Fisher information metric and Poisson kernels

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Abstract

A complete Riemannian manifold X with negative curvature satisfying $-b^2 \leq K_X \leq -a^2 < 0$ for some constants a, b, is naturally mapped in the space of probability measures on the ideal boundary ∂X by assigning the Poisson kernels. We show that this map is embedding and the pull-back metric of the Fisher information metric by this embedding coincides with the original metric of X up to constant provided X is a rank one symmetric space of noncompact type. Furthermore, we give a geometric meaning of the embedding.

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1. Introduction

A Poisson kernel appears in the classical theory of Dirichlet problem for the Euclidean space. As is well known, the Poisson kernel for the n-dimensional unit ball $B^n \subset \mathbf{R}^n$ is given by

$$\Phi(x, \theta) = \frac{1 - |x|^2}{|x - \theta|^n}, \quad x \in B^n,\ \theta \in \partial B^n$$

and the Dirichlet problem for the ordinary Laplacian

$$\Delta = -\sum_{i=1}^{n} \left(\frac{\partial}{\partial x^i} \right)^2$$

on B^n is solved by using the Poisson integral formula as

$$u(x) = \int_{\partial B^n} \Phi(x,\theta)\psi(\theta)\, d\theta,$$

where ψ is a given function on ∂B^n as a boundary condition data. Refer to [5] for the Dirichlet problem and the Poisson kernel in the Euclidean space.

Now, let X be a simply connected, complete n-dimensional Riemannian manifold with sectional curvature K_X satisfying $-b^2 \leq K_X \leq -a^2 < 0$ for constants a, b.

Then, X is equipped with the ideal boundary ∂X, the space of all oriented geodesic rays up to asymptotic equivalence, which is identified with the space of unit vectors tangent to X at a reference point x_o. So ∂X is identified with an $(n-1)$-dimensional sphere. We can therefore consider the Dirichlet problem for this ideal boundary. Like the Euclidean space case, on X a Poisson kernel $\Phi(x,\theta)$ can be defined and moreover any solution of the Dirichlet problem can be written in terms of the Poisson integral. See the detailed argument given by Schoen and Yau in [15].

It is a fundamental and important fact that the Poisson kernels $\Phi(x,\theta)$ are regarded as probability density functions on ∂X, that is,

$$\begin{aligned} \Phi(x,\theta) &> 0, \\ \int_{\partial X} \Phi(x,\theta) d\theta &= 1 \end{aligned}$$

for any fixed $x \in X$. This implies that there is an embedding from the space of Poisson kernels to the space $\mathcal{P}(\partial X)$ of probability measures with positive density function on ∂X.

Consider a compact, connected oriented Riemannian manifold M. The space $\mathcal{P}(M)$ of probability measures whose density function is positive on M has a structure of infinite dimensional manifold. See for this [17] and [19]. On this manifold $\mathcal{P}(M)$ a Riemannian metric g which we call the Fisher information metric is defined as

$$g(\sigma_1, \sigma_2) = \int_M \frac{d\sigma_1}{d\mu}\frac{d\sigma_2}{d\mu}\, d\mu, \quad \sigma_i \in T_\mu\mathcal{P}, \quad i = 1, 2, \quad \mu \in \mathcal{P}(M).$$

Even though $\mathcal{P}(M)$ is infinite dimensional, this metric g possesses fine geometric properties. For instance, any orientation preserving diffeomorphism of M induces an isometry of $(\mathcal{P}(M), g)$ and moreover g is a metric of positive constant sectional curvature.

The Fisher information metric is a natural extension of the Fisher information matrix. The notion of Fisher information matrix is derived from the theory of statistical inference. The Fisher information matrix determines a Riemannian structure on a parametrized space $\mathcal{P}$ of probability measures. Study of geometry of $\mathcal{P}$ with the Riemannian structure, which we call information geometry, contributes greatly to statistical inference. See [1] for details of information geometry.

We are now able to define a map φ from X to $\mathcal{P}(\partial X)$ in terms of the Poisson kernels:

$$\varphi : X \longrightarrow \mathcal{P}(\partial X);\;\; x \longmapsto \varphi(x) = \Phi(x, \theta) d\theta.$$

Remark that Douady and Earle defined the same map for real hyperbolic spaces and discussed the barycenter map associated with this map. See [6] for the detail. Also Besson et al defined in [3] a similar map to the space of L^2-functions on ∂X in order to develop their studies.

The aim of this paper is to investigate geometry of the pull-back metric $\varphi^* g$ of the Fisher information metric g by this map φ.

When X is a rank one symmetric space of noncompact type, the map φ turns out to be an embedding, since the Poisson kernels are expressed as an exponential function of the Busemann functions (see the detailed argument in section 3). We can then make use of an induced action of isometries of X on $\mathcal{P}(\partial X)$ which plays an important role in our investigation.

In fact, the action of $I(X)$, the isometry group of X, on ∂X is naturally defined by the fact that the asymptotical relation on geodesic rays is preserved under the isometric action. Further, we get an action of $I(X)$ on $\mathcal{P}(\partial X)$ by using Bourdon's argument about the Jacobian of the isometric action on the ideal boundary, given in [4], as

$$\mu = m(\theta) d\theta \longmapsto \gamma(\mu) = m(\gamma^{-1}(\theta)) \Phi(\gamma(x_o), \theta) d\theta,$$

where $\gamma \in I(X)$ and x_o is the reference point of X.

Therefore we obtain in Proposition 4.2 in section 4 that the φ is equivariant with respect to the actions of $I(X)$ on X and on $\mathcal{P}(\partial X)$, namely

$$\gamma(\varphi(x)) = \varphi(\gamma(x)),\;\; x \in X,\;\; \gamma \in I(X)$$

Another important fact is that the action of $I(X)$ on $\mathcal{P}(\partial X)$ is isometric with respect to the Fisher information metric, as given in Proposition 4.3.

By making use of these facts together with the homogenuity of our manifold X we obtain the following

Theorem A. *Let (X,h) be an n-dimensional rank one symmetric space of noncompact type. Let $\varphi^* g$ be the pull-back metric of the Fisher information metric via the embedding φ.*

Then $\varphi^* g$ is proportional to the metric h. More explicitly,

$$\varphi^* g = \frac{\rho^2}{n}\ h,$$

where ρ is a constant called the volume entropy of X, the increasing degree of the geodesic volume.

This means that the embedding φ is isometric up to constant factor. Note that $\rho^2 h$ is invariant even if we change conformally the metric h into λh by a constant λ. The following theorem asserts that this isometric embedding is minimal.

Theorem B. *Suppose that (X,h) be a rank one symmetric space of noncompact type. Then, the mean curvature vector of the submanifold $\varphi(X)$ in $\mathcal{P}(\partial X)$ vanishes identically, that is, the φ is minimal.*

The facts upon which this theorem depends crucially are that the Levi-Civita connection of the Fisher information metric has an explicit form as is seen in section 2 and also that the Poisson kernels of rank one symmetric space of noncompact type are in exponential form with exponent of the Busemann functions. Therefore, we consider, conversely, whether this exponential expression of the Poisson kernels characterizes rank one symmetric spaces of noncompact type. We have, in fact, by using the argument of Besson et al given in [3], the following characterization, though under an additional assumption.

Theorem C. *Let (X,h) be a simply connected, complete, n-dimensional Riemannian manifold with sectional curvature satisfying $-b^2 \leq K_X \leq -a^2 < 0$ for some constants a,b. Suppose $n \geq 3$ and that (X,h) admits a compact quotient.*

If the Poisson kernels $\Phi(x,\theta)$ for X are given by

$$\Phi(x,\theta) = \exp(-cB_\theta(x))$$

in terms of the Busemann functions $B_\theta(x)$ and a constant c, then, (X,h) must be a rank one symmetric space of noncompact type and the c is the volume entropy of (X,h).

It is still an interesting, open question whether the above theorem holds without a compact quotient assumption.

As Proposition 3.5 in section 3 indicates, we can characterize the exponential expression of the Poisson kernels $\Phi(x,\theta)$ given as $\Phi(x,\theta) = \exp(-cB_\theta(x))$, by means of the mean curvature of horospheres in X geometrically. We have then

Corollary D. *Let (X,h) be a simply connected, complete, n-dimensional Riemannian manifold with sectional curvature satisfying $-b^2 \leq K_X \leq -a^2 < 0$ for some constants a, b. Suppose that $n \geq 3$ and (X,h) has a compact quotient. If the mean curvature of every horosphere of X is constant c and this constant c takes the same value for all horospheres, then (X,h) is a rank one symmetric space of noncompact type.*

In section 2, we consider the space of probability measures and define the Fisher information metric. In section 3, we give the Poisson kernels on each manifold with negative curvature and characterize a rank one symmetric space in terms of Poisson kernels (Theorem C and Corollary D). We show, in section 4, that the space of probability measures carries the action of the isometry group of X.

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2. Probability measures of positive density function and the Fisher information metric

Let M be a compact, oriented, n-dimensional Riemannian manifold and let dv be the canonical Riemannian volume element of unit volume. Over M we consider the space $\mathcal{P}(M)$ of smooth probability measures whose density function is everywhere positive ;

$$\mathcal{P}(M) = \left\{ \mu = f dv \;\middle|\; f \in C^\infty(M),\ f > 0,\ \int_M \mu = 1 \right\}. \tag{1}$$

It space $\mathcal{P}(M)$ has a C^∞ manifold structure modeled on a Frechét space [17].

We define the Fisher information metric g on $\mathcal{P}(M)$ by

$$g(\sigma_1, \sigma_2) = \int_M \frac{d\sigma_1}{d\mu} \frac{d\sigma_2}{d\mu} \mu \tag{2}$$

at $\mu \in \mathcal{P}(M)$, for $\sigma_1, \sigma_2 \in T_\mu \mathcal{P}(M)$, where $d\sigma_i/d\mu$ denotes the density function of σ_i with respect to μ ($i = 1, 2$). Remark that the tangent space at $\mu \in \mathcal{P}(M)$ is the space of smooth measures whose total measure over M is 0;

$$T_\mu \mathcal{P}(M) = \left\{ \sigma = \frac{d\sigma}{d\mu} \mu \;\middle|\; \frac{d\sigma}{d\mu} \in C^\infty(M),\ \int_M \sigma = 0 \right\}. \tag{3}$$

The metric g is positive definite, because any density function is positive everywhere.

On $\mathcal{P}(M)$ the Levi-Civita connection ∇ for the Fisher information metric g at $\mu \in \mathcal{P}(M)$ is given by

$$(\nabla_{\tau_1} \tau_2)(\mu) = -\frac{1}{2} \left(\frac{d\tau_1}{d\mu} \frac{d\tau_2}{d\mu} - \int_M \frac{d\tau_1}{d\mu} \frac{d\tau_2}{d\mu} \mu \right) \mu, \tag{4}$$

for two vector fields τ_1, τ_2. Refer to [8] for this formula.

For geometrically interesting facts concerning the Fisher information metric, refer to [8, 18, 19]. For example, T. Friedrich showed in [8] that the Riemannian manifold $(\mathcal{P}(M), g)$ is a space of positive constant curvature. Furthermore, an orientation preserving diffeomorphism on M acts on $\mathcal{P}(M)$ by pull-back. Then the action is isometric with respect to the Fisher information metric. This fact plays an important role in studying the space of probability measures.

3. Poisson kernels for a complete manifold with negative curvature

Let X be a simply connected, complete, n-dimensional Riemannian manifold with sectional curvature satisfying $-b^2 \le K_X \le -a^2 < 0$ for constants a, b.

The ideal boundary ∂X is defined as equivalent classes of geodesics. Here two geodesics of unit speed c_1 and c_2 are asymptotic equivalent if $d(c_1(t), c_2(t))$ is a bounded function in t, $t \ge 0$, where d is the Riemannian distance function on X. Fixing $x_0 \in X$, we can identify ∂X with the unit sphere of $T_{x_0}X$. Therefore the ideal boundary ∂X is regarded as the standard sphere S^{n-1}. A natural topology called the cone topology is defined on $\overline{X} = X \cup \partial X$. This topology gives a compactification of X. See [2, 15] for details about the ideal boundary.

Let Δ be the Laplace-Beltrami operator of X. Now we consider the Dirichlet problem for X with respect to ∂X. The existence and uniqueness of a solution to this problem is given in the following theorem;

Theorem 3.1. ([2, 15]). *Let X be a simply connected, complete, Riemannian manifold whose sectional curvature satisfies $-b^2 \le K_X \le -a^2 < 0$. For any $\psi \in C^0(\partial X)$, there exists a unique function $u \in C^\infty(X) \cap C^0(\overline{X})$ such that*

$$\Delta u = 0 \quad \text{in } X, \qquad u\big|_{\partial X} = \psi. \tag{5}$$

Fix $x_0 \in X$ as a base point. Let $d\theta$ be the normalized Riemannian volume form on $\partial X = S^{n-1}$ defined by

$$d\theta = \frac{1}{\mathrm{Vol}(S^{n-1})} d\theta^1 \wedge \cdots \wedge d\theta^{n-1},$$

where $\{d\theta^i\}$ is a local orthonormal frame for the cotangent bundle $T^*(S^{n-1})$.

Definition 3.2. ([2, 15]). Let θ be an element of ∂X. A continuous function $\Phi_\theta : \overline{X} \setminus \theta \to \mathbf{R}$; $x \mapsto \Phi_\theta(x)$ is called the Poisson kernel normalized at x_0 for $\theta \in \partial X$ if it is harmonic in X and satisfies the following properties ;

1. $\Phi_\theta(x) > 0$ for all $x \in X$,

2. $\Phi_\theta(x_0) = 1$,

3. $\lim_{x \to \theta} \Phi_\theta(x) = +\infty$,

4. $\theta' \in \partial X,\ \theta' \neq \theta\ \Rightarrow\ \lim_{x \to \theta'} \Phi_\theta(x) = 0$.

Conventionally we write $\Phi_\theta(x)$ as $\Phi(x, \theta)$.

According to [2, 15], the solution of the Dirichlet problem for X can be written as the Poisson integral formula

$$u(x) = \int_{\partial X} \Phi(x, \theta)\psi(\theta)\, d\theta. \tag{6}$$

We remark here that for every point $x \in X$ the Poisson kernel $\Phi(x, \theta)$ is a probability density function on $\partial X = S^{n-1}$. In fact, assuming $\psi = 1$ on ∂X we have from (5) and (6)

$$\int_{\partial X} \Phi(x, \theta)\, d\theta = 1, \tag{7}$$

So we can define naturally a map from X to the space of probability measures with positive density function on $\partial X = S^{n-1}$;

$$\varphi : X \longrightarrow \mathcal{P}(\partial X);\ x \longmapsto \varphi(x) = \Phi(x, \theta)\, d\theta.$$

When X is a rank one symmetric space of noncompact type, we have the following relation between the Poisson kernel and the Busemann function ;

$$\Phi(x, \theta) = \exp(-\rho B_\theta(x)), \tag{8}$$

where $\rho = n + \dim_{\mathbf{R}}(\mathbf{F}) - 2$ and $\mathbf{F}$ is the field corresponding to X (see [3, 10, 14]).

Here, the Busemann function B_θ is defined by

$$B_\theta(x) = \lim_{t \to \infty} (d(c(t), x) - t), \qquad x \in X, \tag{9}$$

for a point θ of the ideal boundary ∂X, where $c(t)$ is the normalized geodesic such that $c(0) = x_0$, $\lim_{t \to \infty} c(t) = \theta$. For the Busemann function B_θ, the level hypersurfaces $B_\theta^{-1}(k)$ are called horospheres. A horosphere can be regarded as a distance sphere centered at a point of ∂X (See [16], p.232).

We can easily show that the Busemann function $B_\theta(x)$ satisfies the following ;

1. $B_\theta(x_0) = 0$,
2. $\lim_{x \to \theta} B_\theta(x) = -\infty$,
3. $\lim_{x \to \theta'} B_\theta(x) = +\infty, \quad \theta \neq \theta'$.

Moreover it is known that the Busemann function is C^2 in X (see [11]).

If X is a rank one symmetric space of noncompact type, we have the realizations of X by the Poincaré models. For example, when X is a real hyperbolic space $H^n(\mathbf{R})$ or a complex hyperbolic space $H^n(\mathbf{C})$, we have

$$H^n(\mathbf{R}) = \left(D^n(\mathbf{R}),\ h_{H^n(\mathbf{R})} = \frac{4}{(1-|x|^2)^2} dx^2 \right),$$

$$H^n(\mathbf{C}) = \left(D^n(\mathbf{C}),\ h_{H^n(\mathbf{C})} = \frac{4dzd\bar{z}}{1-|z|^2} + \sum_{i,j=1}^{n} \frac{4\bar{z}_i z_j dz_i d\bar{z}_j}{(1-|z|^2)^2} \right),$$

where D^n is the unit disk in the Euclidean space centered at the origin.

The Poisson kernels for a real hyperbolic space or a complex hyperbolic space are given by

$$\Phi(x,\theta) = \left(\frac{1-|x|^2}{|x-\theta|^2} \right)^{n-1}, \qquad X = H^n(\mathbf{R}),$$

$$\Phi(x,\theta) = \left(\frac{1-|x|^2}{|1-x\cdot\theta|^2} \right)^{n}, \qquad X = H^n(\mathbf{C}),$$

where $|\ \ |$ is the norm defined by the canonical inner product in $\mathbf{R}$ or $\mathbf{C}$ and $\cdot$ is the canonical Hermitian inner product in $\mathbf{C}^n$. We refer to [13] for the Poisson kernels for the hyperbolic plane and also to [9] for the complex hyperbolic space.

From the properties of the Busemann functions together with (8), it is clear that the above map φ is injective. Therefore we obtain

Lemma 3.3. *When X is a rank one symmetric space of noncompact type, the map φ is an embedding.*

Now, we characterize a rank one symmetric space of non-compact type by Poisson kernels. To obtain our theorem we make use of a fact which is showed by G. Besson, G. Courtois and S. Gallot;

Theorem 3.4.([3]) *Let X be a compact manifold with negative curvature and $\tilde{X}$ be the universal covering of X. Suppose that $\tilde{X}$ is asymptotic harmonic, i.e., the mean curvature of any horosphere in $\tilde{X}$ is constant. Then $\tilde{X}$ is a rank one symmetric space.*

Proof of Theorem C. Let $e_1, \cdots, e_n$ be a orthonormal basis for T_xX such that $\nabla_{e_i} e_j = 0$ for any i, j at $x \in X$. Now, we calculate the Laplacian of the function $x \mapsto \Phi(x, \theta)$. Then we have from the assumption of the theorem

$$\begin{aligned} \Delta\Phi(x,\theta) &= -\sum_{i=1}^{n} \nabla_{e_i}\nabla_{e_i}\Phi(x,\theta) \\ &= -c\{\Delta B_\theta(x) + c\|\mathrm{grad} B_\theta(x)\|^2\}\Phi(x,\theta). \end{aligned}$$

Since $\|\mathrm{grad} B_\theta(x)\| = 1$ (see [16]), we obtain

$$\Delta\Phi(x,\theta) = -c\{\Delta B_\theta(x) + c\}\Phi(x,\theta). \tag{10}$$

Since the Poisson kernel is a harmonic function, we have $\Delta B_\theta(x) = -c$.

The gradient vector $-\mathrm{grad} B_\theta(x)$ is a unit inward normal vector on the horosphere including θ and x. Remark that the second fundamental form $\Pi(V, W)$ with respect to the normal vector $-\mathrm{grad} B_\theta(x)$ of a horosphere in X is given by

$$\begin{aligned} \Pi(V,W) &= -\langle \nabla_V(\mathrm{grad} B_\theta(x)), W\rangle \\ &= -\mathrm{Hess}\, B_\theta(x)(V,W), \end{aligned} \tag{11}$$

where $\langle\ ,\ \rangle$ is the Riemannian metric on X and ∇ is the Levi-Civita connection of X (see [7]). Thus, $\Delta B_\theta(x) = -c$ implies that the mean curvature of any horosphere in X is constant c/n. Therefore X is a rank one symmetric space of non-compact type by Theorem 3.4. Moreover, from the uniqueness theorem of the Poisson kernel (Theorem 2.8 in [15]), the constant c is the volume entropy. □

We obtain naturally the following result from the caluculation in the above proof.

Proposition 3.5. *Let X be a simply connected, complete, n-dimensional Riemannian manifold whose sectional curvature satisfies $-b^2 \leq K_X \leq -a^2 < 0$ for some constants a, b. If all the horospheres in X have same constant* *mean curvature, then the Poisson kernels of X can be written as exponential functions whose exponents are the Busemann functions.*

Proof. Let c/n be constant mean curvature of horospheres in X. Then, we have $\Delta B_\theta(x) = -c$ from (11). Define $\Phi(x,\theta)$ by $\Phi(x,\theta) = \exp(-cB_\theta(x))$. Since the function $\Phi(x,\theta)$ satisfies the equation (10), $\Phi(x,\theta)$ is harmonic in X.

We can see easily that $\Phi(x,\theta)$ is continuous in $\overline{X} \setminus \{\theta\}$ and also satisfies the conditions of Definition 3.2 because of the properties of the Busemann function. The uniqueness theorem of the Poisson kernels implies that

$$\Phi(x,\theta) = \exp(-cB_\theta(x))$$

is a Poisson kernel for X. $\square$

It is also interesting to consider the following problem ; how do the mappings $\{\varphi_i : X_i \to \mathcal{P}(\partial X_i)\}$ behave in the case of $\sup_{X_i} K_{X_i} \to 0$ or $\inf_{X_i} K_{X_i} \to -\infty$?

4. Isometries of X and the space of probability measures on ∂X

Let $\varphi : X \to \mathcal{P}(\partial X)$ be the embedding. Since the space $\mathcal{P}(\partial X)$ carries the Fisher information metric g, we are interested in geometry of the pull-back metric $\varphi^* g$ of the metric g via φ.

In order to investigate the metric $\varphi^* g$, we rely on the equivariant property of the map φ.

We denote by $I(X)$ the group of isometries of a Riemannian manifold (X,h). Let $\gamma \in I(X)$. Then γ induces naturally an action on the space ∂X. The following proposition shows that the Jacobian of the action of $\gamma \in I(X)$ on ∂X yields the Poisson kernel.

Proposition 4.1 ([3, 4]). *The pull-back of the normalized Riemannian volume form $d\theta \in \mathcal{P}(\partial X)$ by $\gamma \in I(X)$ is*

$$\gamma^*(d\theta) = \Phi(\gamma^{-1}(x_0),\theta)\, d\theta \tag{12}$$

where x_0 is a base point of X.

The action of $I(X)$ on ∂X induces naturally an action on the space $\mathcal{P}(\partial X)$ which we denote by the same symbol γ as

$$\gamma(\mu) = m(\gamma^{-1}(\theta))\Phi(\gamma(x_0), \theta)\, d\theta \tag{13}$$

for $\mu = m(\theta)d\theta \in \mathcal{P}(\partial X)$. Notice that $\gamma(d\theta) = (\gamma^{-1})^*(d\theta)$. Proposition 4.1 asserts that $\gamma(\mu) \in \mathcal{P}(\partial X)$ for an arbitrary $\gamma \in I(X)$ and $\mu \in \mathcal{P}(\partial X)$. Furthermore, we obtain the transition formula of the Poisson kernels ;

$$\Phi(\gamma(x), \theta) = \Phi(x, \gamma^{-1}(\theta))\Phi(\gamma(x_0), \theta) \tag{14}$$

Proposition 4.2. *The embedding φ is $I(X)$-equivariant, namely for all $\gamma \in I(X)$*

$$\varphi(\gamma(x)) = \gamma(\varphi(x)). \tag{15}$$

Proof. Since $\varphi(x) = \Phi(x, \theta)d\theta$, we have

$$\gamma(\varphi(x)) = \gamma(\Phi(x, \theta)d\theta)$$

which reduces from (13) to $\Phi(x, \gamma^{-1}(\theta))\Phi(\gamma(x_0), \theta)d\theta$. This coincides from the transition formula (14) with $\Phi(\gamma(x), \theta)d\theta = \varphi(\gamma(x))$. $\square$

Remark. In [3] Besson, Courtois and Gallot use systematically the notion of Γ-equivariant immersion of a rank-one symmetric space Y of non-compact type into the unit sphere in the L^2-space $L^2(\partial Y)$ over the ideal boundary ∂Y, where Γ is the discrete subgroup of $I(Y)$. By applying such immersions they obtained characterizations of a rank one symmetric spaces of non-compact type in terms of the volume entropy. It is not vague to point out that the framework of our study is similar to theirs, whereas they use the L^2-metric, not Fisher information metric.

Now we consider the Fisher information metric g on $\mathcal{P}(\partial X)$, and we will show that the action of $I(X)$ on $\mathcal{P}(\partial X)$ preserves g.

Let $\Omega(\partial X)$ be the space of smooth $(n-1)$-forms on ∂X. We define the action of $\gamma \in I(X)$ on $\Omega(\partial X)$ by

$$\gamma(\sigma) = f(\gamma^{-1}(\theta))\Phi(\gamma(x_0), \theta)d\theta \tag{16}$$

for $\sigma = f(\theta)d\theta \in \Omega(\partial X)$. This action is a natural extension of the action of $\gamma \in I(X)$ on $\mathcal{P}(\partial X) \subset \Omega(\partial X)$.

Proposition 4.3. *The action of $I(X)$ on $\mathcal{P}(\partial X)$ is isometric with respect to the Fisher information metric, that is, any $\gamma \in I(X)$ satisfies*

$$g(d\gamma(\sigma_1), d\gamma(\sigma_2))_{\gamma(\mu)} = g(\sigma_1, \sigma_2)_\mu, \tag{17}$$

at any $\mu \in \mathcal{P}(\partial X)$ and for any $\sigma_1, \sigma_2 \in T_\mu(\mathcal{P}(\partial X))$.

Proof. Write $\mu = m(\theta)d\theta \in \mathcal{P}(\partial X)$ and let $\sigma_i = f_i(\theta)\mu \in T_\mu\mathcal{P}(\partial X)$ $(i = 1, 2)$ be two tangent vectors. Let $\gamma \in I(X)$. Since the differential map of γ is given by (16), we have

$$\begin{aligned} d\gamma(\sigma_i) = \gamma(\sigma_i) &= f_i(\gamma^{-1}(\theta))m(\gamma^{-1}(\theta))\Phi(\gamma(x_0), \theta)d\theta \\ &= f_i(\gamma^{-1}(\theta))\gamma(\mu). \end{aligned}$$

Therefore we have

$$\begin{aligned} g(d\gamma(\sigma_1), d\gamma(\sigma_2))_{\gamma(\mu)} &= \int_{\partial X} \frac{d(\gamma(\sigma_1))}{d(\gamma(\mu))} \frac{d(\gamma(\sigma_2))}{d(\gamma(\mu))} \gamma(\mu) \\ &= \int_{\partial X} f_1(\gamma^{-1}(\theta))\, f_2(\gamma^{-1}(\theta))\, m(\gamma^{-1}(\theta))\, \Phi(\gamma(x_0), \theta)\, d\theta. \end{aligned}$$

From (12), $\Phi(\gamma(x_0), \theta)d\theta = (\gamma^{-1})^*(d\theta)$. So, the above reduces to

$$\int_{\partial X} f_1(\gamma^{-1}(\theta))\, f_2(\gamma^{-1}(\theta))\, m(\gamma^{-1}(\theta))\, (\gamma^{-1})^*(d\theta).$$

We put $\theta' = \gamma^{-1}(\theta)$. Then, this integration is $\int_{\partial X} f_1(\theta')\, f_2(\theta')\, m(\theta')\, d\theta'$ and thus coincides with $g(\sigma_1, \sigma_2)_\mu$. $\square$

5. Proofs of Theorems

Let (X, h) be a rank one symmetric space of noncompact type of dimension n, and x_0 be the origin of X. X is two-point homogeneous so that $I(X)_{x_0}$ acts transitively on the unit sphere $U_{x_0}X$ in $T_{x_0}X$ (see [12] p355), where $I(X)_{x_0} = \{\gamma \in I(X) \mid \gamma x_0 = x_0\}$ be the isotropy subgroup of $I(X)$ at x_0. Therefore, from Propositions 4.2 and 4.3 it is sufficient to consider at the single point $x_0 = \mathbf{0} \in X$,

Proof of Theorem A.

Let $u \in T_{\mathbf{0}}X$ be an unit vector with respect to the original metric h. From $\Phi(x,\theta) = \exp(-\rho B_\theta(x))$, the formula of the differential map of the embedding φ is given by

$$d\varphi(u) = -\rho \, dB_\theta(u) \, d\theta.$$

Then we have

$$\begin{aligned} \varphi^* g(u,u) &= \rho^2 \int_{\partial X} (dB_\theta(u))^2 d\theta \\ &= \rho^2 \int_{\partial X} \Big(h(u, c'(0)) dB_\theta(c'(0)) \Big)^2 d\theta, \end{aligned}$$

where $c(t)$ is the normalized geodesic passing through $\mathbf{0}$ and θ. In fact, the differential at $\mathbf{0}$ of the Busemann function vanishes except the direction of $c'(0)$.

Remark that $dB_\theta(c'(0)) = -1$. For any unit vector $u \in T_{\mathbf{0}}X$, we can take an orthonormal basis $e_1, \cdots, e_n$ such that $u = e_1$, then we have

$$\begin{aligned} \varphi^* g(u,u) &= \rho^2 \int_{\partial X} (h(u, c'(0)))^2 d\theta \\ &= \rho^2 \int_{w \in \partial X = S^{n-1}} h(u,w)^2 dw \\ &= \rho^2 \int_{w \in \partial X = S^{n-1}} (w_1)^2 dw \\ &= \frac{\rho^2}{n} = \frac{\rho^2}{n} h(u,u). \quad \Box \end{aligned}$$

Proof of Theorem B

Let $e_1, \cdots, e_n$ be the basis for $T_{\mathbf{0}}X$. Since $d\varphi(u) = -\rho \, dB_\theta(u) \, d\theta$, we have $d\varphi_0(e_i) = -\rho \, \theta_i d\theta$.

By the formula (4) of the Levi-Civita connection of the Fisher information metric g we obtain

$$\nabla_{d\varphi_{\mathbf{0}}(e_i)} d\varphi_{\mathbf{0}}(e_j) = -\frac{1}{2} \left(\rho^2 \theta_i \theta_j - \rho^2 \int_{S^{n-1}} \theta_i \theta_j \, d\theta \right) d\theta.$$

Then one can easily see that

$$\sum_{i=1}^{n} \nabla_{d\varphi_{\mathbf{0}}(e_i)} d\varphi_{\mathbf{0}}(e_i) = 0.$$

Since the trace of normal part of $\nabla_{d\varphi_{\mathbf{0}}(e_i)} d\varphi_{\mathbf{0}}(e_j)$ vanishes, we complete the proof of Theorem B. □

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