

The General Jacobi Identity
Revisited

By

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Abstract

In our previous paper (Nishimura, 1997) we probed into a deeper structure of the Jacobi identity of vector fields with respect Lie brackets in the realm of synthetic differential geometry to find out what might be called the general Jacobi identity of microcubes. The main objective of this paper is to present a less esoteric and more lucid proof of it.

§0. Introduction

Kock and Lavendhomme (1984) have developed a theory of microsquares, in which the Lie bracket of vector fields on a microlinear space M can be expressed as the strong difference of their associated microsquares on M^M . Nishimura (1997) took a step forward to find out that the Jacobi identity of vector fields on M with respect to Lie brackets is a reverberation of a deeper identity of microcubes on M^M , which might be called the *general Jacobi identity*. Though its proof there was thoroughly correct and exact, the exposition might appear precipitous and more esoteric than it was to be. The principal objective of this paper is to elaborate it into a less esoteric and more comprehensible one.

The main text of the paper consists of three sections, the first two of which are a hasty review of Kock and Lavendhomme (1984) and Nishimura (1997) and are intended mainly for fixing our notation and preparing the reader for more advanced quasi-colimit diagrams in the last section. The first and the second sections are devoted to simplicial objects and strong differences respectively. The gigantic quasi-colimit diagram of small objects in our previous paper (Nishimura, 1997, Lemma 3.3) is successfully divided into a few more manageable and more accessible ones in the last section. In particular, the core of the proof of the general Jacobi identity is crystalized as an elegant quasi-colimit diagram of small objects in Theorem 3.6.

We assume that the reader is well familiar with Lavendhomme (1996) up to Chapter 3. We choose, once and for all, a microlinear space M . The extended set of real numbers including infinitesimal ones is denoted by \mathbb{R} and is expected to satisfy the general Kock axiom. We denote $\{d \in \mathbb{R} \mid d^2 = 0\}$ by D as usual. Elements of D are usually denoted by d with or without subscripts. As is usual in synthetic differential geometry, the reader should presume that we are working in a non-Boolean topos, so that the principle of excluded middle and Zorn's lemma should be avoided. But for these two points, we could feel that we are working in the standard universe of sets.

§1. Simplicial Objects

In this section we distinguish a clear-cut class of small objects. Let n be a natural number and \mathfrak{n} the set consisting exactly of $1, 2, \dots, n$. Let Δ_n be the set of finite sequences (i_1, \dots, i_k) in \mathfrak{n} with $i_1 < \dots < i_k$. Given a finite subset p of Δ_n , we define a small object $D^n\{p\}$ as follows:

$$(1.1) \quad D^n\{p\} = \{(d_1, \dots, d_n) \in D^n \mid d_{i_1} \dots d_{i_k} = 0 \text{ for any } (i_1, \dots, i_k) \in p\}$$

If p is empty, $D^n\{p\}$ is D^n itself. If p is Δ_n , then $D^n\{p\}$ is $D(n)$ in standard terminology. Small objects of the form $D^n\{p\}$ are called *simplicial objects of degree n* . If $p \subset q \subset \Delta_n$, then $D^n\{q\}$ is a subset of $D^n\{p\}$, in which the canonical injection of $D^n\{q\}$ into $D^n\{p\}$ is generally denoted by ι . Given two simplicial objects $D^m\{p\}$ and $D^n\{q\}$ of degrees m and n respectively, we define a simplicial object $D^m\{p\} \oplus D^n\{q\}$ to be $D^{m+n}\{p \oplus q\}$, where

$$(1.2) \quad p \oplus q = p \cup \{(j_1+m, \dots, j_k+m) \mid (j_1, \dots, j_k) \in q\} \cup \{(i, j+m) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

By way of example, $D(m) \oplus D(n)$ is $D(m+n)$. Simplicial objects $D^m\{p\}$ and $D^n\{q\}$ can naturally be regarded as subsets of

$D^m\{p\} \oplus D^n\{q\}$. Given functions $\beta_i: D^{m_i}\{p_i\} \rightarrow D^m\{p\}$ ($1 \leq i \leq n$) of simplicial objects with $\beta_i(0, \dots, 0) = \beta_j(0, \dots, 0)$ for any i, j , there exists a unique function $\beta: D^{m_1}\{p_1\} \oplus \dots \oplus D^{m_n}\{p_n\} \rightarrow D^m\{p\}$ whose restriction to $D^{m_i}\{p_i\}$ coincides with β_i for each i . We denote this β by $\beta_1 \oplus \dots \oplus \beta_n$.

Given a simplicial object $D^n\{p\}$, we denote by $T^{D^n\{p\}}(M)$ the set of all functions from $D^n\{p\}$ to M . In particular, $T^D(M)$ is the set of tangent vectors to M , $T^{D^2}(M)$ is the set of microsquares on M , and $T^{D^3}(M)$ is the set of microcubes on M . It is well known that, given tangent vectors t_i to M ($1 \leq i \leq n$) with $t_i(0) = t_j(0)$, there exists unique $\ell(t_1, \dots, t_n): D(n) \rightarrow M$ whose restriction to i -th D coincides with t_i ($1 \leq i \leq n$).

We note in passing that Lavendhomme and Nishimura (1997) have developed a synthetic theory of differential forms based on simplicial objects.

§2. Strong Differences

The following proposition is borrowed from Lavendhomme (1996, §3.4).

Proposition 2.1. The diagram

$$\begin{array}{ccc}
 D(2) & \xrightarrow{\iota} & D^2 \\
 \iota \downarrow & & \downarrow \psi \\
 D^2 & \xrightarrow{\varphi} & D^3\{(1,3), (2,3)\}
 \end{array}$$

is a quasi-colimit diagram of small objects, where

$$(2.1) \quad \varphi(d_1, d_2) = (d_1, d_2, 0), \text{ and}$$

$$(2.2) \quad \psi(d_1, d_2) = (d_1, d_2, d_1 d_2)$$

for any $(d_1, d_2) \in D^2$. □

As a direct corollary of the above proposition we have

Proposition 2.2. For any $\gamma_1, \gamma_2 \in T^2(M)$, if

$\gamma_1|_{D(2)} = \gamma_2|_{D(2)}$, then there exists unique

$\gamma: D^3\{(1,3), (2,3)\} \rightarrow M$ with $\gamma \circ \varphi = \gamma_1$ and $\gamma \circ \psi = \gamma_2$. □

We will write $\mathcal{F}_{(\gamma_1, \gamma_2)}$ for γ in the above proposition.

The strong difference $\gamma_2 \dot{-} \gamma_1$ is defined to be the tangent

vector $d \in D \mapsto \mathcal{F}_{(\gamma_1, \gamma_2)}(0, 0, d)$ to M .

By relativizing Proposition 2.1 we have

Proposition 2.3. The diagram

$$\begin{array}{ccc}
 D^3\{(2,3)\} & \xrightarrow{\iota} & D^3 \\
 \iota \downarrow & & \downarrow \psi_1^3 \\
 D^3 & \xrightarrow{\varphi_1^3} & D^4\{(2,4), (3,4)\}
 \end{array}$$

is a quasi-colimit diagram of small objects, where

$$(2.3) \quad \varphi_1^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0), \text{ and}$$

$$(2.4) \quad \psi_1^3(d_1, d_2, d_3) = (d_1, d_2, d_3, d_2 d_3)$$

for any $(d_1, d_2, d_3) \in D^3$. □

As a direct corollary of the above proposition we have

Proposition 2.4. For any $\gamma_1, \gamma_2 \in T^3(M)$, if

$\gamma_1 \upharpoonright_{D^3\{(2,3)\}} = \gamma_2 \upharpoonright_{D^3\{(2,3)\}}$, then there exists unique

$\gamma: D^4\{(2,4), (3,4)\} \rightarrow M$ with $\gamma \cdot \varphi_1^3 = \gamma_1$ and $\gamma \cdot \psi_1^3 = \gamma_2$. □

We will write $\mathcal{F}_{(\gamma_1, \gamma_2)}^1$ for γ in the above proposition.

The strong difference $\gamma_2 \overset{\cdot}{\underset{1}{-}} \gamma_1$ is defined to be the

microsquare $(d_1, d_2) \in D^2 \mapsto \mathcal{F}_{(\gamma_1, \gamma_2)}^1(d_1, 0, 0, d_2)$ on M .

An appropriate variant of Proposition 2.3 readily yields

Proposition 2.5. For any $\gamma_1, \gamma_2 \in \Gamma^3(M)$, if

$\gamma_1 \upharpoonright_{D^3\{(1,3)\}} = \gamma_2 \upharpoonright_{D^3\{(1,3)\}}$, then there exists unique $\gamma: D^4\{(1,4), (3,4)\} \rightarrow M$ with $\gamma \circ \varphi_2^3 = \gamma_1$ and $\gamma \circ \psi_2^3 = \gamma_2$, where functions $\varphi_2^3, \psi_2^3: D^3 \rightarrow D^4\{(1,4), (3,4)\}$ go as follows:

$$(2.5) \quad \varphi_2^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0)$$

$$(2.6) \quad \psi_2^3(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1 d_3) \quad \square$$

We will write $\mathcal{F}_{(\gamma_1, \gamma_2)}^2$ for γ in the above proposition.

The strong difference $\gamma_2 \stackrel{\cdot}{\sim} \gamma_1$ is defined to be the microsquare $(d_1, d_2) \in D^2 \mapsto \mathcal{F}_{(\gamma_1, \gamma_2)}^2(0, d_1, 0, d_2)$ on M .

An appropriate variant of Proposition 2.3 readily yields

Proposition 2.6. For any $\gamma_1, \gamma_2 \in \Gamma^3(M)$, if

$\gamma_1 \upharpoonright_{D^3\{(1,2)\}} = \gamma_2 \upharpoonright_{D^3\{(1,2)\}}$, then there exists unique $\gamma: D^4\{(1,4), (2,4)\} \rightarrow M$ with $\gamma \circ \varphi_3^3 = \gamma_1$ and $\gamma \circ \psi_3^3 = \gamma_2$, where functions $\varphi_3^3, \psi_3^3: D^3 \rightarrow D^4\{(1,4), (2,4)\}$ go as follows:

$$(2.7) \quad \varphi_3^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0)$$

$$(2.8) \quad \psi_3^3(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1 d_2) \quad \square$$

We will write $\mathcal{F}_{(\gamma_1, \gamma_2)}^3$ for γ in the above proposition.

The strong difference $\gamma_2 \overset{\cdot}{\underset{3}{\dashv}} \gamma_1$ is defined to be the microsquare $(d_1, d_2) \in D^2 \mapsto \mathcal{F}_{(\gamma_1, \gamma_2)}^2(0, 0, d_1, d_2)$ on M .

The general Jacobi identity goes as follows:

Theorem 2.7. Let $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in \Gamma^3(M)$.

As long as the following three expressions are well defined, they sum up only to vanish:

$$(2.9) \quad (\gamma_{123} \overset{\cdot}{\underset{1}{\dashv}} \gamma_{132}) \overset{\cdot}{\underset{2}{\dashv}} (\gamma_{231} \overset{\cdot}{\underset{1}{\dashv}} \gamma_{321})$$

$$(2.10) \quad (\gamma_{231} \overset{\cdot}{\underset{2}{\dashv}} \gamma_{213}) \overset{\cdot}{\underset{3}{\dashv}} (\gamma_{312} \overset{\cdot}{\underset{2}{\dashv}} \gamma_{132})$$

$$(2.11) \quad (\gamma_{312} \overset{\cdot}{\underset{3}{\dashv}} \gamma_{321}) \overset{\cdot}{\underset{1}{\dashv}} (\gamma_{123} \overset{\cdot}{\underset{3}{\dashv}} \gamma_{213})$$

§3. The General Jacobi Identity

This section is devoted completely to a proof of Theorem 2.7.

Proposition 3.1. The diagram

$$\begin{array}{ccc}
 D(2) & \xrightarrow{\iota_{14}} & D^4\{(2,4), (3,4)\} \\
 \iota_{14} \downarrow & & \downarrow \eta_2 \\
 D^4\{(2,4), (3,4)\} & \xrightarrow{\eta_1} & E
 \end{array}$$

is a quasi-colimit diagram of small objects with its quasi-colimit E , where

$$(3.1) \quad E \text{ is } D^7\{(2,6), (3,6), (4,6), (5,6), (1,7), (2,7), (3,7), (4,7), (5,7), (6,7), (2,4), (2,5), (3,4), (3,5)\}.$$

$$(3.2) \quad \iota_{14}(d_1, d_2) = (d_1, 0, 0, d_2) \text{ for any } (d_1, d_2) \in D(2).$$

$$(3.3) \quad \eta_1(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, 0, 0, d_4, 0) \text{ for any } (d_1, d_2, d_3, d_4) \in D^4\{(2,4), (3,4)\}.$$

$$(3.4) \quad \eta_2(d_1, d_2, d_3, d_4) = (d_1, 0, 0, d_2, d_3, d_4, d_1 d_4) \text{ for any } (d_1, d_2, d_3, d_4) \in D^4\{(2,4), (3,4)\}.$$

proof. The so-called general Kock axiom warrants that functions $\gamma_1, \gamma_2: D^4\{(2,4), (3,4)\} \rightarrow \mathbb{R}$ and $\gamma: E \rightarrow \mathbb{R}$ should be polynomials of infinitesimals in D with coefficients in \mathbb{R} of the following forms:

$$(3.5) \quad \gamma_1(d_1, d_2, d_3, d_4) = a + a_1 d_1 + a_2 d_2 + a_3 d_3 + a_4 d_4 + \\ a_{12} d_1 d_2 + a_{13} d_1 d_3 + a_{14} d_1 d_4 + a_{23} d_2 d_3 + a_{123} d_1 d_2 d_3$$

$$(3.6) \quad \gamma_2(d_1, d_2, d_3, d_4) = b + b_1 d_1 + b_2 d_2 + b_3 d_3 + b_4 d_4 + \\ b_{12} d_1 d_2 + b_{13} d_1 d_3 + b_{14} d_1 d_4 + b_{23} d_2 d_3 + b_{123} d_1 d_2 d_3$$

$$(3.7) \quad \gamma(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = c + c_1 d_1 + c_2 d_2 + c_3 d_3 + \\ c_4 d_4 + c_5 d_5 + c_6 d_6 + c_7 d_7 + c_{12} d_1 d_2 + c_{13} d_1 d_3 + \\ c_{14} d_1 d_4 + c_{15} d_1 d_5 + c_{16} d_1 d_6 + c_{23} d_2 d_3 + c_{45} d_4 d_5 + \\ c_{123} d_1 d_2 d_3 + c_{145} d_1 d_4 d_5$$

The condition that $\gamma_1 \circ \iota_{14} = \gamma_2 \circ \iota_{14}$ is equivalent to the following condition:

$$(3.8) \quad a = b, \quad a_1 = b_1 \quad \text{and} \quad a_4 = b_4.$$

Therefore it is not difficult to see that, as long as

$\gamma_1 \circ \iota_{14} = \gamma_2 \circ \iota_{14}$, it is the case that $\gamma \circ \eta_1 = \gamma_1$ and $\gamma \circ \eta_2 = \gamma_2$ exactly when γ is of the following form:

$$(3.9) \quad \gamma(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = a + a_1 d_1 + a_2 d_2 + a_3 d_3 + \\ b_2 d_4 + b_3 d_5 + a_4 d_6 + (b_{14} - a_{14}) d_7 + a_{12} d_1 d_2 + \\ a_{13} d_1 d_3 + b_{12} d_1 d_4 + b_{13} d_1 d_5 + a_{14} d_1 d_6 + a_{23} d_2 d_3 + \\ b_{23} d_4 d_5 + a_{123} d_1 d_2 d_3 + b_{123} d_1 d_4 d_5$$

This means that the above diagram is a quasi-colimit diagram of small objects. □

Proposition 3.2. The diagram consisting of objects

$$(3.10) \ E_1, E_2, E_3, E_4, \text{ all of which are equal to } D^3$$

$$(3.11) \ F_{12}, F_{34}, \text{ all of which are equal to } D^3\{(2,3)\}$$

$$(3.12) \ E_{12}, E_{34}, \text{ all of which are equal to } D^4\{(2,4), (3,4)\}$$

$$(3.13) \ F_{1234}, \text{ which is equal to } D(2)$$

$$(3.14) \ E_{1234}, \text{ which is equal to } D^7\{(2,6), (3,6), (4,6), (5,6), (1,7), (2,7), (3,7), (4,7), (5,7), (6,7), (2,4), (2,5), (3,4), (3,5)\}$$

and of morphisms

$$(3.15) \ \iota:F_{12} \rightarrow E_1, \ \iota:F_{12} \rightarrow E_2, \ \iota:F_{34} \rightarrow E_3, \ \iota:F_{34} \rightarrow E_4$$

$$(3.16) \ f_1^{12}:E_1 \rightarrow E_{12}, \ f_3^{34}:E_3 \rightarrow E_{34}, \text{ all of which are equal to } \varphi_1^3:D^3 \rightarrow D^4\{(2,4), (3,4)\}$$

$$(3.17) \ f_2^{12}:E_2 \rightarrow E_{12}, \ f_4^{34}:E_4 \rightarrow E_{34}, \text{ all of which are equal to } \psi_1^3:D^3 \rightarrow D^4\{(2,4), (3,4)\}$$

$$(3.18) \ \iota_{14}:F_{1234} \rightarrow E_{12}, \ \iota_{14}:F_{1234} \rightarrow E_{34}$$

$$(3.19) \ \eta_1:E_{12} \rightarrow E_{1234}, \ \eta_2:E_{34} \rightarrow E_{1234}$$

is a quasi-colimit diagram of small objects with its quasi-colimit E_{1234} .

proof. It suffices to note that the above diagram is a hybrid of the following three diagrams:

$$\begin{array}{ccc}
F_{12} & \xrightarrow{\iota} & E_2 \\
\iota \downarrow & & \downarrow f_2^{12} \\
E_1 & \xrightarrow{f_1^{12}} & E_{12} \\
\\
F_{34} & \xrightarrow{\iota} & E_4 \\
\iota \downarrow & & \downarrow f_4^{34} \\
E_3 & \xrightarrow{f_3^{34}} & E_{34} \\
\\
F_{1234} & \xrightarrow{\iota_{14}} & E_{34} \\
\iota_{14} \downarrow & & \downarrow \eta_2 \\
E_{12} & \xrightarrow{\eta_1} & E_{1234}
\end{array}$$

Each of them is a quasi-colimit diagram of small objects by Proposition 2.3 or Proposition 3.1. \square

We will write $E[1]$ for $D^7\{(2,6),(3,6),(4,6),(5,6),(1,7), (2,7),(3,7),(4,7),(5,7),(6,7),(2,4),(2,5),(3,4),(3,5)\}$. We will write $\iota_1^1, \iota_2^1, \iota_3^1$ and ι_4^1 for $\eta_1 \circ \varphi_1^3, \eta_1 \circ \psi_1^3, \eta_2 \circ \varphi_1^3$ and $\eta_2 \circ \psi_1^3$ respectively. That is to say, for any $(d_1, d_2, d_3) \in D^3$, we have

$$(3.20) \quad \iota_1^1(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, 0, 0)$$

$$(3.21) \quad \iota_2^1(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, d_2 d_3, 0)$$

$$(3.22) \quad \iota_3^1(d_1, d_2, d_3) = (d_1, 0, 0, d_2, d_3, 0, 0)$$

$$(3.22) \quad \iota_4^1(d_1, d_2, d_3) = (d_1, 0, 0, d_2, d_3, d_2 d_3, d_1 d_2 d_3)$$

As a direct corollary of Proposition 3.2 we have

Proposition 3.3. For any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in T^3(M)$, if the expression $(\gamma_4 \overset{\cdot}{\underset{1}{\dashv}} \gamma_3) \overset{\cdot}{\underset{1}{\dashv}} (\gamma_2 \overset{\cdot}{\underset{1}{\dashv}} \gamma_1)$ is well defined, then there exists unique $\gamma \in T^{E[1]}(M)$ such that $\gamma \circ \iota_i^1 = \gamma_i$ ($i = 1, 2, 3, 4$). \square

We will write $\hbar^1_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}$ for γ in the above proposition. We note that for any $(d_1, d_2, d_3, d_4) \in D^4\{(2, 4), (3, 4)\}$,

$$(3.23) \quad \begin{aligned} & \mathcal{F}^1_{(\gamma_1, \gamma_2)}(d_1, d_2, d_3, d_4) \\ &= \hbar^1_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}(d_1, d_2, d_3, 0, 0, d_4, 0) \end{aligned}$$

$$(3.24) \quad \begin{aligned} & \mathcal{F}^1_{(\gamma_3, \gamma_4)}(d_1, d_2, d_3, d_4) \\ &= \hbar^1_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}(d_1, 0, 0, d_2, d_3, d_4, d_1 d_4) \end{aligned}$$

Therefore, for any $(d_1, d_2) \in D^2$, we have

$$(3.25) \quad \begin{aligned} & (\gamma_2 \overset{\cdot}{\underset{1}{\dashv}} \gamma_1)(d_1, d_2) \\ &= \hbar^1_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}(d_1, 0, 0, 0, 0, d_2, 0) \end{aligned}$$

$$(3.26) \quad \begin{aligned} & (\gamma_4 \overset{\cdot}{\underset{1}{\dashv}} \gamma_3)(d_1, d_2) \\ &= \hbar^1_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}(d_1, 0, 0, 0, 0, d_2, d_1 d_2) \end{aligned}$$

(3.25) and (3.26) imply that for any

$(d_1, d_2, d_3) \in D^3\{(1, 3), (2, 3)\}$,

$$(3.27) \quad \mathcal{F}(\gamma_2 \overset{\cdot}{\underset{1}{\dashv}} \gamma_1, \gamma_4 \overset{\cdot}{\underset{1}{\dashv}} \gamma_3)(d_1, d_2, d_3)$$

$$= \mathcal{L}^1_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}(d_1, 0, 0, 0, 0, d_2, d_3)$$

Therefore, for any $d \in D$, we have

$$(3.28) \quad ((\gamma_4 \overset{\cdot}{\underset{1}{\rceil}} \gamma_3) - (\gamma_2 \overset{\cdot}{\underset{1}{\rceil}} \gamma_1))(d) \\ = \mathcal{L}^1_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}(0, 0, 0, 0, 0, 0, d)$$

We will write $E[2]$ for $D^7 \setminus \{(1,6), (3,6), (4,6), (5,6), (1,7), (2,7), (3,7), (4,7), (5,7), (6,7), (1,4), (1,5), (3,4), (3,5)\}$. We define functions ι_1^2 , ι_2^2 , ι_3^2 and ι_4^2 from D^3 to $E[2]$ as follows:

$$(3.29) \quad \iota_1^2(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, 0, 0)$$

$$(3.30) \quad \iota_2^2(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, d_1 d_3, 0)$$

$$(3.31) \quad \iota_3^2(d_1, d_2, d_3) = (0, d_2, 0, d_3, d_1, 0, 0)$$

$$(3.32) \quad \iota_4^2(d_1, d_2, d_3) = (0, d_2, 0, d_3, d_1, d_1 d_3, d_1 d_2 d_3)$$

By the same token as in Proposition 3.3 we have

Proposition 3.4. For any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in T^3(M)$, if the expression $(\gamma_4 \overset{\cdot}{\underset{2}{\rceil}} \gamma_3) \overset{\cdot}{\underset{2}{\rceil}} (\gamma_2 \overset{\cdot}{\underset{2}{\rceil}} \gamma_1)$ is well defined, then there exists unique $\gamma \in T^{E[2]}(M)$ such that $\gamma \circ \iota_i^2 = \gamma_i$ ($i = 1, 2, 3, 4$). \square

We will write $\mathcal{F}^2_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}$ for γ in the above proposition. By the same token as in (3.28) we have that for any $d \in D$,

$$(3.33) \quad ((\gamma_4 \overset{\cdot}{\underset{2}{\gamma}} \gamma_3) - (\gamma_2 \overset{\cdot}{\underset{2}{\gamma}} \gamma_1))(d) \\ = \mathfrak{k}^2_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}(0, 0, 0, 0, 0, 0, d)$$

We will write $E[3]$ for $D^7\{(1,6), (2,6), (4,6), (5,6), (1,7), (2,7), (3,7), (4,7), (5,7), (6,7), (1,4), (1,5), (2,4), (2,5)\}$. We define functions i_1^3 , i_2^3 , i_3^3 and i_4^3 from D^3 to $E[3]$ as follows:

$$(3.34) \quad i_1^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, 0, 0)$$

$$(3.35) \quad i_2^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, d_1 d_2, 0)$$

$$(3.36) \quad i_3^3(d_1, d_2, d_3) = (0, 0, d_3, d_1, d_2, 0, 0)$$

$$(3.37) \quad i_4^3(d_1, d_2, d_3) = (0, 0, d_3, d_1, d_2, d_1 d_2, d_1 d_2 d_3)$$

By the same token as in Proposition 3.3 we have

Proposition 3.5. For any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in T^3(M)$, if the expression $(\gamma_4 \overset{\cdot}{\underset{3}{\gamma}} \gamma_3) \overset{\cdot}{\underset{3}{\gamma}} (\gamma_2 \overset{\cdot}{\underset{3}{\gamma}} \gamma_1)$ is well defined, then there exists unique $\gamma \in T^{E[3]}(M)$ such that $\gamma \circ i_i^3 = \gamma_i$ ($i = 1, 2, 3, 4$). \square

We will write $\mathfrak{F}^3_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}$ for γ in the above proposition. By the same token as in (3.28) we have that for any $d \in D$,

$$(3.38) \quad ((\gamma_4 \overset{\cdot}{\underset{2}{\gamma}} \gamma_3) - (\gamma_2 \overset{\cdot}{\underset{2}{\gamma}} \gamma_1))(d) \\ = \mathfrak{k}^2_{(\gamma_1, \gamma_2, \gamma_3, \gamma_4)}(0, 0, 0, 0, 0, 0, d)$$

The crucial step in the proof of Theorem 2.7 is

epitomized by the following theorem.

Theorem 3.6. The diagram consisting of objects

$$(3.39) \quad E[1], E[2], E[3]$$

$$(3.40) \quad H_{12}, H_{23}, H_{31}, \text{ all of which are equal to } D^3 \oplus D^3$$

$$(3.41) \quad G, \text{ which is equal to } D^8 \{(2,4), (3,4), (1,5), (3,5), \\ (1,6), (2,6), (4,5), (4,6), (5,6), (1,7), (2,7), (3,7), \\ (4,7), (5,7), (6,7), (1,8), (2,8), (3,8), (4,8), (5,8), \\ (6,8), (7,8)\}$$

and of morphisms

$$(3.42) \quad h_{12}^1: H_{12} \rightarrow E[1], \quad h_{12}^2: H_{12} \rightarrow E[2], \quad h_{23}^2: H_{23} \rightarrow E[2], \\ h_{23}^3: H_{23} \rightarrow E[3], \quad h_{31}^3: H_{31} \rightarrow E[3], \quad h_{31}^1: H_{31} \rightarrow E[1]$$

$$(3.43) \quad k_1: E[1] \rightarrow G, \quad k_2: E[2] \rightarrow G, \quad k_3: E[3] \rightarrow G$$

is a quasi-colimit diagram of small objects, where

$$(3.44) \quad h_{12}^1 = \iota_2^1 \oplus \iota_3^1, \quad h_{12}^2 = \iota_4^2 \oplus \iota_1^2, \quad h_{23}^2 = \iota_2^2 \oplus \iota_3^2, \\ h_{23}^3 = \iota_4^3 \oplus \iota_1^3, \quad h_{31}^3 = \iota_2^3 \oplus \iota_3^3, \quad h_{31}^1 = \iota_4^1 \oplus \iota_1^1$$

$$(3.45) \quad k_1(d_1, d_2, d_3, d_4, d_5, d_6, d_7) \\ = (d_1, d_2 + d_4, d_3 + d_5, d_6, -d_1 d_5, d_1 d_4, d_7, 0) \\ \text{for any } (d_1, d_2, d_3, d_4, d_5, d_6, d_7) \in E[1],$$

$$(3.46) \quad k_2(d_1, d_2, d_3, d_4, d_5, d_6, d_7) \\ = (d_1 + d_5, d_2, d_3 + d_4, d_2 d_4, d_6, -d_2 d_5, 0, d_7) \\ \text{for any } (d_1, d_2, d_3, d_4, d_5, d_6, d_7) \in E[2], \text{ and}$$

$$\begin{aligned}
(3.47) \quad & k_3(d_1, d_2, d_3, d_4, d_5, d_6, d_7) \\
& = (d_1 + d_4, d_2 + d_5, d_3, -d_3 d_5, d_3 d_4, d_6, -d_7, -d_7) \\
& \text{for any } (d_1, d_2, d_3, d_4, d_5, d_6, d_7) \in E[3].
\end{aligned}$$

proof. The so-called general Kock axiom warrants that functions $\gamma_1: E[1] \rightarrow \mathbb{R}$, $\gamma_2: E[2] \rightarrow \mathbb{R}$, $\gamma_3: E[3] \rightarrow \mathbb{R}$ and $\gamma: G \rightarrow \mathbb{R}$ should be polynomials of infinitesimals in D with coefficients in \mathbb{R} of the following forms:

$$\begin{aligned}
(3.48) \quad & \gamma_1(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = a^1 + a_1^1 d_1 + a_2^1 d_2 + a_3^1 d_3 \\
& + a_4^1 d_4 + a_5^1 d_5 + a_6^1 d_6 + a_7^1 d_7 + a_{12}^1 d_1 d_2 + a_{13}^1 d_1 d_3 + \\
& a_{14}^1 d_1 d_4 + a_{15}^1 d_1 d_5 + a_{16}^1 d_1 d_6 + a_{23}^1 d_2 d_3 + a_{45}^1 d_4 d_5 + \\
& a_{123}^1 d_1 d_2 d_3 + a_{145}^1 d_1 d_4 d_5
\end{aligned}$$

$$\begin{aligned}
(3.49) \quad & \gamma_2(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = a^2 + a_1^2 d_1 + a_2^2 d_2 + a_3^2 d_3 \\
& + a_4^2 d_4 + a_5^2 d_5 + a_6^2 d_6 + a_7^2 d_7 + a_{12}^2 d_1 d_2 + a_{13}^2 d_1 d_3 + \\
& a_{23}^2 d_2 d_3 + a_{24}^2 d_2 d_4 + a_{25}^2 d_2 d_5 + a_{26}^2 d_2 d_6 + a_{45}^2 d_4 d_5 + \\
& a_{123}^2 d_1 d_2 d_3 + a_{245}^2 d_2 d_4 d_5
\end{aligned}$$

$$\begin{aligned}
(3.50) \quad & \gamma_3(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = a^3 + a_1^3 d_1 + a_2^3 d_2 + a_3^3 d_3 \\
& + a_4^3 d_4 + a_5^3 d_5 + a_6^3 d_6 + a_7^3 d_7 + a_{12}^3 d_1 d_2 + a_{13}^3 d_1 d_3 + \\
& a_{23}^3 d_2 d_3 + a_{34}^3 d_3 d_4 + a_{35}^3 d_3 d_5 + a_{36}^3 d_3 d_6 + a_{45}^3 d_4 d_5 + \\
& a_{123}^3 d_1 d_2 d_3 + a_{345}^3 d_3 d_4 d_5
\end{aligned}$$

$$\begin{aligned}
(3.51) \quad & \gamma(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8) = b + b_1 d_1 + b_2 d_2 + \\
& b_3 d_3 + b_4 d_4 + b_5 d_5 + b_6 d_6 + b_7 d_7 + b_8 d_8 + b_{12} d_1 d_2 \\
& + b_{13} d_1 d_3 + b_{14} d_1 d_4 + b_{23} d_2 d_3 + b_{25} d_2 d_5 + b_{36} d_3 d_6
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
(3.52) \quad & (\gamma_1 \circ h_{12}^1)(d_1, d_2, d_3, d_4, d_5, d_6) \\
& = a^1 + a_1^1 d_1 + a_2^1 d_2 + a_3^1 d_3 + a_6^1 d_2 d_3 + a_{12}^1 d_1 d_2 + \\
& \quad a_{13}^1 d_1 d_3 + a_{16}^1 d_1 d_2 d_3 + a_{23}^1 d_2 d_3 + a_{123}^1 d_1 d_2 d_3 + \\
& \quad a_4^1 d_4 + a_5^1 d_5 + a_6^1 d_6 + a_{14}^1 d_4 d_5 + a_{15}^1 d_4 d_6 + a_{45}^1 d_5 d_6 \\
& \quad + a_{145}^1 d_4 d_5 d_6 \\
& = a^1 + a_1^1 d_1 + a_2^1 d_2 + a_3^1 d_3 + a_{12}^1 d_1 d_2 + a_{13}^1 d_1 d_3 + \\
& \quad (a_6^1 + a_{23}^1) d_2 d_3 + (a_{16}^1 + a_{123}^1) d_1 d_2 d_3 + a_4^1 d_4 + \\
& \quad a_5^1 d_5 + a_6^1 d_6 + a_{14}^1 d_4 d_5 + a_{15}^1 d_4 d_6 + a_{45}^1 d_5 d_6 + \\
& \quad + a_{145}^1 d_4 d_5 d_6
\end{aligned}$$

$$\begin{aligned}
(3.53) \quad & (\gamma_2 \circ h_{12}^2)(d_1, d_2, d_3, d_4, d_5, d_6) \\
& = a^2 + a_2^2 d_2 + a_4^2 d_3 + a_5^2 d_1 + a_6^2 d_1 d_3 + a_7^2 d_1 d_2 d_3 + \\
& \quad a_{24}^2 d_2 d_3 + a_{25}^2 d_1 d_2 + a_{26}^2 d_1 d_2 d_3 + a_{45}^2 d_1 d_3 + \\
& \quad a_{245}^2 d_1 d_2 d_3 + a_1^2 d_4 + a_2^2 d_5 + a_3^2 d_6 + a_{12}^2 d_4 d_5 + \\
& \quad a_{13}^2 d_4 d_6 + a_{23}^2 d_5 d_6 + a_{123}^2 d_4 d_5 d_6 \\
& = a^2 + a_5^2 d_1 + a_2^2 d_2 + a_4^2 d_3 + a_{25}^2 d_1 d_2 + \\
& \quad (a_6^2 + a_{45}^2) d_1 d_3 + a_{24}^2 d_2 d_3 + \\
& \quad (a_7^2 + a_{26}^2 + a_{245}^2) d_1 d_2 d_3 + a_1^2 d_4 + a_2^2 d_5 + a_3^2 d_6 + \\
& \quad a_{12}^2 d_4 d_5 + a_{13}^2 d_4 d_6 + a_{23}^2 d_5 d_6 + a_{123}^2 d_4 d_5 d_6
\end{aligned}$$

Therefore the condition that $\gamma_1 \circ h_{12}^1 = \gamma_2 \circ h_{12}^2$ is equivalent to the following conditions:

$$(3.54) \quad a^1 = a^2$$

$$\begin{aligned}
(3.55) \quad & a_1^1 = a_5^2, \quad a_2^1 = a_2^2, \quad a_3^1 = a_4^2, \quad a_6^1 = a_1^2, \quad a_4^1 = a_2^2, \\
& a_5^1 = a_3^2
\end{aligned}$$

$$\begin{aligned}
(3.56) \quad & a_{12}^1 = a_{25}^2, \quad a_{13}^1 = a_6^2 + a_{45}^2, \quad a_6^1 + a_{23}^1 = a_{24}^2, \\
& a_{14}^1 = a_{12}^2, \quad a_{15}^1 = a_{13}^2, \quad a_{45}^1 = a_{23}^2
\end{aligned}$$

$$(3.57) \quad a_{16}^1 + a_{123}^1 = a_7^2 + a_{26}^2 + a_{245}^2, \quad a_{145}^1 = a_{123}^2$$

By the same token the condition that $\gamma_2 \circ h_{23}^2 = \gamma_3 \circ h_{23}^3$ is equivalent to the following conditions:

$$(3.58) \quad a^2 = a^3$$

$$(3.59) \quad a_2^2 = a_5^3, \quad a_3^2 = a_3^3, \quad a_1^2 = a_4^3, \quad a_2^2 = a_2^3, \quad a_4^2 = a_3^3, \\ a_5^2 = a_1^3$$

$$(3.60) \quad a_{23}^2 = a_{35}^3, \quad a_{12}^2 = a_6^3 + a_{45}^3, \quad a_6^2 + a_{13}^2 = a_{34}^3, \\ a_{24}^2 = a_{23}^3, \quad a_{25}^2 = a_{12}^3, \quad a_{45}^2 = a_{13}^3$$

$$(3.61) \quad a_{26}^2 + a_{123}^2 = a_7^3 + a_{36}^3 + a_{345}^3, \quad a_{245}^2 = a_{123}^3$$

By the same token again the condition that $\gamma_3 \circ h_{31}^3 = \gamma_1 \circ h_{31}^1$ is equivalent to the following conditions:

$$(3.62) \quad a^3 = a^1$$

$$(3.63) \quad a_3^3 = a_5^1, \quad a_1^3 = a_1^1, \quad a_2^3 = a_4^1, \quad a_3^3 = a_3^1, \quad a_4^3 = a_1^1, \\ a_5^3 = a_2^1$$

$$(3.64) \quad a_{13}^3 = a_{15}^1, \quad a_{23}^3 = a_6^1 + a_{45}^1, \quad a_6^3 + a_{12}^3 = a_{14}^1, \\ a_{34}^3 = a_{13}^1, \quad a_{35}^3 = a_{23}^1, \quad a_{45}^3 = a_{12}^1$$

$$(3.65) \quad a_{36}^3 + a_{123}^3 = a_7^1 + a_{16}^1 + a_{145}^1, \quad a_{345}^3 = a_{123}^1$$

Three conditions (3.54), (3.58) and (3.62) can be combined into

$$(3.66) \quad a^1 = a^2 = a^3$$

Three conditions (3.55), (3.59) and (3.63) are to be superseded by the following three conditions:

$$(3.67) \quad a_1^1 = a_1^2 = a_1^3 = a_5^2 = a_4^3$$

$$(3.68) \quad a_2^1 = a_2^2 = a_2^3 = a_4^1 = a_5^3$$

$$(3.69) \quad a_3^1 = a_3^2 = a_3^3 = a_5^1 = a_4^2$$

Three conditions (3.56), (3.60) and (3.64) are equivalent to the following six conditions:

$$(3.70) \quad a_{12}^1 = a_{12}^2 = a_{12}^3$$

$$(3.71) \quad a_{13}^1 = a_{13}^2 = a_{13}^3$$

$$(3.72) \quad a_{23}^1 = a_{23}^2 = a_{23}^3$$

$$(3.73) \quad a_{14}^1 = a_{12}^1 + a_6^3, \quad a_{15}^1 = a_{13}^1 - a_6^2, \quad a_{45}^1 = a_{23}^1$$

$$(3.74) \quad a_{24}^2 = a_{23}^2 + a_6^1, \quad a_{25}^2 = a_{12}^2 - a_6^3, \quad a_{45}^2 = a_{13}^2$$

$$(3.75) \quad a_{34}^3 = a_{13}^3 + a_6^2, \quad a_{35}^3 = a_{23}^3 - a_6^1, \quad a_{45}^3 = a_{12}^3$$

Conditions (3.57), (3.61) and (3.65) imply that

$$\begin{aligned}
 (3.76) \quad & a_7^1 + a_7^2 + a_7^3 \\
 &= (a_{36}^3 + a_{123}^3 - a_{16}^1 - a_{145}^1) + (a_{16}^1 + a_{123}^1 - a_{26}^2 - \\
 & \quad a_{245}^2) + (a_{26}^2 + a_{123}^2 - a_{36}^3 - a_{345}^3) \\
 &= (a_{36}^3 + a_{123}^3 - a_{16}^1 - a_{123}^2) + (a_{16}^1 + a_{123}^1 - a_{26}^2 - \\
 & \quad a_{123}^3) + (a_{26}^2 + a_{123}^2 - a_{36}^3 - a_{123}^1) \\
 &= 0
 \end{aligned}$$

Now it is not difficult to see that $\gamma_1 h_{12}^1 = \gamma_2 h_{12}^2$.

$\gamma_2 \circ h_{23}^2 = \gamma_3 \circ h_{23}^3$ and $\gamma_3 \circ h_{31}^3 = \gamma_1 \circ h_{31}^1$ exactly when there exists $\gamma: G \rightarrow \mathbb{R}$ with $\gamma_i = \gamma \circ k_i$ ($i = 1, 2, 3$), in which γ is to be of the following form:

$$(3.77) \quad \gamma(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8) = a^1 + a_1^1 d_1 + a_2^1 d_2 + a_3^1 d_3 + a_4^1 d_4 + a_5^2 d_5 + a_6^3 d_6 + a_7^1 d_7 + a_8^2 d_8 + a_{12}^1 d_1 d_2 + a_{13}^1 d_1 d_3 + a_{16}^1 d_1 d_4 + a_{23}^1 d_2 d_3 + a_{26}^2 d_2 d_5 + a_{36}^3 d_3 d_6$$

This completes the proof of the theorem. □

As a direct corollary of Theorem 3.6 we have

Theorem 3.7. For any $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in T^3(M)$, if all the expressions (2.9)-(2.11) are well defined, then there exists unique $\gamma \in T^G(M)$ such that

$$(3.78) \quad \gamma \circ k_1 = \mathcal{h}^1(\gamma_{321}, \gamma_{231}, \gamma_{132}, \gamma_{123})$$

$$(3.79) \quad \gamma \circ k_2 = \mathcal{h}^2(\gamma_{132}, \gamma_{312}, \gamma_{213}, \gamma_{231})$$

$$(3.80) \quad \gamma \circ k_3 = \mathcal{h}^3(\gamma_{213}, \gamma_{123}, \gamma_{321}, \gamma_{312})$$

□

We will write $m(\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321})$ or m for short for the above γ .

Once the above theorem is established, we can proceed in the same line as in our previous paper (Nishimura, 1997, pp.1117-1118) so as to get the general Jacobi identity. Indeed we note that

$$(3.81) \quad ((\gamma_{123} \overset{\cdot}{1} \gamma_{132}) \overset{\cdot}{-} (\gamma_{231} \overset{\cdot}{1} \gamma_{321}))(d) \\ = m(0,0,0,0,0,0,d,0) \text{ for any } d \in D.$$

$$(3.82) \quad ((\gamma_{231} \overset{\cdot}{2} \gamma_{213}) \overset{\cdot}{-} (\gamma_{312} \overset{\cdot}{2} \gamma_{132}))(d) \\ = m(0,0,0,0,0,0,0,d) \text{ for any } d \in D.$$

$$(3.83) \quad ((\gamma_{312} \overset{\cdot}{3} \gamma_{321}) \overset{\cdot}{-} (\gamma_{123} \overset{\cdot}{3} \gamma_{213}))(d) \\ = m(0,0,0,0,0,0,-d,-d) \text{ for any } d \in D.$$

Therefore, letting t_1 , t_2 and t_3 denote expressions (2.9)-(2.11) in order, we have

$$(3.84) \quad \ell(t_1, t_2, t_3)(d_1, d_2, d_3) \\ = m(0,0,0,0,0,0,d_1 - d_3, d_2 - d_3) \text{ for any } \\ (d_1, d_2, d_3) \in D(3).$$

This means that for any $d \in D$,

$$(3.85) \quad (t_1 + t_2 + t_3)(d) \\ = \ell(t_1, t_2, t_3)(d, d, d) \\ = m(0,0,0,0,0,0,d - d, d - d) \\ = m(0,0,0,0,0,0,0,0)$$

This completes the proof of Theorem 2.7.

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