Universal behavior of correlations between eigenvalues of random matrices

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Recently, the smoothed correlation between the density of eigenvalues of Hermitian random matrices was found to be universal, that is, independent of the probability distribution from which the random matrices are taken. We study this universal correlation numerically by ensemble averaging, using the Monte Carlo sampling method. Although the density of eigenvalues and the "bare" correlation between the density of eigenvalues are certainly not universal, we find that the smoothed correlation indeed shows a universal behavior. The orthogonal and the symplectic ensembles are also studied numerically. The smoothed correlation is shown to be universal in these cases.

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We examine numerically the universal behavior of the correlation between the density of eigenvalues of random matrices discovered recently by Brézin and Zee [1–3]. Let ϕ be $N \times N$ Hermitian matrices taken from the probability distribution

$$P(\phi) = \frac{1}{Z}\exp[-N\mathrm{Tr}V(\phi)], \tag{1}$$

where $V(\phi)$ is an even polynomial of ϕ. Namely, the unitary ensemble is considered here. The density of the eigenvalues $\rho(\lambda)$ and the correlation of the eigenvalues $\rho(\mu,\nu)$ are defined respectively by

$$\rho(\lambda) = \left\langle \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\right\rangle, \tag{2}$$

$$\rho(\mu,\nu) = \left\langle \frac{1}{N}\sum_{i=1}^{N}\delta(\mu-\lambda_i)\frac{1}{N}\sum_{j=1}^{N}\delta(\nu-\lambda_i)\right\rangle, \tag{3}$$

where $\langle\,\rangle$ is an ensemble average and λ_i denotes the ith eigenvalue of ϕ. The connected part of the correlation function is defined by

$$\rho_c(\mu,\nu) = \rho(\mu,\nu) - \rho(\mu)\rho(\nu). \tag{4}$$

The density $\rho(\lambda)$ has long been known to depend in detail on the distribution $P(\phi)$, i.e., $V(\phi)$ [4], as might be expected. Brézin and Zee [1] proposed an ansatz for the asymptotic behavior of the orthogonal polynomials for any V and as a result were able to determine the connected correlation function in the large N limit. They found that $\rho_c(\mu,\nu)$ oscillates wildly as a function of μ and ν and depends on V in detail, again as might be expected. But perhaps surprisingly, they found that when these oscillations were smoothed over, the smoothed connected correlation ${\rho_c}^{\text{smooth}}(\mu,\nu)$ had the universal form

$${\rho_c}^{\text{smooth}}(\mu,\nu) = -\frac{1}{2N^2\pi^2a^2}f(\mu/a,\nu/a), \tag{5}$$

where the universal function

$$f(x,y) = \frac{1}{(x-y)^2}\frac{(1-xy)}{\sqrt{(1-x^2)(1-y^2)}} \tag{6}$$

did not depend on V. Here a denotes half of the width of the spectrum of eigenvalues. [For simplicity, we assume here that $V(\phi)$ is an even polynomial in ϕ and so the spectrum is symmetric about the origin.] The width of the spectrum depends in a rather complicated way on V [4].

More precisely, the smoothed connected correlation ${\rho_c}^{\text{smooth}}(\mu,\nu)$ is defined as the connected correlation averaged over intervals $\delta\mu$ and $\delta\nu$ much less than $O(1)$ but larger than $O(1/N)$, so that Eq. (5) is valid where $|\mu-\nu|, |\pm a-\mu|$, and $|\pm a-\nu|$ are larger than $O(1/N)$. The universality of the function f has since been verified by Beenakker [5], by Eynard [6], and by Forrester *et al.* [7] using other methods.

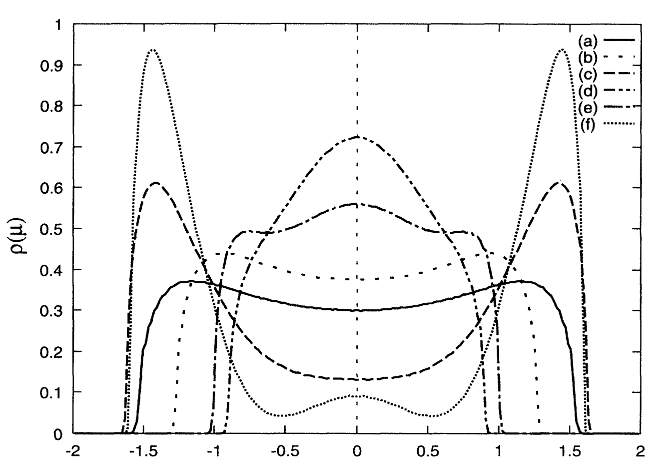

FIG. 1. Density of the eigenvalues for the probability distribution (1). The parameters (v_2, v_4, v_6) are (a) (0.46, 0.25, 0.21), (b) (0.87, 0.36, 0.68), (c) (−0.70, −0.89, 0.91), (d) (5.77, −5.37, 5.62), (e) (3.32, −3.72, 5.20), and (f) (0.0271, −4.63, 2.55).

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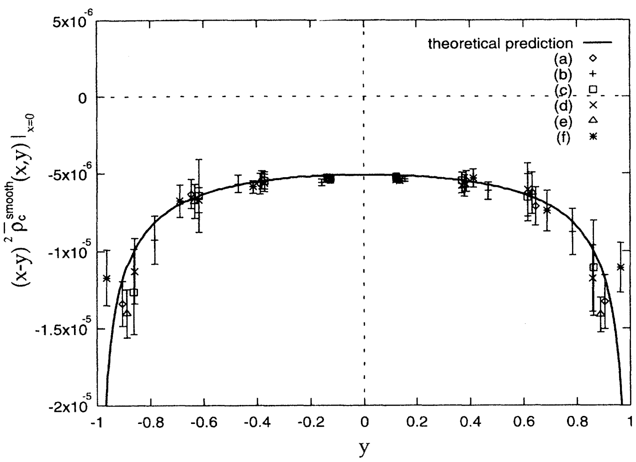

FIG. 2. Smoothed connected correlation

$$(x-y)^2\bar{\rho}_c{}^{\text{smooth}}(x,y)|_{x=0},$$

where x and y are the scaling variables. The parameters (v_2, v_4, v_6) are the same as in Fig. 1. The solid line represents Eq. (5).

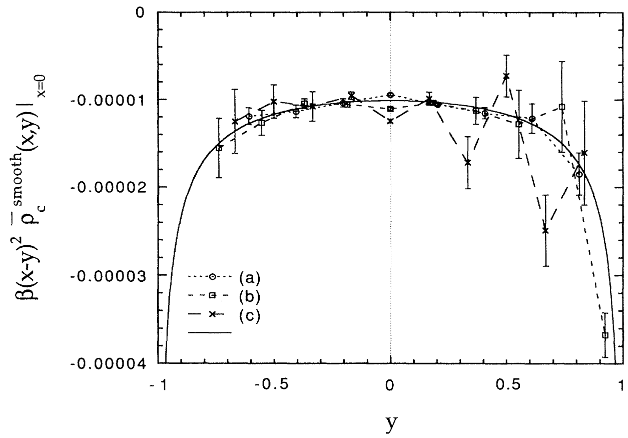

FIG. 4. Smoothed connected correlations

$$\beta(x-y)^2\bar{\rho}_c{}^{\text{smooth}}(x,y)|_{x=0}$$

for (a) orthogonal (β=1), (b) unitary (β=2), and (c) symplectic (β=4) ensembles, where x and y are the scaling variables. The unitary ensemble is plotted for comparison to other ensembles. The parameters (v_2, v_4, v_6) are (0.0757, −2.24, 6.15) and the matrix size N is 100. The solid line represents Eq. (9).

In this paper we study numerically the correlation between the density of eigenvalues of large random matrices. The purpose of this study consists of (i) verifying the conclusion of Brézin and Zee and (ii) analyzing those situations not tractable numerically.

We will take $V(\phi)$ to be

$$V(\phi) = v_2\phi^2 + v_4\phi^4 + v_6\phi^6. \tag{7}$$

We choose the matrix size N to be 100 and performed the ensemble average by Monte Carlo (MC) method with importance sampling. In the calculation, 10^6 MC samples are taken.

First, we calculate the density and the correlation for three sets of $v_2, v_4,$ and v_6. The result for the density is displayed in Fig. 1, which shows clearly that it is not universal. The smoothed connected correlation function are also not universal. Now let us multiply the smoothed connected correlation by a^2 and express the result in terms of the scaling variables $x = \mu/a$ and $y = \nu/a$,

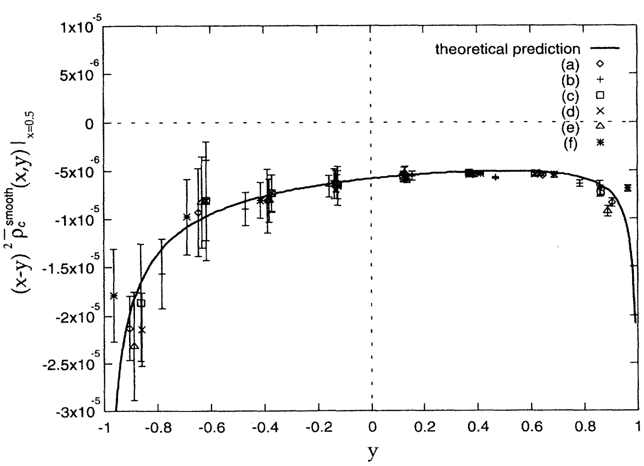

FIG. 3. Smoothed connected correlation

$$(x-y)^2\bar{\rho}_c{}^{\text{smooth}}(x,y)|_{x=0.5},$$

where x and y are the scaling variables. The parameters (v_2, v_4, v_6) are the same as in Fig. 1. The solid line represents Eq. (5).

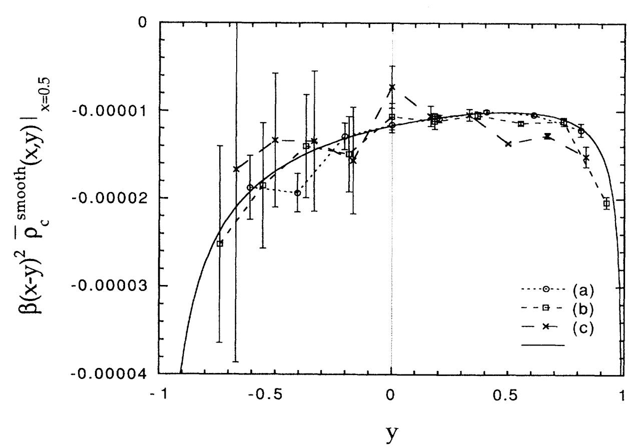

FIG. 5. Smoothed connected correlations

$$\beta(x-y)^2\bar{\rho}_c{}^{\text{smooth}}(x,y)|_{x=0.5}$$

for (a) orthogonal (β=1), (b) unitary (β=2), and (c) symplectic (β=4) ensembles, where x and y are the scaling variables. The unitary ensemble is plotted for comparison to other ensembles. The parameters (v_2, v_4, v_6) and N are the same as in Fig. 4. The solid line represents Eq. (9).

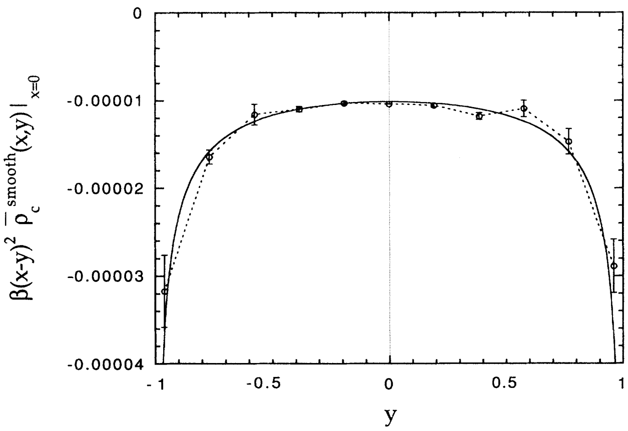

FIG. 6. Smoothed connected correlation

$$\beta(x-y)^2\bar{\rho}_c{}^{\mathrm{smooth}}(x,y)|_{x=0}$$

for the probability distribution (10), where x and y are the scaling variables. β is 1.5, the parameters (v_2, v_4, v_6) are (0.0757, −2.24, 6.15), and the matrix size N is 100. The solid line represents Eq. (9).

$$\bar{\rho}_c{}^{\mathrm{smooth}}(x,y) = a^2\rho_c{}^{\mathrm{smooth}}(\mu,\nu). \tag{8}$$

Quite dramatically, all the curves now fall on the same universal curve, as shown in Figs. 2 and 3. The numerical errors grow near the edge of the spectra. We take smoothing intervals $\delta\mu$ and $\delta\nu$ to be around 0.25, which satisfy the smoothing conditions. We estimate the error bars to be the standard deviation of the results divided into several parts. The error bars are of the MC sampling since they decrease as $1/\sqrt{N}$.

Analytically, it is easiest to deal with Hermitian matrices (unitary ensembles). Numerically, however, we can easily treat the cases of real symmetric matrices (orthogonal ensembles) and quaternion-real, self-dual matrices

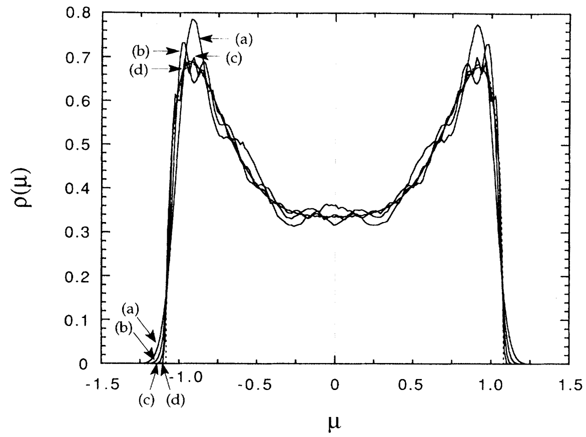

FIG. 8. Density of the eigenvalues for the unitary ensemble (β=2) with the probability distribution (1). The parameters (v_2, v_4, v_6) are (0.0757, −2.24, 6.15). The matrix size N is (a) 5, (b) 10, (c) 25, and (d) 100. The dashed line is the analytical result in the large N limit [4].

(symplectic ensembles). Using the functional derivative method Beenakker [5] obtained

$$\rho_c{}^{\mathrm{smooth}}(\mu,\nu) = -\frac{1}{\beta N^2\pi^2a^2}f(\mu/a,\nu/a), \tag{9}$$

where β is 1, 2, and 4 for orthogonal, unitary, and symplectic ensembles, respectively. It cannot be justified easily since there is no explicit procedure for smoothing. Our numerical results presented in Figs. 4 and 5, however, support his prediction. Again, we choose $\delta\mu$ and $\delta\nu$ to be around 0.25. As for the β dependence, the MC errors become large as β increases.

Indeed, random matrix models, as pointed out by Dyson and others, have a gas analog. The distribution of eigenvalues has the form

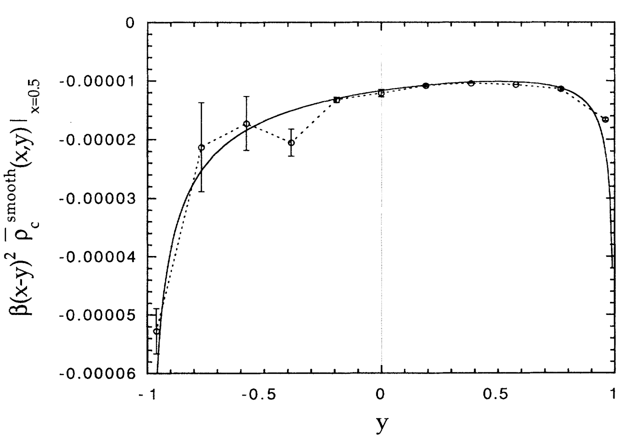

FIG. 7. Smoothed connected correlation

$$\beta(x-y)^2\bar{\rho}_c{}^{\mathrm{smooth}}(x,y)|_{x=0.5}$$

for the probability distribution (10), where x and y are the scaling variables. The parameters β, (v_2, v_4, v_6), and N are the same as in Fig. 6. The solid line represents Eq. (9).

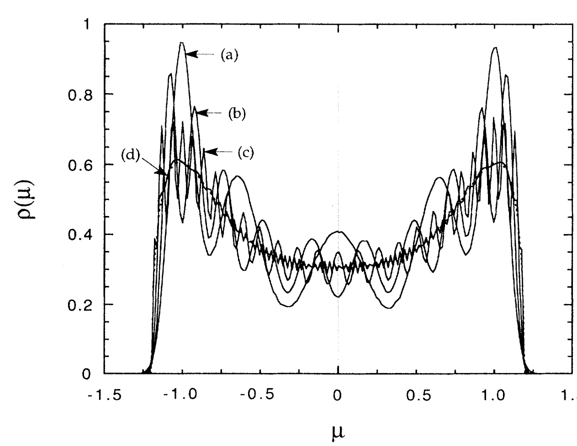

FIG. 9. Density of the eigenvalues for the symplectic ensemble (β=4) with the probability distribution (1). The parameters (v_2, v_4, v_6) are the same as in Fig. 8. The matrix size N is (a) 5, (b) 10, (c) 25, and (d) 100. The dashed line is the analytical result in the large N limit [4].

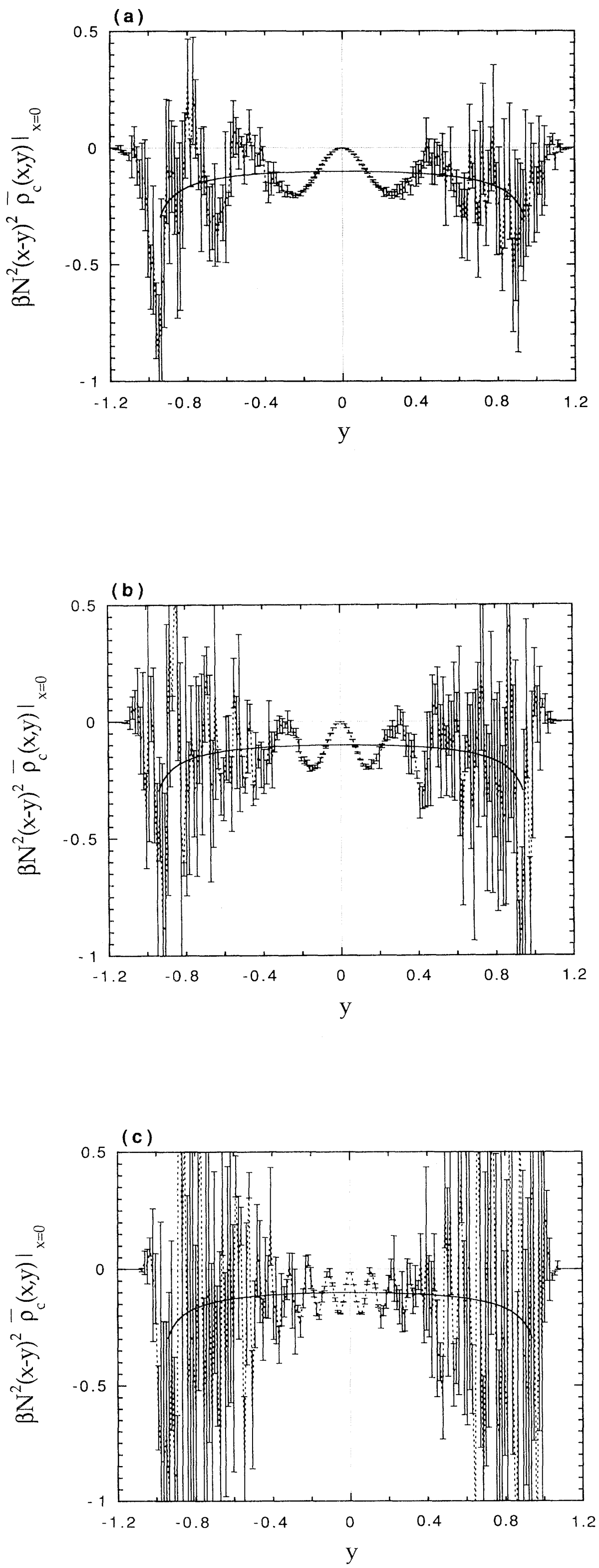

FIG. 10. Connected correlations $\beta N^2(x-y)^2\bar{\rho}_c(x,y)|_{x=0}$ for the unitary ensemble (β=2) with the probability distribution (1). The parameters (v_2, v_4, v_6) are the same as in Fig. 8. The matrix size N is (a) 5, (b) 10, and (c) 25. The solid line represents Eq. (9).

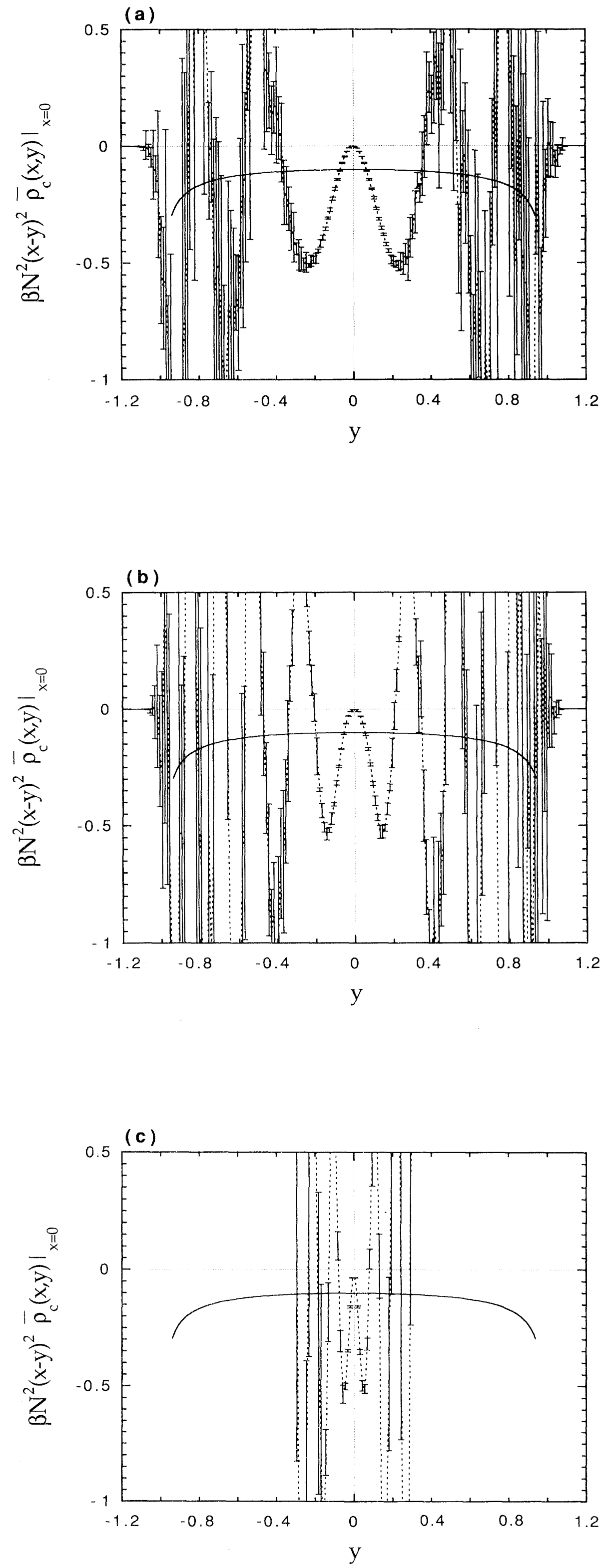

FIG. 11. Connected correlations $\beta N^2(x-y)^2\bar{\rho}_c(x,y)|_{x=0}$ for the symplectic ensemble (β=4) with the probability distribution (1). The parameters (v_2, v_4, v_6) are the same as in Fig. 8. The matrix size N is (a) 5, (b) 10, and (c) 25 (data outside $y = \pm 0.3$ are omitted). The solid line represents Eq. (9).

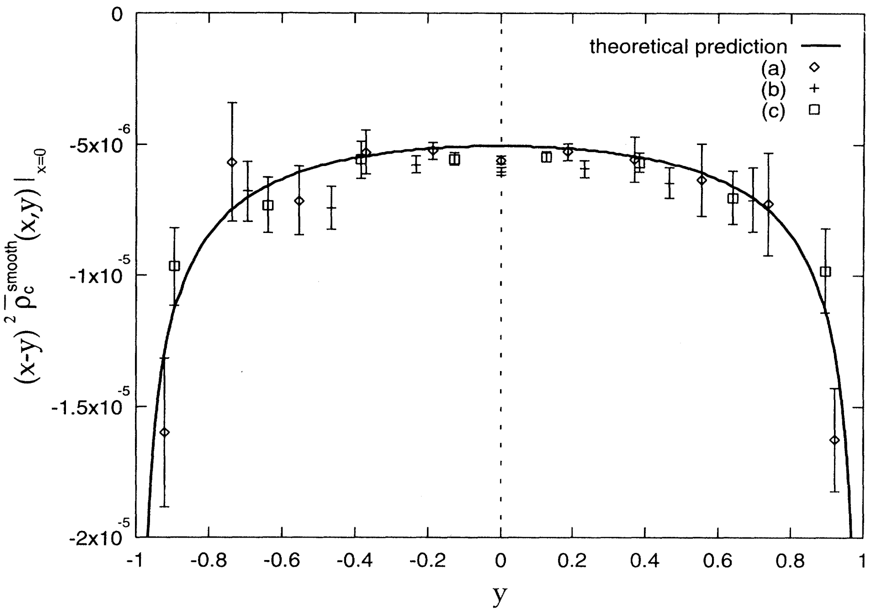

FIG. 12. Smoothed connected correlation $(x-y)^2\bar{\rho}_c{}^{\rm smooth}(x,y)|_{x=0}$ for the probability distribution (11), where x and y are the scaling variables. The parameters $(v_2, v_4, v_6, w_2, w_4, w_6)$ are (a) (1.23, 7.90, 0.290, −0.385, 2.57, 2.80), (b) (3.19, −5.35, 5.66, −6.28, −9.55, 6.77) and (c) (7.45, 5.67, 8.94, 3.14, −0.0231, 8.75). The solid line represents Eq. (5).

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N)$$

$$= \frac{1}{Z}\exp\left(-N\sum_i V(\lambda_i) + \beta\sum_{i<j}\ln|\lambda_i - \lambda_j|\right). \quad (10)$$

The variables λ_i may then be interpreted as the position of the ith gas molecule on the real axis. We can study the density and the density-density correlation of this gas without necessarily restricting to the values β =1, 2, and 4. For general β, however, we are not aware of any correspondence to the matrix models. In Figs. 6 and 7, we show the results for $\beta = 1.5$, which support Eq. (9).

Numerically, we can also study the approach to the large N limit. We study the density and the correlation for some relatively small values of N. Analytic results for such small N are evidently difficult, if not impossible, to obtain. We calculate the eigenvalue density and correlations for N=5, 10, 25, and 100. We performed the MC calculations for β=1, 2, and 4 and the results for the density are displayed in Figs. 8 and 9, for β=2 and 4, respectively. They show that, as N becomes large, the oscillation amplitudes of the eigenvalue density decrease (including the case $\beta = 1$). The oscillations seem to vanish in the large N limit, as is known analytically. Concerning the β dependence, the oscillations become large as β increases (including the case $\beta = 1$). Therefore, when considering the large N limit, we may have to choose larger N for β=4 to have the same accuracy as for β=1 or 2.

The results for the connected correlations are plotted in Figs. 10 and 11 for β=2 and 4, respectively. They show

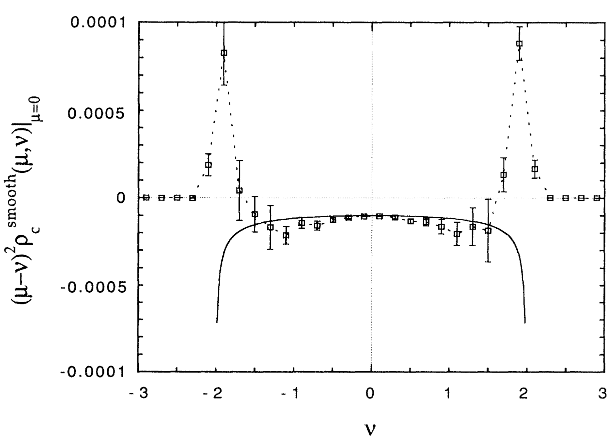

FIG. 13. Smoothed connected correlation $(\mu-\nu)^2\rho_c{}^{\rm smooth}(\mu,\nu)|_{\mu=0}$ for the Wigner class distribution. The solid line represents Eq. (9).

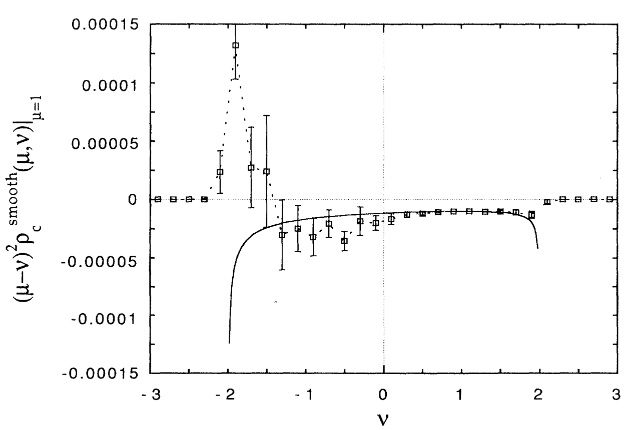

FIG. 14. Smoothed connected correlation $(\mu-\nu)^2\rho_c{}^{\rm smooth}(\mu,\nu)|_{\mu=1}$ for the Wigner class distribution. The solid line represents Eq. (9).

oscillatory behavior for all N examined. The oscillation amplitude remains finite, except the factor N^{-2} (including the case $\beta = 1$). As for particular case β=2, the oscillation period is predicted to be of the order N^{-1} from the orthogonal polynomial method. [1] Our results shown in Fig. 10 are consistent with the prediction. The oscillations become large as β increases. It may be interesting to study and understand these oscillations analytically.

Brézin and Zee [2] have also considered the more general probability distribution

$$P(\phi) = \frac{1}{Z}\exp\{-N\mathrm{Tr}V(\phi) - [\mathrm{Tr}W(\phi)]^2\} \tag{11}$$

with polynomials of the Hermitian matrices $V(\phi)$ and $W(\phi)$. They showed that in the large N limit the distribution of eigenvalues for the ensemble with the distribution (11) can be mapped onto that for the ensemble with the distribution (1).

We verify this universal behavior numerically. We take $V(\phi)$ and $W(\phi)$ to

$$\begin{aligned} V(\phi) &= v_2\phi^2 + v_4\phi^4 + v_6\phi^6, \\ W(\phi) &= w_2\phi^2 + w_4\phi^4 + w_6\phi^6. \end{aligned} \tag{12}$$

Again, we choose N to be 100 and take 10^6 MC samples. We calculate for three sets of these parameters. The results are shown in Fig. 12. It supports the universal correlation (5), although the accuracy is not as good as the distribution (1).

The density of the eigenvalue is separated into two parts in some parameter region. If only quadratic and quartic terms of V in the distribution (1) are considered, the separation occurs when $v_2 < -2\sqrt{v_4}$ for the unitary ensemble [8]. Moreover, within one region, the connected correlations of the eigenvalues was shown to obey the universal behavior. The numerical calculations in this parameter region agree with the theoretical prediction to the extent that they agreed in the previous nonseparated cases.

Brézin and Zee [3] have also discussed the correlation for what they called the Wigner class [9]. Here the random matrices are constructed by letting each element of the matrix be taken from a random distribution. As an example, we may consider a random matrix whose elements are chosen to be either $+1/\sqrt{N}$ or $-1/\sqrt{N}$, subject to the requirement that the matrix is real symmetric. The density of eigenvalues is known to obey semicircle law. The correlation function for this class of random matrices was shown to be universal, that is, independent of the random distribution, by a diagrammatic method. Here we study this universality numerically. The results are shown in Figs. 13 and 14. The smoothing intervals $\delta\mu$ and $\delta\nu$ are 0.2. Again, the numerical results do not follow the universal correlation (9) near the edge as expected. The MC errors are less than those of the previous non-Wigner class. Quite probably, the connected correlations near the edge are positive.

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[1] E. Brézin and A. Zee, Nucl. Phys. **402(FS)**, 613 (1993).
[2] E. Brézin and A. Zee, C. R. Acad. Sci. **317**, 735 (1993).
[3] E. Brézin and A. Zee (unpublished).
[4] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, Commun. Math. Phys. **59**, 35 (1978).
[5] C. W. J. Beenakker, Nucl. Phys. B **422(FS)**, 515 (1994).
[6] B. Eynard, Nucl. Phys. (to be published).
[7] P. J. Forrester, Phys. Lett. A **179**, 127 (1993); B. Jancovici and P. J. Forrester, Phys. Rev. B (to be published).
[8] Y. Morita, Y. Hatsugai, and M. Kohmoto (unpublished).
[9] E. P. Wigner, in *Can. Math. Congr. Proc.* (University of Toronto Press, Toronto, 1957), p. 174.