particles sitting on the same position and P_N is the permutation group for particle labels. The many-body wave function for anyons is known to form a one-dimensional representation of the first homotopy group (or the fundamental group) of C, which is the well-known braid group $B_N(\mathbb{R}^2)$. The generators of the latter consist of the pair-exchange operators σ_i $(i=1,\ldots,N-1)$, which interchanges the ith and $(i+1)$th particles along a counterclockwise loop without enclosing any other particles. These generators satisfy the defining relations

$$\begin{aligned}&\sigma_i\sigma_j=\sigma_j\sigma_i \quad (i\neq j\pm 1)\\ &\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}\,.\end{aligned}\tag{1}$$

The one-dimensional representations of σ_i's are labeled by an angular statistics parameter θ $(0\le\theta<2\pi)$. In this paper we use the convention that the quantum-mechanical anyon Hamiltonian is the same as the classical Hamiltonian and the statistics are represented by the boundary conditions satisfied by the anyon wave functions. As shown in Ref. 9, for a generic θ the boundary conditions are such that the wave function is single valued on the universal covering $\tilde{C}$ of C, which is related to the latter by $C=\tilde{C}/B_N(\mathbb{R}^2)$. Since $B_N(\mathbb{R}^2)$ is an infinite non-Abelian group, the universal covering $\tilde{C}$ has infinitely many sheets.

However, a great simplification occurs when $\theta/2\pi$ is fractional. To describe anyons with $\theta/2\pi=p/q$ (p,q positive integers), one may use a q-sheet wave function on the many-body configuration space C, or, more precisely, a single-valued wave function on the q-fold covering C_q of C defined as follows: Any element of $B_N(\mathbb{R}^2)$ is a product of the generators σ_i $(i=1,2,\ldots,N-1)$ and their inverse. We call the sum of the powers of the generators in the product modulo q as the index of the element. It is easy to see that the subgroup consisting of all elements with index 0, which we denote as $B_{N,q}(\mathbb{R}^2)$, is a normal subgroup of $B_N(\mathbb{R}^2)$. And

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$$B_N(\mathbb{R}^2)/B_{N,q}(\mathbb{R}^2)=\mathbb{Z}_q ,$$

the cyclic group of order q. The quotient space $\tilde{C}/B_{N,q}(\mathbb{R}^2)$ is a q-sheet covering of C, which we define as C_q. It is a bundle based on C with a discrete fiber $\mathbb{Z}_q$, and its fundamental group is known to be $B_{N,q}(\mathbb{R}^2)$. Thus one can easily see that the boundary conditions for anyons with $\theta=2\pi p/q$ are such that the wave function is always single valued on C_q: Each generator σ_i gives rise to the same phase $e^{i\theta}=e^{i2\pi p/q}$, thus every element in $B_{N,q}(\mathbb{R}^2)$ gives rise to a factor of unity. The action of the braid-group generator σ_i in C_q is the same as the action of the generator of $\mathbb{Z}_q$ on a fiber—namely, σ_i moves one point in C_q from one sheet to the corresponding point in the next sheet on the same fiber; in particular, the action of $(\sigma_i)^q$ on C_q is effectively the same as the identity. Thus the anyon wave function can be viewed as the eigenfunction of these sheet-shifting operators σ_i in C_q with eigenvalue $e^{i2\pi p/q}$. This is the desired geometric characterization of fractional statistics as the quantum number related to sheet-shifting operators in the q-sheet configuration space C_q.

More explicitly, we may introduce a multicomponent many-body wave function $\Psi(r_1,r_2,\ldots,r_N;\alpha)$ in the configuration space C, where $r_1,r_2,\ldots,r_N$ are anyon coordinates and α is the sheet index taking values in the set $(0,1,\ldots,q-1)$, i.e., the set of non-negative integers modulo q. We require that this wave function be totally symmetric in $r_1,r_2,\ldots,$ and r_N, but its sheet index is shifted by 1 $(\alpha\to\alpha+1)$ when r_i exchanges with r_j along a counterclockwise loop without enclosing other particles. Denote by T the unitary operator that shifts α by 1. Obviously, T satisfies $(T)^q=1$. Furthermore, the many-body wave function is required to be an eigenfunction of the operator T with eigenvalue $e^{i2\pi p/q}$, then it describes a system of anyons with $\theta=2\pi p/q$.

Since the exchange operation σ_i is realized by the sheet-shifting operator T in this formalism, the consistency requires that the many-body wave function $\Psi(r_1,r_2,\ldots,r_N;\alpha)$ should go to zero whenever any two coordinates tend to coincide. (This is the same condition for the Hamiltonian to have a self-adjoint extension.) Physically, the wave function can be thought of as a q-component wave function for *hard-core bosons.*

What is more interesting is that this formalism will give a unifying treatment of anyons with different statistics, if we do not require Ψ to be an eigenstate with a definite eigenvalue of T. In this case, we actually have a q-dimensional representation of the braid group $B_N(\mathbb{R}^2)$ with all σ_i's realized by the one and same operator T, given by the following matrix (acting on the sheet index α):

$$T=\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ 0 & & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{2}$$

which satisfies $(T)^q=1$. Obviously, the braid-group relations (1) are satisfied. It is easy to see that T commutes with the classical Hamiltonian (without statistical vector potentials), which is symmetric in $r_1,r_2,\ldots,$ and r_N and does not contain any sheet index. So T must be conserved and its eigenvalues are of the form $e^{i2\pi p/q}$ $(p=0,1,\ldots,q-1)$. Notably, T gives rise to superselection rules. Any physical observable A that is symmetric in $r_1,r_2,\ldots,r_N$ and does not involve the sheet index must commute with T; it cannot have nonvanishing matrix elements between two states $|p\rangle$ and $|p'\rangle$ having different eigenvalues of T:

$$\begin{aligned}\langle p'|A|p\rangle&=\langle p'|T^{-1}AT|p\rangle\\&=e^{-i2\pi(p'-p)/q}\langle p'|A|p\rangle=0 ,\end{aligned} \tag{3}$$

unless $p=p'$. So the Hilbert space is divided into subspaces belonging to different eigenvalues of T which satisfy the superselection rule (3). The eigenvalues of T should be interpreted as the statistics parameter of the particles. Thus, if we diagonalize the Hamiltonian for such a system, the energy spectrum consists of all possible levels of N identical anyons with all different $\theta=2\pi p/q$ $(p=0,1,\ldots,q-1)$. The superselection rule (3) guarantees that there is no mixing between states of different statistics.

Analytically, it is hard to work with the appropriate boundary conditions (or strength-free vector potentials) for $\Psi(r_1,r_2,\ldots,r_N;\alpha)$ with $N>2$ that realize the local exchanges σ_i by the matrix T, but is easy to realize them on a square lattice for exact diagonalization by computer. To do this, we just attach a string to each anyon in the usual way (see Fig. 1): Every string starts from, say, the lower-right plaquette of the anyon site and runs horizontally in parallel to each other to the right until hitting the boundary of the lattice. But the rule for anyon hopping across the string is unusual: Instead of giving a phase,

$$\alpha\to\alpha+1 \quad \text{for hopping upward across a string .} \tag{4}$$

More precisely, when an anyon hops upwards across the string of another anyon from below (or downwards from above), the sheet index of many-body wave function changes by $+1$ (or -1) without any extra statistical phase factor. Our previous conclusion asserts that by diagonalizing a Hamiltonian with such a hopping rule (4), one will obtain a spectrum composed of all levels of N identical anyons with all possible statistics

$$\theta=0,2\pi/q,\ldots,2\pi(q-1)/q .$$

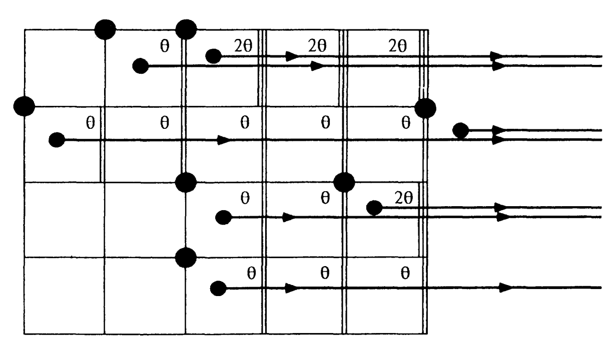

FIG. 1. A string configuration representing anyons.

TABLE I. Twenty-five lowest energies of a three-anyon system on a 4×3 lattice with five sheets.

	Energy	p (mod q)
E_1	-7.4529	0
E_2	-7.1072	1,4
E_3	-7.1072	1,4
E_4	-6.6113	2,3
E_5	-6.6113	2,3
E_6	-6.2875	0
E_7	-6.2298	2,3
E_8	-6.2298	2,3
E_9	-6.1391	1,4
E_{10}	-6.1391	1,4
E_{11}	-5.8507	1,4
E_{12}	-5.8507	1,4
E_{13}	-5.8390	0
E_{14}	-5.8175	1,4
E_{15}	-5.8175	1,4
E_{16}	-5.6658	2,3
E_{17}	-5.6658	2,3
E_{18}	-5.5494	0
E_{19}	-5.5395	1,4
E_{20}	-5.5395	1,4
E_{21}	-5.4892	0
E_{22}	-5.4506	2,3
E_{23}	-5.4506	2,3
E_{24}	-5.2129	2,3
E_{25}	-5.2129	2,3

TABLE III. Twenty-five lowest energies of a three-anyon system on a 4×3 lattice with five sheets in a uniform external magnetic field with flux $\phi_P=2\pi/50$ per plaquette.

	Energy	p (mod q)
E_1	-7.4407	0
E_2	-7.1974	1
E_3	-6.9946	4
E_4	-6.6477	3
E_5	-6.5514	2
E_6	-6.3857	2
E_7	-6.2876	0
E_8	-6.2286	4
E_9	-6.1122	1
E_{10}	-6.0727	3
E_{11}	-5.8634	4
E_{12}	-5.8221	0
E_{13}	-5.8100	4
E_{14}	-5.8083	1
E_{15}	-5.7530	1
E_{16}	-5.6835	2
E_{17}	-5.6502	0
E_{18}	-5.6439	3
E_{19}	-5.5930	1
E_{20}	-5.4808	3
E_{21}	-5.4507	4
E_{22}	-5.4144	2
E_{23}	-5.3774	0
E_{24}	-5.3264	0
E_{25}	-5.2385	2

We have done a lattice computation to verify the above results. We have taken $q=5$ sheets and $N=3$ free anyons on a 4×3 square lattice and have computed all energy levels by using the above rules. The results agree with what we predicted before. For the purpose of illustration, we list the lowest 25 levels in column 1 of Table I. To identify the corresponding statistics parameters, we also did the usual (one-sheet) computations for anyons with fixed $\theta=2\pi/5$, $4\pi/5$, $6\pi/5$, $8\pi/5$, and $10\pi/5$ (hard-core boson), respectively. The results are shown in Table II. It is clear that the energy levels coincide exactly with those obtained from the five-sheet calculation; this gives the identification of θ for the five-sheet levels, shown in the second column in Table I. In this way, we have explicitly verified that with one calculation using q sheets, one really obtains the energy levels for all species of anyons with different fractional values of θ of the same denominator q. We note the unlike usual one-sheet calculations for anyons, which inevitably deal with complex hopping matrix elements, all matrix elements involved in the q-sheet calculation are real.

One may note the double degeneracies in the five-sheet case, which arise from the symmetry of free anyons under $\theta\rightarrow(-\theta)$. This symmetry can be broken by imposing an external uniform magnetic field on the lattice. Taking the flux per plaquette to be $\Phi_B=2\pi/50$, we have repeated the above five-sheet and one-sheet computations and obtain the levels and identifications in Table III and IV. The double degeneracies between $\theta=2\pi p/5$ and $\theta=-2\pi p/5$ $(p\neq0,5)$ in the multisheet treatment have disappeared.

We emphasize that the sheet index α is assigned to the many-body state, but not to each individual anyon. It is easy to see that the assignment of a sheet index α_i to each anyon would not give us fractional statistics, even if the wave function is required to depend only on the sum of all sheet indices α_i.

We also note that our q-sheet treatment is valid with

TABLE II. Five lowest energies of a three-anyon system on a 4×3 lattice with fixed $\theta=2\pi/5$, $4\pi/5$, $6\pi/5$, $8\pi/5$, $10\pi/5$ (hard-core boson), respectively.

Statistics	$1\cdot2\pi/5$	$2\cdot2\pi/5$	$3\cdot2\pi/5$	$4\cdot2\pi/5$	$0\cdot2\pi/5$
E_1	-7.1072	-6.6113	-6.6113	-7.1072	-7.4529
E_2	-6.1391	-6.2298	-6.2298	-6.1391	-6.2875
E_3	-5.8507	-5.6658	-5.6658	-5.8507	-5.8390
E_4	-5.8175	-5.4506	-5.4506	-5.8175	-5.5494
E_5	-5.5395	-5.2129	-5.2129	-5.5395	-5.4892

TABLE IV. Five lowest energies of a three-anyon system on a 4×3 lattice with one-sheet in a uniform external magnetic field with flux $\phi_P=2\pi/50$ per plaquette, with fixed $\theta=2\pi/5$, $4\pi/5$, $6\pi/5$, $8\pi/5$, $10\pi/5$ (hard-core boson), respectively.

Statistics	$1\cdot 2\pi/5$	$2\cdot 2\pi/5$	$3\cdot 2\pi/5$	$4\cdot 2\pi/5$	$0\cdot 2\pi/5$
E_1	-7.1974	-6.5514	-6.6477	-6.9946	-7.4407
E_2	-6.1122	-6.3857	-6.0727	-6.2286	-6.2876
E_3	-5.8083	-5.6835	-5.6439	-5.8634	-5.8221
E_4	-5.7530	-5.4144	-5.4808	-5.8100	-5.6502
E_5	-5.5930	-5.2385	-5.2112	-5.4507	-5.3774

an arbitrary classical interacting Hamiltonian that is symmetric in $r_1, r_2, \ldots,$ and r_N (without statistical vector potentials). In particular, our hopping rule (4) for putting q sheets on a lattice can be used in combination with other lattice rules[11–13] for putting anyons on a periodic lattice appropriate for a cylinder or a torus with or without Aharanov-Bohm flux through the holes.

The typical feature of the above q-sheet treatment is the use of a multicomponent wave function corresponding to a higher-dimensional braid-group representation. The use of such multicomponent wave functions has been made previously in a number of topologically nontrivial situations, such as anyons on a torus[14,12,13] or fractionally charged quasiparticles on a cylinder.[15] For anyons on a plane, our multicomponent wave function corresponds to a reducible braid-group representation and can be interpreted as the use of a multivalued wave function. One is naturally concerned about whether it is legitimate to use a multivalued wave function. Certainly an arbitrary multivaluedness would violate the probability interpretation. However, path-integral formalism on a multiply connected configuration space[5,16,17] tells us that multivalued wave functions be properly used, if and only if the wave function provides a representation of the fundamental group of the configuration space. In other words, the multivaluedness should be available as holonomies[18] of a flat connection (i.e., gauge potential with zero field strength) in the configuration space.[19] In short, the allowed multivaluedness of a wave function in C must be such that it can be eliminated by introducing a flat connection in C, or it is equivalent to a single-valued wave function on a covering of C.

We also note the formal similarity of the multisheet interpretation of θ statistics to that of the vacuum θ parameter in QCD.[20,21] There in the temporal gauge $A_0=0$, the configuration space of QCD is the orbit space $A^{(3)}/G^{(3)}$, where $G^{(3)}$ is the group of gauge transformations in three-space and $A^{(3)}$ is the space of all gauge potentials in three-space, which is nothing but the universal covering of the gauge-orbit space. The wave functional is always single valued in $A^{(3)}$, but may be multivalued in the orbit space, and the multivaluedness is characterized by the vacuum θ parameter, which also satisfies a superselection rule[20,21] similar to our Eq. (3).

We stress that our discussion of q-sheet (or q-component) wave functions is not purely academic; they may well be related to realistic *planar* systems. Recently it has been argued that both fractionally quantized Hall conductance[22–25] and fractional charge of quasiparticles[15] in a cylindrical fractional quantum Hall system require the existence of a set of low-lying states which flow into each other as the magnetic flux threading through the hole varies. It seems plausible that these states on a cylinder may have counterparts in certain two-dimensional planar systems of topological order,[26] in the sense that it is more appropriate to describe these planar states in terms of a q-component wave function as we did for the cylinder case.[15] The number q of the components is presumably determined by (or closely related to) the topological order. It is well known that if one introduces axions[27,28] in QCD, then the θ parameter becomes a dynamically determined parameter. The above-mentioned analogy between the θ parameter and θ statistics suggests the interesting possibility that in certain cases the statistics parameter may be dynamically determined to be some preferred value in a way similar to the preferred value $\theta=0$ in QCD with axions. Our above q-sheet formalism using the T matrix (2) for local exchanges would be particularly suitable for such a situation.

In conclusion, we have given fractional quantum statistics a geometric characterization by using a multisheet configuration space. Namely, fractional statistics can be viewed as a topological quantum number associated with the sheet-shifting operator. Theoretically, this point of view may be useful for the geometrization of physics. In practice, it may be useful to unify all anyons with different fractional statistics or to determine dynamically the preferred value of the statistics parameter of quasiparticles. The multisheet formalism is generalizable to vortex like quasiparticles in three-dimensional systems.

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