Synthetic Differential Geometry of Higher-Order Total Differentials

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RESUME. Etant donné des espaces microlinéaires M, N avec $x \in M$ et $y \in N$, nous avons étudié dans un article précédent [Beiträge zur Algebra und Geometrie, 45 (2004), 677-696] un certain type d'applications de la totalité T_x^{Dn} des D^n-microcubes sur M en x vers la totalité T_y^{Dn} des D^n-microcubes sur N en y, appelées alors pré-connexions d'ordre n, et appelées ici D^n-tangentielles, qui donnent une généralisation sans-germe des différentielles totales d'ordre n. Dans cet article, après avoir étudié de manière plus approfondie cette généralisation, nous proposons un certain type d'applications de T_x^{Dn} vers T_y^{Dn}, appelées D_n-tangentielles, qui donnent une autre généralisation sans-germe des différentielles totales d'ordre n. Nous étudions alors la relation entre D^n-tangentielles et D_n-tangentielles, d'abord dans le cas où des coordonnées ne sont pas accessibles (i.e., M et N sont des espaces microlinéaires généaux sans condition supplémentaire), puis lorsqu'il y a des coordonnées (i.e., M et N sont des variétés formelles). Dans le premier cas, nous avons une application naturelle des D^n-tangentielles dans les D_n-tangentielles, et dans le deuxième cas nous montrons que cette application naturelle est injective. Nos idées sont présentées dans notre cadre préféré la géométrie différentielle synthétique, mais elles sont facilement applicables, avec quelques modifications, à des généalisations des variétés différentiables telles que les espaces différentiables et des variétés de dimension infinie appropriées. Cet article peut être vu comme donnant une généralisation microlinéaire des intéressantes considérations de Kock [1978] sur le calcul des séries de Taylor.

1 Introduction

In teaching differential calculus of several variables, mathematicians are expected to exhort freshmen or sophomores majoring in science, engineering etc. to understand that it is not partial derivatives but total differentials that are of intrinsic meaning, while partial derivatives are used for computational purposes. If we want to discuss not only first-order total differentials but higher-order ones, we have to resort to the theory of jets initiated by Ehresmann, though it is not easy to generalize it beyond the scope of finite-dimensional smooth manifolds so as to encompass differentiable spaces and suitable infinite-dimensional manifolds, for which the reader is referred, e.g., to Navarro and Sancho de Salas [8] and Libermann [6].

The then moribund notion of nilpotent infinitesimals in differential geometry was retrieved by Lawvere in the middle of the preceding century, while Robinson revived invertible infinitesimals in analysis, and Grothendieck authenticated nilpotent infinitesimals in algebraic geometry. Kock [2, 3], following the new directions in differential geometry enunciated by Lawvere as *synthetic differential geometry* (usually abbreviated to SDG), has investigated differential calculus from this noble standpoint as the foundations of SDG. For readable textbooks on SDG, the reader is referred to Kock [4], Lavendhomme [5] and Moerdijk and Reyes [7].

Kock [3] has shown that the infinitesimal space D_n captures n-th order differential calculus. To show this, he had to exploit the fact that another infinitesimal space $D^n = D \times ... \times D$ (the product of n copies of D) has a good grasp of n-th order differential calculus. In our previous paper (Nishimura [13]) we have demonstrated that, given microlinear spaces M, N with $x \in M$ and $y \in N$, n–th order total differentials can be captured as a certain kind of mappings from the totality $\mathbf{T}_x^{D^n}(M)$ of D^n-microcubes on M at x to the totality $\mathbf{T}_y^{D^n}(N)$ of D^n-microcubes on N at y, which were called n–th order preconnections there and are to be called D^n-tangentials here. In this paper we propose another generalization of n–th order total differentials as a certain kind of mappings from the totality $\mathbf{T}_x^{D_n}(M)$ of D_n-microcubes on M at x to the totality $\mathbf{T}_y^{D_n}(N)$ of D_n-microcubes on N at y, which are to be called D_n-tangentials. Then we study the relationship between D^n-tangentials and D_n-tangentials, firstly in case that coordinates are not available (i.e., M and N are general microlinear spaces without further conditions imposed) and secondly in case that coordinates are available (i.e., M and N are formal manifolds). In the former case we have a natural mapping from D^n-tangentials into D_n-tangentials, while in the latter case the natural mapping is shown to be bijective. Since we have shown in our previous paper that our notion of D^n-tangentials is a generalization of Ehresmann's classical notion of jets, this means that not only D^n-tangentials but also D_n-tangentials are a generalization of jets. Our ideas will be presented within our favorite framework of synthetic differential geometry, but they are readily applicable to such generalizations of smooth manifolds as differentiable spaces and suitable infinite-dimensional manifolds with due modifications. This paper is to be looked upon as a microlinear generalization of Kock's [3] perspicacious considerations on Taylor series calculus. The exact relationship between D^n-tangentials and D_n-tangentials in the general setting, besides mere existence of a canonical mapping from the former to the latter, remains an open problem for the competent and inspired reader. Last but not least, we gladly acknowledge our indebtedness to the anonymous referee, who has made many constructive suggestions, without which the paper would not have been completed.

2 Preliminaries

2.1 Microcubes

Let $\mathbb{R}$ be the extended set of real numbers with cornucopia of nilpotent infinitesimals, which is expected to acquiesce in the so-called general Kock axiom (cf. Lavendhomme [5, p.42]. We denote by D_1 or D the totality of elements of $\mathbb{R}$ whose squares vanish. More generally, given a natural number n, we denote by D_n the set

$$\{d \in \mathbb{R} | d^{n+1} = 0\}.$$

Given natural numbers m, n, we denote by $D(m)_n$ the set

$$\{(d_1, ..., d_m) \in D^m | d_{i_1} ... d_{i_{n+1}} = 0\},$$

where $i_1, ..., i_{n+1}$ shall range over natural numbers between 1 and m including both ends. We will often write $D(m)$ for $D(m)_1$. By convention $D^0 = D_0 = \{0\}$. A polynomial ρ of $d \in D_n$ is called a *simple* polynomial of $d \in D_n$ if every coefficient of ρ is either 1 or 0, and if the constant term is 0. A simple polynomial ρ of $d \in D_n$ is said to be of dimension m, in notation $\dim(\rho) = m$, provided that m is the least integer with $\rho^{m+1} = 0$. By way of example, letting $d \in D_3$, we have $\dim(d) = \dim(d + d^2) = \dim(d + d^3) = 3$ and $\dim(d^2) = \dim(d^3) = \dim(d^2 + d^3) = 1$.

Simplicial infinitesimal spaces are spaces of the form

$$D(m; \mathcal{S}) = \{(d_1, ..., d_m) \in D^m | d_{i_1} ... d_{i_k} = 0 \text{ for any } (i_1, ..., i_k) \in \mathcal{S}\},$$

where $\mathcal{S}$ is a finite set of sequences $(i_1, ..., i_k)$ of natural numbers with $1 \leq i_1 < ... < i_k \leq m$. A simplicial infinitesimal space $D(m; \mathcal{S})$ is said to be *symmetric* if $(d_1, ..., d_m) \in D(m; \mathcal{S})$ and $\sigma \in \mathfrak{S}_m$ always imply $(d_{\sigma(1)}, ..., d_{\sigma(m)}) \in D(m; \mathcal{S})$. To give an example of simplicial infinitesimal spaces, we have $D(2) = D(2; (1, 2))$ and $D(3) = D(3; (1, 2), (1, 3), (2, 3))$, which are all symmetric. The number m is called the *degree* of $D(m; \mathcal{S})$, in notation: $m = \deg D(m; \mathcal{S})$, while the maximum number n such that there exists a sequence $(i_1, ..., i_n)$ of natural numbers of length n with $1 \leq i_1 < ... < i_n \leq m$ containing no subsequence in $\mathcal{S}$ is called the *dimension* of $D(m; \mathcal{S})$, in notation: $n = \dim D(m; \mathcal{S})$. By way of example, $\deg D(3) = \deg D(3; (1, 2)) = \deg D(3; (1, 2), (1, 3)) = \deg D^3 = 3$, while $\dim D(3) = 1$, $\dim D(3; (1, 2)) = \dim D(3; (1, 2), (1, 3)) = 2$ and $\dim D^3 = 3$. It is easy to see that if $n = \dim D(m; \mathcal{S})$, then $d_1 + ... + d_m \in D_n$ for any $(d_1, ..., d_m) \in D(m; \mathcal{S})$. Infinitesimal spaces of the form D^m are called *basic infinitesimal spaces.* Given two simplicial infinitesimal spaces $D(m; \mathcal{S})$ and $D(m'; \mathcal{S}')$, a mapping $\varphi = (\varphi_1, ..., \varphi_{m'}) : D(m; \mathcal{S}) \to D(m'; \mathcal{S}')$ is called a *monomial mapping* if every φ_j is a monomial in $d_1, ..., d_m$ with coefficient 1.

Given a microlinear space M and an infinitesimal space $\mathbb{E}$, a mapping γ from $\mathbb{E}$ to M is called an $\mathbb{E}$-*microcube* on M. D^n-microcubes are often called *n-microcubes*. In particular, 1-microcubes are usually called *tangent vectors*, and 2-microcubes are often referred to as *microsquares*. We denote by $\mathbf{T}^{\mathbb{E}}(M)$ the totality of $\mathbb{E}$-microcubes on M. Given $x \in M$, we denote by $\mathbf{T}_x^{\mathbb{E}}(M)$ the totality of $\mathbb{E}$-microcubes γ on M with $\gamma(0, ..., 0) = x$.

We denote by $\mathfrak{S}_n$ the symmetric group of the set $\{1, ..., n\}$, which is well known to be generated by $n-1$ transpositions $< i, i+1 >$ exchanging i and $i+1 (1 \leq i \leq n-1)$ while keeping the other elements fixed. Given $\sigma \in \mathfrak{S}_n$ and $\gamma \in \mathbf{T}_x^{D^n}(M)$, we define $\Sigma_\sigma(\gamma) \in \mathbf{T}_x^{D^n}(M)$ to be

$$\Sigma_\sigma(\gamma)(d_1, ..., d_n) = \gamma(d_{\sigma(1)}, ..., d_{\sigma(n)})$$

for any $(d_1, ..., d_n) \in D^n$. Given $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{T}_x^{D^n}(M)$, we define $\alpha \underset{i}{\cdot} \gamma \in \mathbf{T}_x^{D^n}(M)$ $(1 \leq i \leq n)$ to be

$$(\alpha \underset{i}{\cdot} \gamma)(d_1, ..., d_n) = \gamma(d_1, ..., d_{i-1}, \alpha d_i, d_{i+1}, ..., d_n)$$

for any $(d_1, ..., d_n) \in D^n$. Given $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{T}_x^{D_n}(M)$, we define $\alpha\gamma \in \mathbf{T}_x^{D_n}(M)$ $(1 \leq i \leq n)$ to be

$$(\alpha\gamma)(d) = \gamma(\alpha d)$$

for any $d \in D_n$. The restriction mapping $\gamma \in \mathbf{T}_x^{D_{n+1}}(M) \mapsto \gamma|_{D_n} \in \mathbf{T}_x^{D_n}(M)$ is often denoted by $\pi_{n+1,n}$.

Between $\mathbf{T}_x^{D^n}(M)$ and $\mathbf{T}_x^{D^{n+1}}(M)$ there are $2n+$ 2 canonical mappings:

$$\mathbf{T}_x^{D^{n+1}}(M) \underset{\mathbf{s}_i}{\overset{\mathbf{d}_i}{\rightleftarrows}} \mathbf{T}_x^{D^n}(M) \qquad (1 \leq i \leq n+1)$$

For any $\gamma \in \mathbf{T}_x^{D^n}(M)$, we define $\mathbf{s}_i(\gamma) \in \mathbf{T}_x^{D^{n+1}}(M)$ to be

$$\mathbf{s}_i(\gamma)(d_1, ..., d_{n+1}) = \gamma(d_1, ..., d_{i-1}, d_{i+1}, ..., d_{n+1})$$

for any $(d_1, ..., d_{n+1}) \in D^{n+1}$. For any $\gamma \in \mathbf{T}_x^{D^{n+1}}(M)$, we define $\mathbf{d}_i(\gamma) \in$

$\mathbf{T}_x^{D^n}(M)$ to be

$$\mathbf{d}_i(\gamma)(d_1, ..., d_n) = \gamma(d_1, ..., d_{i-1}, 0, d_i, ..., d_n)$$

for any $(d_1, ..., d_n) \in D^n$. These operations satisfy the so-called simplicial identities (cf. Goerss and Jardine [1, p.4]).

For any $\gamma \in \mathbf{T}_x^{D_n}(M)$ and any $d \in D_n$, we define $\mathbf{i}_d(\gamma) \in \mathbf{T}_x^{D_{n+1}}(M)$ to be

$$\mathbf{i}_d(\gamma)(d') = \gamma(dd')$$

for any $d' \in D_{n+1}$.

2.2 Quasi-Colimit Diagrams

Proposition 1 $\mathbb{R}$ *believes that the multiplication* $\mathfrak{m}_n : D_n \times D_n \to D_n$*, given by* $\mathfrak{m}_n(d_1, d_2) = d_1 d_2$ *for any* $(d_1, d_2) \in D_n \times D_n$*, is surjective.*

Proof. By the same token as in the proof of Proposition 1 of Lavendhomme [5, Section 2.2]. ■

Proposition 2 $\mathbb{R}$ *believes that the addition* $\mathfrak{a}_n : D^n \to D_n$*, given by* $\mathfrak{a}_n(d_1, ..., d_n) = d_1 + ... + d_n$ *for any* $(d_1, ..., d_n) \in D^n$*, is surjective.*

Proof. By the same token as in the proof of Proposition 2 of Lavendhomme [5, Section 2.2]. ■

Corollary 3 $\mathbb{R}$ *believes that the mapping* $\mathfrak{ma}_n : D_n \times D^n \to D_n$*, given by* $\mathfrak{ma}_n(d, d_1, ..., d_n) = d(d_1 + ... + d_n)$ *for any* $(d, d_1, ..., d_n) \in D_n \times D^n$*, is surjective.*

Proof. This follows from Propositions 1 and 2. ■

Proposition 4 $\mathbb{R}$ *perceives the addition* $\mathfrak{a}_n : D^n \to D_n$ *as a coequalizer of* n *mappings* $\mathrm{id}_{D^n}, \tau_1, ..., \tau_{n-1}$ *of* D^n *into itself, where* $\tau_i : D^n \to D^n$ *is the mapping permuting the* i*-th and* $(i+1)$*-th components of* D^n *while fixing the other components.*

Proof. By the same token as in the proof of Proposition 3 of Lavendhomme [5, Section 2.2]. ■

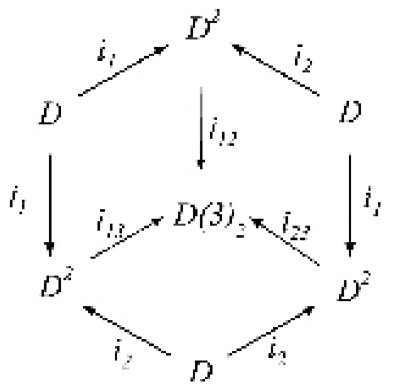
Proposition 5 *$\mathbb{R}$ perceives the multiplication $\mathfrak{m}_{n,n+1} : D_n \times D_{n+1} \rightarrow D_n$, given by $\mathfrak{m}_{n,n+1}(d_1, d_2) = d_1 d_2$ for any $(d_1, d_2) \in D_n \times D_{n+1}$, as a coequalizer of mappings $\mathfrak{m}_{n,n+1} \times \mathrm{id}_{D_{n+1}}$ and $\mathrm{id}_{D_n} \times \mathfrak{m}_{n+1}$ of $D_n \times D_{n+1} \times D_{n+1}$ into $D_n \times D_{n+1}$.*

Proof. By the same token as in the proof of Proposition 5 of Lavendhomme [5, Section 2.2]. ■

The following theorem will play a predominant role in this paper.

Theorem 6 *Any simplicial infinitesimal space $\mathfrak{D}$ of dimension n is the quasi-colimit of a finite diagram whose objects are of the form D^k's $(0 \leq k \leq n)$ and whose arrows are natural injections.*

Proof. Let $\mathfrak{D}$=$D(m; \mathcal{S})$. For any maximal sequence $1 \leq i_1 < ... < i_k \leq m$ of natural numbers containing no subsequence in $\mathcal{S}$ (maximal in the sense that it is not a proper subsequence of such a sequence), we have a natural injection of D^k into $\mathfrak{D}$. By collecting all such D^k's together with their natural injections into $\mathfrak{D}$, we have an overlapping representation of $\mathfrak{D}$ in terms of basic infinitesimal spaces. This representation is completed into a quasi-colimit representation of $\mathfrak{D}$ by taking D^l together with its natural injections into D^{k_1} and D^{k_2} for any two basic infinitesimal spaces D^{k_1} and D^{k_2} in the overlapping representation of $\mathfrak{D}$, where if D^{k_1} and D^{k_2} come from the sequences $1 \leq i_1 < ... < i_{k_1} \leq m$ and $1 \leq \bar{i}_1 < ... < \bar{i}_{k_2} \leq m$ in the above manner, then D^l together with its natural injections into D^{k_1} and D^{k_2} comes from the maximal common subsequence $1 \leq \widetilde{i}_1 < ... < \widetilde{i}_l \leq m$ of both the preceding sequences of natural numbers in the above manner. By way of example, the method leads to the following quasi-colimit representation of $\mathfrak{D}$=$D(3)_2$:



In the above representation i_{jk}'s and i_j's are as follows:

1. the j-th and k-th components of $i_{jk}(d_1, d_2) \in D(3)_2$ are d_1 and d_2 respectively, while the remaining component is 0;
2. the j-th component of $i_j(d) \in D^2$ is d, while the other component is 0.

■

The quasi-colimit representation of $\mathfrak{D}$ depicted in the proof of the above theorem is called *standard*. Generally speaking, there are multiple ways of quasi-colimit representation of a given simplicial infinitesimal space. By way of example, two quasi-colimit representations of $D(3; (1,3), (2,3))$ $(= (D \times D) \vee D)$ were given in Lavendhomme [5, pp.92-93], only the second one being standard.

2.3 Convention

Unless stated to the contrary, M and N are microlinear spaces with $x \in M$ and $y \in N$.

3 The First Kind of Tangentials

Let n be a natural number. A *D^n-pseudotangential* from (M, x) to (N, y) is a mapping $\mathsf{f} : \mathbf{T}_x^{D^n}(M) \to \mathbf{T}_y^{D^n}(N)$ such that for any $\gamma \in$

$\mathbf{T}_x^{D^n}(M)$, any $\alpha \in \mathbb{R}$ and any $\sigma \in \mathfrak{S}_n$, we have the following:

$$\begin{aligned} \mathsf{f}(\alpha \underset{i}{\cdot} \gamma) &= \alpha \underset{i}{\cdot} \mathsf{f}(\gamma) \qquad (1 \leq i \leq n) & (1) \\ \mathsf{f}(\Sigma_\sigma(\gamma)) &= \Sigma_\sigma(\mathsf{f}(\gamma)) & (2) \end{aligned}$$

We denote by $\hat{\mathbf{J}}^n(M, x; N, y)$ the totality of D^n-pseudotangentials from (M, x) to (N, y). We have the canonical projection $\hat{\pi}_{n+1,n} : \hat{\mathbf{J}}^{n+1}(M, x; N, y) \to \hat{\mathbf{J}}^n(M, x; N, y)$, so that

$$\mathsf{f}(\mathbf{s}_{n+1}(\gamma)) = \mathbf{s}_{n+1}(\hat{\pi}_{n+1,n}(\mathsf{f})(\gamma))$$

for any $\mathsf{f} \in \hat{\mathbf{J}}^{n+1}(M, x; N, y)$ and any $\gamma \in \mathbf{T}_x^{D^n}(M)$. For any natural numbers n, m with $m \leq n$, we define $\hat{\pi}_{n,m} : \hat{\mathbf{J}}^n(M, x; N, y) \to \hat{\mathbf{J}}^m(M, x; N, y)$ to be $\hat{\pi}_{m+1,m} \circ ... \circ \hat{\pi}_{n,n-1}$.

Interestingly enough, any D^n-pseudotangential naturally gives rise to what might be called a $\mathfrak{D}$-*pseudotangential* for any simplicial infinitesimal space $\mathfrak{D}$ of dimension less than or equal to n.

Theorem 7 *Let n be a natural number. Let $\mathfrak{D}$ be a simplicial infinitesimal space of dimension less than or equal to n. Any D^n-pseudotangential f from (M, x) to (N, y) naturally induces a mapping $\mathsf{f}_{\mathsf{D}} : \mathbf{T}_x^{\mathsf{D}}(M) \to \mathbf{T}_y^{\mathsf{D}}(N)$ abiding by the following condition:*

$$\mathsf{f}_{\mathsf{D}}(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \mathsf{f}_{\mathsf{D}}(\gamma)$$

for any $\alpha \in \mathbb{R}$ and any $\gamma \in \mathbf{T}_x^{\mathsf{D}}(M)$. If the simplicial infinitesimal space $\mathfrak{D}$ is symmetric, the induced mapping $\mathsf{f}_{\mathsf{D}} : \mathbf{T}_x^{\mathsf{D}}(M) \to \mathbf{T}_y^{\mathsf{D}}(N)$ acquiesces in the following condition of symmetry besides the above one:

$$\mathsf{f}_{\mathsf{D}}(\Sigma_\sigma(\gamma)) = \Sigma_\sigma(\mathsf{f}_{\mathsf{D}}(\gamma))$$

for any $\sigma \in \mathfrak{S}_n$ and any $\gamma \in \mathbf{T}_x^{\mathsf{D}}(M)$.

Proof. For the sake of simplicity, we deal only with the case that $\mathfrak{D} = D(3)_2$, for which the standard quasi-colimit representation was given in the proof of Theorem 6. Therefore, giving $\gamma \in \mathbf{T}_x^{D(3)_2}(M)$ is equivalent to giving $\gamma_{12}, \gamma_{13}, \gamma_{23} \in \mathbf{T}_x^{D^2}(M)$ with $\mathbf{d}_2(\gamma_{12}) = \mathbf{d}_2(\gamma_{13})$, $\mathbf{d}_1(\gamma_{12}) =$

$\mathbf{d}_2(\gamma_{23})$ and $\mathbf{d}_1(\gamma_{13}) = \mathbf{d}_1(\gamma_{23})$. By Proposition 1.3 of Nishimura [13], we have

$$\begin{aligned}
\mathbf{d}_2(\mathsf{f}(\gamma_{12})) &= \mathsf{f}(\mathbf{d}_2(\gamma_{12})) = \mathsf{f}(\mathbf{d}_2(\gamma_{13})) = \mathbf{d}_2(\mathsf{f}(\gamma_{13})), \\
\mathbf{d}_1(\mathsf{f}(\gamma_{12})) &= \mathsf{f}(\mathbf{d}_1(\gamma_{12})) = \mathsf{f}(\mathbf{d}_2(\gamma_{23})) = \mathbf{d}_2(\mathsf{f}(\gamma_{23})), \text{ and} \\
\mathbf{d}_1(\mathsf{f}(\gamma_{13})) &= \mathsf{f}(\mathbf{d}_1(\gamma_{13})) = \mathsf{f}(\mathbf{d}_1(\gamma_{23})) = \mathbf{d}_1(\mathsf{f}(\gamma_{23})),
\end{aligned}$$

which determines a unique $\mathsf{f}_{D(3)_2}(\gamma) \in \mathbf{T}_y^{D(3)_2}(N)$ with $\mathbf{d}_1(\mathsf{f}_{D(3)_2}(\gamma)) = \mathsf{f}(\gamma_{23})$, $\mathbf{d}_2(\mathsf{f}_{D(3)_2}(\gamma)) = \mathsf{f}(\gamma_{13})$ and $\mathbf{d}_3(\mathsf{f}_{D(3)_2}(\gamma)) = \mathsf{f}(\gamma_{12})$. The proof that $\mathsf{f}_{D(3)_2}$ acquiesces in the desired two properties is safely left to the reader.

■

Remark 8 *The reader should note that the induced mapping* f_{D} *is defined in terms of the standard quasi-colimit representation of* $\mathfrak{D}$*. The concluding corollary of this section will show that the induced mapping* f_{D} *is independent of our choice of a quasi-colimit representation of* $\mathfrak{D}$ *to a large extent, whether it is standard or not, as long as* f *is not only a* D^n*-pseudotangential but also a* D^n*-tangential (to be defined just below). We note in passing that* $\hat{\pi}_{n,m}(\mathsf{f})$ *with* $m \leq n$ *is no other than* f_{D^m}*.*

The notion of a D^n-*tangential* from (M, x) to (N, y) is defined inductively on n. The notion of a D^0-*tangential* from (M, x) to (N, y) and that of a D^1-*tangential* from (M, x) to (N, y) shall be identical with that of a D^0-pseudotangential from (M, x) to (N, y) and that of a D^1-pseudotangential from (M, x) to (N, y) respectively. Now we proceed by induction. A D^{n+1}-pseudotangential $\mathsf{f} : \mathbf{T}_x^{D^{n+1}}(M) \to \mathbf{T}_y^{D^{n+1}}(N)$ from (M, x) to (N, y) is called a D^{n+1}-*tangential* from (M, x) to (N, y) if it acquiesces in the following two conditions:

1. $\hat{\pi}_{n+1,n}(\mathsf{f})$ is a D^n-tangential from (M, x) to (N, y).
2. For any $\gamma \in \mathbf{T}_x^{D^n}(M)$, we have

$$\begin{aligned}
\mathsf{f}((d_1, ..., d_{n+1}) &\in D^{n+1} \longmapsto \gamma(d_1, ..., d_{n-1}, d_n d_{n+1}) \in M) \\
&= (d_1, ..., d_{n+1}) \in D^{n+1} \longmapsto \\
&\quad \hat{\pi}_{n+1,n}(\mathsf{f})(\gamma)(d_1, ..., d_{n-1}, d_n d_{n+1}) \in N \quad (3)
\end{aligned}$$

We denote by $\mathbf{J}^n(M,x;N,y)$ the totality of D^n-tangentials from (M,x) to (N,y). By the very definition of a D^n-tangential, the projection $\hat{\pi}_{n+1,n}:\hat{\mathbf{J}}^{n+1}(M,x;N,y)\to\hat{\mathbf{J}}^n(M,x;N,y)$ is naturally restricted to a mapping $\pi_{n+1,n}:\mathbf{J}^{n+1}(M,x;N,y)\to\mathbf{J}^n(M,x;N,y)$. Similarly for $\pi_{n,m}:\mathbf{J}^n(M,x;N,y)\to\mathbf{J}^m(M,x;N,y)$ with $m\leq n$.

Proposition 9 *Let L,M,N be microlinear spaces with $x\in L,y\in M$ and $z\in N$. If f is a D^n-tangential from (L,x) to (M,y) and g is a D^n-tangential from (M,y) to (N,z), then the composition $\mathsf{g}\circ\mathsf{f}$ is a D^n-tangential from (L,x) to (N,z), and $\pi_{n,n-1}(\mathsf{g}\circ\mathsf{f})=\pi_{n,n-1}(\mathsf{g})\circ\pi_{n,n-1}(\mathsf{f})$ provided that $n\geq 1$.*

Proof. In case of $n=0$, there is nothing to prove. It is easy to see that if f is a D^n-tangential from (L,x) to (M,y) and g is a D^n-tangential from (M,y) to (N,z), then the composition $\mathsf{g}\circ\mathsf{f}$ satisfies conditions (1) and (2). For any $\gamma\in\mathbf{T}_x^{D^n}(M)$, if f is a D^{n+1}-tangential from (L,x) to (M,y) and g is a D^{n+1}-tangential from (M,y) to (N,z), we have

$$\begin{aligned}
&\mathsf{g}\circ\mathsf{f}(\mathbf{s}_{n+1}(\gamma))=\mathsf{g}(\mathsf{f}(\mathbf{s}_{n+1}(\gamma)))\\
=\ &\mathsf{g}(\mathbf{s}_{n+1}(\pi_{n+1,n}(\mathsf{f})(\gamma)))=\mathbf{s}_{n+1}(\pi_{n+1,n}(\mathsf{g})\circ\pi_{n+1,n}(\mathsf{f})(\gamma)),
\end{aligned}$$

which implies that $\pi_{n+1,n}(\mathsf{g}\circ\mathsf{f})=\pi_{n+1,n}(\mathsf{g})\circ\pi_{n+1,n}(\mathsf{f})$. Therefore we have

$$\begin{aligned}
\mathsf{g}\circ\mathsf{f}((d_1,...,d_{n+1})\ &\in\ D^{n+1}\mapsto\gamma(d_1,...,d_{n-1},d_nd_{n+1})\in L)\\
&=\ \mathsf{g}(\mathsf{f}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto\\
&\quad\ \gamma(d_1,...,d_{n-1},d_nd_{n+1})\in L))\\
&=\ \mathsf{g}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto\\
&\quad\ \pi_{n+1,n}(\mathsf{f})(\gamma)(d_1,...,d_{n-1},d_nd_{n+1})\in M)\\
&=\ (d_1,...,d_{n+1})\in D^{n+1}\longmapsto\\
&\quad\ \pi_{n+1,n}(\mathsf{g})\circ\pi_{n+1,n}(\mathsf{f})(\gamma)(d_1,...,d_{n-1},d_nd_{n+1})\in N\\
&=\ (d_1,...,d_{n+1})\in D^{n+1}\longmapsto\\
&\quad\ \pi_{n+1,n}(\mathsf{g}\circ\mathsf{f})(\gamma)(d_1,...,d_{n-1},d_nd_{n+1})\in N,
\end{aligned}$$

which implies that the composition $\mathsf{g}\circ\mathsf{f}$ satisfies condition (3). Now we can prove by induction on n that $\hat{\pi}_{n+1,n}(\mathsf{g} \circ \mathsf{f})$ is a D^n-tangential from (L, x) to (N, z), so that it is a D^{n+1}-tangential from (L, x) to (N, z). ■

The following simple proposition may help the reader understand where our locution of D^n-tangential has originated.

Proposition 10 *Let M, N be microlinear spaces with $x \in M$ and $y \in N$. If f is a mapping from (M, x) to (N, y), then the assignment of $f \circ \gamma \in \mathbf{T}_y^n(N)$ to each $\gamma \in \mathbf{T}_x^n(M)$, denoted by $\mathbf{D}^n f$ and called the D^n-prolongation of f, is a D^n-tangential from (M, x) to (N, y). We have $\mathbf{D}^n f = \pi_{n+1,n}(\mathbf{D}^{n+1} f)$. If L is another microlinear space with $z \in L$ and g is a mapping from (N, y) to (L, z), then we have $\mathbf{D}^n(g \circ f) = (\mathbf{D}^n g) \circ (\mathbf{D}^n f)$.*

Proof. It is easy to see that $\mathbf{D}^n f$ abides by conditions (1) and (2). Trivially $\mathbf{D}^n f = \pi_{n+1,n}(\mathbf{D}^{n+1} f)$ and $\mathbf{D}^n(g \circ f) = (\mathbf{D}^n g) \circ (\mathbf{D}^n f)$. For any $\gamma \in \mathbf{T}_x^n(M)$, we have

$$
\begin{aligned}
\mathbf{D}^{n+1} f((d_1, ..., d_{n+1}) &\in D^{n+1} \mapsto \gamma(d_1, ..., d_{n-1}, d_n d_{n+1})) \\
&= (d_1, ..., d_{n+1}) \in D^{n+1} \mapsto \\
&\quad f(\gamma(d_1, ..., d_{n-1}, d_n d_{n+1})) \\
&= (d_1, ..., d_{n+1}) \in D^{n+1} \mapsto \\
&\quad \mathbf{D}^n f(\gamma)(d_1, ..., d_{n-1}, d_n d_{n+1}) \\
&= (d_1, ..., d_{n+1}) \in D^{n+1} \mapsto \\
&\quad \pi_{n+1,n}(\mathbf{D}^{n+1} f)(\gamma)(d_1, ..., d_{n-1}, d_n d_{n+1}),
\end{aligned}
$$

which implies that $\mathbf{D}^{n+1} f$ abides by condition 3 for any natural number n. By dint of $\mathbf{D}^n f = \pi_{n+1,n}(\mathbf{D}^{n+1} f)$ again, we can prove by induction on n that $\hat{\pi}_{n+1,n}(\mathbf{D}^{n+1} f)$ is a D^n-tangential from (M, x) to (N, y), so that $\mathbf{D}^{n+1} f$ is a D^{n+1}-tangential from (M, x) to (N, y). ■

With due regard to Theorem 7, we have the following far-flung generalization of Proposition 1.5 of Nishimura [13]:

Theorem 11 *Let f be a D^n-tangential from (M, x) to (N, y). Let $\mathfrak{D}$ and $\mathfrak{D}'$ be simplicial infinitesimal spaces of dimension less than or equal to n. Let χ be a monomial mapping from $\mathfrak{D}$ to $\mathfrak{D}'$. Let $\gamma \in \mathbf{T}_x^{\mathsf{D}'}(M)$. Then we have*

$$\mathsf{f}_{\mathsf{D}}(\gamma \circ \chi) = \mathsf{f}_{\mathsf{D}'}(\gamma) \circ \chi$$

Remark 12 *The reader should note that the above far-flung generalization of Proposition 1.5 of Nishimura [13] subsumes not only Proposition 1.5 of Nishimura [13] (subsuming (2) and (3)) but also Proposition 1.3 of Nishimura [13].*

Proof. In place of giving a general proof with formidable notation, we satisfy ourselves with an illustration. Here we deal only with the case that $\mathfrak{D} = D^3$, $\mathfrak{D}' = D(3)$ and

$$\chi(d_1, d_2, d_3) = (d_1 d_2, d_1 d_3, d_2 d_3)$$

for any $(d_1, d_2, d_3) \in D^3$, assuming that $n \geq 3$. We note first that the monomial mapping $\chi : D^3 \to D(3)$ is the composition of two monomial mappings
$\chi_1 : D^3 \to D(6; (1, 2), (3, 4), (5, 6))$
and
$\chi_2 : D(6; (1, 2), (3, 4), (5, 6)) \to D(3)$
with $\chi_1(d_1, d_2, d_3) = (d_1, d_1, d_2, d_2, d_3, d_3)$ for any $(d_1, d_2, d_3) \in D^3$ and $\chi_2(d_1, d_2, d_3, d_4, d_5, d_6) = (d_1 d_3, d_2 d_5, d_4 d_6)$ for any $(d_1, d_2, d_3, d_4, d_5, d_6) \in D(6; (1, 2), (3, 4), (5, 6))$, while the former monomial mapping $\chi_1 : D^3 \to D(6; (1, 2), (3, 4), (5, 6))$ is in turn the composition of three monomial mappings $\chi_1^1 : D^3 \to D(4; (1, 2))$, $\chi_1^2 : D(4; (1, 2)) \to D(5; (1, 2), (3, 4))$ and $\chi_1^3 : D(5; (1, 2), (3, 4)) \to D(6; (1, 2), (3, 4), (5, 6))$ with

$$\chi_1^1(d_1, d_2, d_3) = (d_1, d_1, d_2, d_3)$$

for any $(d_1, d_2, d_3) \in D^3$,

$$\chi_1^2(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, d_3, d_4)$$

for any $(d_1, d_2, d_3, d_4) \in D(4; (1, 2))$ and

$$\chi_1^3(d_1, d_2, d_3, d_4, d_5) = (d_1, d_2, d_3, d_4, d_5, d_5)$$

for any $(d_1, d_2, d_3, d_4, d_5) \in D(5; (1, 2), (3, 4))$. Therefore it suffices to prove that

$$\mathsf{f}(\gamma' \circ \chi_1^1) = \mathsf{f}_{D(4;(1,2))}(\gamma') \circ \chi_1^1 \tag{4}$$

for any $\gamma' \in \mathbf{T}_x^{D(4;(1,2))}(M)$, that

$$\mathsf{f}_{D(4;(1,2))}(\gamma'' \circ \chi_1^2) = \mathsf{f}_{D(5;(1,2),(3,4))}(\gamma'') \circ \chi_1^2 \tag{5}$$

for any $\gamma'' \in \mathbf{T}_x^{D(5;(1,2),(3,4))}(M)$, that

$$\mathsf{f}_{D(5;(1,2),(3,4))}(\gamma''' \circ \chi_1^3) = \mathsf{f}_{D(6;(1,2),(3,4),(5,6))}(\gamma''') \circ \chi_1^3 \tag{6}$$

for any $\gamma''' \in \mathbf{T}_x^{D(6;(1,2),(3,4),(5,6))}(M)$, and that

$$\mathsf{f}_{D(6;(1,2),(3,4),(5,6))}(\gamma'''' \circ \chi_2) = \mathsf{f}_{D(3)}(\gamma'''') \circ \chi_2 \tag{7}$$

for any $\gamma'''' \in \mathbf{T}_x^{D(3)}(M)$. Since $D(4;(1,2)) = D(2) \times D^2$, it is easy to see that

$$\mathsf{f}(\gamma' \circ \chi_1^1) = \mathsf{f}(\gamma_1' \underset{1}{+} \gamma_2') = \mathsf{f}(\gamma_1') + \mathsf{f}(\gamma_2')$$

where $\gamma_1' = \gamma' \circ (i_1 \times \mathrm{id}_{D^2})$ and $\gamma_2' = \gamma' \circ (i_2 \times \mathrm{id}_{D^2})$ with $i_1(d) = (d, 0) \in D(2)$ and $i_2(d) = (0, d) \in D(2)$ for any $d \in D$. On the other hand, we have

$$\mathsf{f}_{D(4;(1,2))}(\gamma') \circ \chi_1^1 = \mathbf{l}_{(\mathsf{f}(\gamma_1'),\mathsf{f}(\gamma_2'))} \circ \chi_1^1 = \mathsf{f}(\gamma_1') + \mathsf{f}(\gamma_2')$$

where $\mathbf{l}_{(\mathsf{f}(\gamma_1'),\mathsf{f}(\gamma_2'))} : D(2) \times D^2 \to N$ is the unique function with $\mathbf{l}_{(\mathsf{f}(\gamma_1'),\mathsf{f}(\gamma_2'))} \circ (i_1 \times \mathrm{id}_{D^2}) = \mathsf{f}(\gamma_1')$ and $\mathbf{l}_{(\mathsf{f}(\gamma_1'),\mathsf{f}(\gamma_2'))} \circ (i_2 \times \mathrm{id}_{D^2}) = \mathsf{f}(\gamma_2')$. Thus we have established (4). By the same token we can establish (5) and (6). In order to prove (7), it suffices to note that

$$\mathsf{f}_{D(6;(1,2),(3,4),(5,6))}(\gamma'''' \circ \chi_2) \circ i_{135} = \mathsf{f}_{D(3)}(\gamma'''') \circ \chi_2 \circ i_{135}$$

together with the seven similar identities obtained from the above by replacing i_{135} by seven other $i_{jkl} : D^3 \to D(6;(1,2),(3,4),(5,6))$ in the standard quasi-colimit representation of $D(6;(1,2),(3,4),(5,6))$, where $i_{jkl} : D^3 \to D(6;(1,2),(3,4),(5,6))$ $(1 \leq j < k < l \leq 6)$ is a mapping with $i_{jkl}(d_1, d_2, d_3) = (..., \underset{j}{d_1}, ..., \underset{k}{d_2}, ..., \underset{l}{d_3}, ...)$ (d_1, d_2 and d_3 are inserted at the j-th, k-th and l-th positions respectively, while the other components are fixed at 0). Its proof goes as follows. Since

$$\mathsf{f}_{D(6;(1,2),(3,4),(5,6))}(\gamma'''' \circ \chi_2) \circ i_{135} = \mathsf{f}(\gamma'''' \circ \chi_2 \circ i_{135}),$$

it suffices to show that

$$\mathsf{f}(\gamma'''' \circ \chi_2 \circ i_{135}) = \mathsf{f}_{D(3)}(\gamma'''') \circ \chi_2 \circ i_{135}.$$

However the last identity follows at once by simply observing that the mapping $\chi_2 \circ i_{135} : D^3 \to D(3)$ is the mapping

$$(d_1, d_2, d_3) \in D^3 \longmapsto (d_1 d_2, 0, 0) \in D(3),$$

which is the successive composition of the following three mappings:

$$\begin{aligned} (d_1, d_2, d_3) &\in D^3 \longmapsto (d_1, d_2) \in D^2 \\ (d_1, d_2) &\in D^2 \longmapsto d_1 d_2 \in D \\ d &\in D \longmapsto (d, 0, 0) \in D(3). \end{aligned}$$

■

Corollary 13 *Let* f *be a* D^n*-tangential from* (M, x) *to* (N, y)*. Let* $\mathfrak{D}$ *be a simplicially infinitesimal spaces of dimension less than or equal to* n*. Any nonstandard quasi-colimit representation of* $\mathfrak{D}$*, if any mapping into* $\mathfrak{D}$ *in the representation is monomial, induces the same mapping as* f_{D} *(induced by the standard quasi-colimit representation of* $\mathfrak{D}$*) by the method in the proof of Theorem 7.*

Proof. It suffices to note that

$$\mathsf{f}_{D^m}(\gamma \circ \chi) = \mathsf{f}_{\mathsf{D}}(\gamma) \circ \chi$$

for any mapping $\chi : D^m \to \mathfrak{D}$ in the given nonstandard quasi-colimit representation of $\mathfrak{D}$, which follows directly from the above theorem. ■

4 The Second Kind of Tangentials

Let n be a natural number. A D_n*-pseudotangential* from (M, x) to (N, y) is a mapping $\mathfrak{f} : \mathbf{T}_x^{D_n}(M) \to \mathbf{T}_y^{D_n}(N)$ such that for any $\gamma \in \mathbf{T}_x^{D_n}(M)$ and any $\alpha \in \mathbb{R}$, we have the following:

$$\mathfrak{f}(\alpha\gamma) = \alpha\mathfrak{f}(\gamma) \tag{8}$$

We denote by $\hat{\mathbb{J}}^n(M,x;N,y)$ the totality of D_n-pseudotangentials from (M,x) to (N,y).

Lemma 14 *Let $\mathfrak{f}$ be a D_{n+1}-pseudotangential from (M,x) to (N,y) and $\gamma \in \mathbf{T}_x^{D_n}(M)$. Then there exists a unique $\gamma' \in \mathbf{T}_y^{D_n}(N)$ such that for any $d \in D_n$, we have*

$$\mathfrak{f}(\mathbf{i}_d(\gamma)) = \mathbf{i}_d(\gamma')$$

Proof. For any $d' \in D_{n+1}$, we have

$$\begin{aligned} & \mathfrak{f}(\mathbf{i}_{d'd}(\gamma)) = \mathfrak{f}(d'(\mathbf{i}_d(\gamma))) \\ = \;& d'(\mathfrak{f}(\mathbf{i}_d(\gamma))) \end{aligned}$$

so that the lemma follows from Proposition 5. ■

Proposition 15 *The assignment $\gamma \in \mathbf{T}_x^{D_n}(M) \longmapsto \gamma' \in \mathbf{T}_y^{D_n}(N)$ in the above lemma is a D_n-pseudotangential from (M,x) to (N,y).*

Proof. For any $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} & \mathbf{i}_d(\alpha(\hat{\pi}_{n+1,n}(\mathfrak{f})(\gamma))) = \alpha(\mathbf{i}_d(\hat{\pi}_{n+1,n}(\mathfrak{f})(\gamma))) \\ = \;& \alpha(\mathfrak{f}(\mathbf{i}_d(\gamma))) = \mathfrak{f}(\alpha(\mathbf{i}_d(\gamma))) = \mathfrak{f}(\mathbf{i}_d(\alpha\gamma)), \end{aligned}$$

which establishes the desired proposition. ■

By the above proposition we have the canonical projection $\hat{\pi}_{n+1,n} : \hat{\mathbb{J}}^{n+1}(M,x;N,y) \to \hat{\mathbb{J}}^n(M,x;N,y)$, so that

$$\mathfrak{f}(\mathbf{i}_d(\gamma)) = \mathbf{i}_d(\hat{\pi}_{n+1,n}(\mathfrak{f})(\gamma))$$

for any $\mathfrak{f} \in \hat{\mathbb{J}}^{n+1}(M,x;N,y)$, any $d \in D_n$ and any $\gamma \in \mathbf{T}_x^{D_n}(M)$. For any natural numbers n, m with $m \leq n$, we define $\hat{\pi}_{n,m} : \hat{\mathbb{J}}^n(M,x;N,y) \to \hat{\mathbb{J}}^m(M,x;N,y)$ to be $\hat{\pi}_{m+1,m} \circ ... \circ \hat{\pi}_{n,n-1}$.

Proposition 16 *Let $\mathfrak{f}$ be a D_{n+1}-pseudotangential from (M,x) to (N,y) and $d \in D_n$. Then the following diagrams are commutative:*

$$\begin{tikzcd} \mathbf{T}_x^{D_{n+1}}(M) \arrow[r, "\mathfrak{f}"] & \mathbf{T}_y^{D_{n+1}}(N) \\ \mathbf{T}_x^{D_n}(M) \arrow[u, "\mathbf{i}_d"] \arrow[r, "\hat{\pi}_{n+1,n}(\mathfrak{f})"'] & \mathbf{T}_y^{D_n}(N) \arrow[u, "\mathbf{i}_d"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{T}_x^{D_{n+1}}(M) \arrow[r, "\mathfrak{f}"] \arrow[d, "\boldsymbol{\pi}_{n+1,n}"'] & \mathbf{T}_y^{D_{n+1}}(N) \arrow[d, "\boldsymbol{\pi}_{n+1,n}"] \\ \mathbf{T}_x^{D_n}(M) \arrow[r, "\hat{\pi}_{n+1,n}(\mathfrak{f})"'] & \mathbf{T}_y^{D_n}(N) \end{tikzcd}$$

Proof. The commutativity of the first diagram is exactly the definition of $\hat{\pi}_{n+1,n}(\mathfrak{f})$. For the sake of commutativity of the second diagram, it suffices to note by dint of Proposition 1 that for any $d \in D_n$, we have

$$\begin{aligned} & \mathbf{i}_d(\hat{\pi}_{n+1,n}(\mathfrak{f})(\boldsymbol{\pi}_{n+1,n}(\gamma))) = \mathfrak{f}(\mathbf{i}_d(\boldsymbol{\pi}_{n+1,n}(\gamma))) \\ = \ & \mathfrak{f}(d\gamma) = d(\mathfrak{f}(\gamma)) = \mathbf{i}_d(\boldsymbol{\pi}_{n+1,n}(\mathfrak{f}(\gamma))). \end{aligned}$$

■

Corollary 17 *Let $\mathfrak{f}$ be a D_{n+1}-pseudotangential from (M,x) to (N,y). For any $\gamma, \gamma' \in \mathbf{T}_x^{D_{n+1}}(M)$, if $\gamma|_{D_n} = \gamma'|_{D_n}$, then $\mathfrak{f}(\gamma)|_{D_n} = \mathfrak{f}(\gamma')|_{D_n}$.*

Proof. By the above proposition, we have

$$\begin{aligned} & \boldsymbol{\pi}_{n+1,n}(\mathfrak{f}(\gamma)) = \hat{\pi}_{n+1,n}(\mathfrak{f})(\boldsymbol{\pi}_{n+1,n}(\gamma)) \\ = \ & \hat{\pi}_{n+1,n}(\mathfrak{f})(\boldsymbol{\pi}_{n+1,n}(\gamma')) = \boldsymbol{\pi}_{n+1,n}(\mathfrak{f}(\gamma')), \end{aligned}$$

which establishes the desired proposition. ■

The notion of a D_n-*tangential* from (M,x) to (N,y) is defined inductively on n. The notion of a D_0-*tangential* from (M,x) to (N,y) and that of a D_1-*tangential* from (M,x) to (N,y) shall be identical with that of a D_0-pseudotangential from (M,x) to (N,y) and that of a D_1-pseudotangential from (M,x) to (N,y) respectively. Now we proceed by induction on n. A D_{n+1}-pseudotangential $\mathfrak{f} : \mathbf{T}_x^{D_{n+1}}(M) \to \mathbf{T}_y^{D_{n+1}}(N)$ from (M,x) to (N,y) is called a D_{n+1}-*tangential* from (M,x) to (N,y) if it acquiesces in the following two conditions:

1. $\hat{\pi}_{n+1,n}(\mathfrak{f})$ is a D_n-tangential from (M,x) to (N,y).

2. For any simple polynomial ρ of $d \in D_{n+1}$ with $l = \dim\rho$ and any $\gamma \in \mathbf{T}_x^{D_l}(M)$, we have

$$\mathfrak{f}(\gamma \circ \rho) = (\pi_{n+1,l}(\mathfrak{f})(\gamma)) \circ \rho \tag{9}$$

We denote by $\mathbb{J}^n(M,x;N,y)$ the totality of D_n-tangentials from (M,x) to (N,y). By the very definition of a D_n-tangential, the projection $\hat{\pi}_{n+1,n} : \hat{\mathbb{J}}^{n+1}(M,x;N,y) \to \hat{\mathbb{J}}^n(M,x;N,y)$ is naturally restricted to a mapping $\pi_{n+1,n} : \mathbb{J}^{n+1}(M,x;N,y) \to \mathbb{J}^n(M,x;N,y)$. Similarly for $\pi_{n,m} : \mathbb{J}^n(M,x;N,y) \to \mathbb{J}^m(M,x;N,y)$ with $m \leq n$. We note in passing that Propositions 9 and 10 together with their proofs can be modified easily for D_n-tangentials.

5 The Relationship between the Two Kinds of Tangentials without Coordinates

The principal objective in this section is to define a mapping φ_n : $\mathbf{J}^n(M,x;N,y) \to \mathbb{J}^n(M,x;N,y)$. Let us begin with

Lemma 18 *Let* f *be a* D^n*-pseudotangential from* (M,x) *to* (N,y) *and* $\gamma \in \mathbf{T}_x^{D_n}(M)$. *Then there exists a unique* $\gamma' \in \mathbf{T}_y^{D_n}(N)$ *such that*

$$\begin{aligned} \mathsf{f}((d_1,...,d_n) &\in D^n \longmapsto \gamma(d_1+...+d_n) \in M) \\ &= (d_1,...,d_n) \in D^n \longmapsto \gamma'(d_1+...+d_n) \in N \end{aligned}$$

Proof. By Proposition 4. ■

We will denote by $\hat{\varphi}_n(\mathsf{f})(\gamma)$ the unique γ' in the above lemma, thereby getting a function $\hat{\varphi}_n(\mathsf{f}) : \mathbf{T}_x^{D_n}(M) \to \mathbf{T}_y^{D_n}(N)$.

Proposition 19 *For any* $\mathsf{f} \in \hat{\mathbf{J}}^n(M,x;N,y)$, *we have* $\hat{\varphi}_n(\mathsf{f}) \in \hat{\mathbb{J}}^n(M,x;N,y)$.

Proof. It suffices to note that for any $\alpha \in \mathbb{R}$ and any $\gamma \in \mathbf{T}_x^{D_n}(M)$, we have

$$\begin{aligned}
&(d_1, ..., d_n) \in D^n \longmapsto \hat{\varphi}_n(\mathsf{f})(\alpha\gamma)(d_1 + ... + d_n) \in N \\
&\quad = \mathsf{f}((d_1, ..., d_n) \in D^n \longmapsto (\alpha\gamma)(d_1 + ... + d_n) \in M) \\
&\quad = \mathsf{f}((d_1, ..., d_n) \in D^n \longmapsto \gamma(\alpha d_1 + ... + \alpha d_n) \in M) \\
&\quad = \mathsf{f}(\alpha \underset{1}{\cdot} ...\alpha \underset{n}{\cdot} (d_1, ..., d_n) \in D^n \longmapsto \gamma(d_1 + ... + d_n) \in M)) \\
&\quad = \alpha \underset{1}{\cdot} ...\alpha \underset{n}{\cdot} (\mathsf{f}((d_1, ..., d_n) \in D^n \longmapsto \gamma(d_1 + ... + d_n) \in M)) \\
&\quad = \alpha \underset{1}{\cdot} ...\alpha \underset{n}{\cdot} ((d_1, ..., d_n) \in D^n \longmapsto \hat{\varphi}_n(\mathsf{f})(\gamma)(d_1 + ... + d_n) \in N) \\
&\quad = (d_1, ..., d_n) \in D^n \longmapsto \hat{\varphi}_n(\mathsf{f})(\gamma)(\alpha d_1 + ... + \alpha d_n) \in N \\
&\quad = (d_1, ..., d_n) \in D^n \longmapsto \alpha(\hat{\varphi}_n(\mathsf{f})(\gamma))(d_1 + ... + d_n) \in N,
\end{aligned}$$

which implies that

$$\hat{\varphi}_n(\mathsf{f})(\alpha\gamma) = \alpha(\hat{\varphi}_n(\mathsf{f})(\gamma))$$

■

Proposition 20 *The diagram*

$$\begin{array}{ccc}
\hat{\mathbf{J}}^{n+1}(M, x; N, y) & \xrightarrow{\qquad\hat{\varphi}_{n+1}\qquad} & \hat{\mathbb{J}}^{n+1}(M, x; N, y) \\
\hat{\pi}_{n+1,n} \downarrow & & \downarrow \hat{\pi}_{n+1,n} \\
\hat{\mathbf{J}}^n(M, x; N, y) & \xrightarrow[\qquad\hat{\varphi}_n\qquad]{} & \hat{\mathbb{J}}^n(M, x; N, y)
\end{array}$$

is commutative.

Proof. For any $d \in D_n$, we have

$$\begin{aligned}
&(d_1,..., d_n) \in D^n \longmapsto \mathbf{i}_d(\hat{\pi}_{n+1,n}(\hat{\varphi}_{n+1}(\mathsf{f}))(\gamma))(d_1 + ... + d_n) \in N \\
&\quad = \mathbf{d}_{n+1}((d_1, ..., d_{n+1}) \in D^{n+1} \longmapsto \\
&\qquad \mathbf{i}_d(\hat{\pi}_{n+1,n}(\hat{\varphi}_{n+1}(\mathsf{f}))(\gamma))(d_1 + ... + d_{n+1}) \in N)
\end{aligned}$$

$$
\begin{aligned}
&=\mathbf{d}_{n+1}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto\\
&\quad\hat{\varphi}_{n+1}(\mathsf{f})(\mathbf{i}_d(\gamma))(d_1+...+d_{n+1})\in N)\\
&=\mathbf{d}_{n+1}(\mathsf{f}((d_1,...,d_{n+1})\in D^{n+1}\longmapsto\mathbf{i}_d(\gamma)(d_1+...+d_{n+1})\in M))\\
&=\hat{\pi}_{n+1,n}(\mathsf{f})((d_1,...,d_n)\in D^n\longmapsto\mathbf{i}_d(\gamma)(d_1+...+d_n)\in M)\\
&=\hat{\pi}_{n+1,n}(\mathsf{f})((d_1,...,d_n)\in D^n\longmapsto(d\gamma)(d_1+...+d_n)\in M)\\
&=(d_1,...,d_n)\in D^n\longmapsto\hat{\varphi}_n(\hat{\pi}_{n+1,n}(\mathsf{f}))(d\gamma)(d_1+...+d_n)\in N\\
&=(d_1,...,d_n)\in D^n\longmapsto d(\hat{\varphi}_n(\hat{\pi}_{n+1,n}(\mathsf{f}))(\gamma))(d_1+...+d_n)\in N\\
&=(d_1,...,d_n)\in D^n\longmapsto\mathbf{i}_d(\hat{\varphi}_n(\hat{\pi}_{n+1,n}(\mathsf{f}))(\gamma))(d_1+...+d_n)\in N,
\end{aligned}
$$

which implies by Proposition 2 that

$$\hat{\pi}_{n+1,n}(\hat{\varphi}_{n+1}(\mathsf{f}))=\hat{\varphi}_n(\hat{\pi}_{n+1,n}(\mathsf{f}))$$

■

Proposition 21 *Let $\mathfrak{D}$ be a simplicial infinitesimal space of dimension n and degree m. Let f be a D^n-pseudotangential from (M,x) to (N,y) and $\gamma\in\mathbf{T}_x^{D_n}(M)$. Then we have*

$$
\begin{aligned}
\mathsf{f}_{\mathsf{D}}((d_1,...,d_m) &\in \mathfrak{D}\longmapsto\gamma(d_1+...+d_m)\in M)\\
&= (d_1,...,d_m)\in\mathfrak{D}\longmapsto\hat{\varphi}_n(\mathsf{f})(d_1+...+d_m)\in N
\end{aligned}
$$

Proof. It suffices to note that for any $i:D^k\to\mathfrak{D}$ in the standard quasi-colimit representation of $\mathfrak{D}$, we have

$$
\begin{aligned}
\mathsf{f}_{\mathsf{D}}((d_1,...,d_m) &\in \mathfrak{D}\longmapsto\gamma(d_1+...+d_m)\in M)\circ i\\
&= \mathsf{f}_{D^k}((d_1,...,d_k)\in D^k\longmapsto\gamma(d_1+...+d_k)\in M)\\
&= (d_1,...,d_k)\in D^k\longmapsto\hat{\varphi}_k(\mathsf{f}_{D^k})(d_1+...+d_k)\in N\\
&\quad\text{[by Lemma 18]}\\
&= ((d_1,...,d_m)\in\mathfrak{D}\longmapsto\hat{\varphi}_n(\mathsf{f})(d_1+...+d_m)\in N)\circ i\\
&\quad\text{[by Proposition 20]}
\end{aligned}
$$

■

Theorem 22 *For any* $\mathsf{f} \in \mathbf{J}^n(M, x; N, y)$, *we have* $\hat{\varphi}_n(\mathsf{f}) \in \mathbb{J}^n(M, x; N, y)$.

Proof. In view of Proposition 19, it suffices to show that $\hat{\varphi}_n(\mathsf{f})$ satisfies the condition (9). Here we deal only with the case that $n = 3$ and the simple polynomial ρ at issue is $d \in D_3 \longmapsto d^2 \in D$, leaving the general treatment safely to the reader. Since

$$(d_1 + d_2 + d_3)^2 = 2(d_1 d_2 + d_1 d_3 + d_2 d_3)$$

for any $(d_1, d_2, d_3) \in D^3$, we have the following commutative diagram:

$$\begin{array}{ccc} D^3 & \xrightarrow{\chi} & D(6) \\ +_{D^3} \downarrow & & \downarrow +_{D(6)} \\ D_3 & \underset{\rho}{\rightarrow} & D \end{array} \tag{10}$$

where $+_{D^3} : D^3 \to D_3$ and $+_{D(6)} : D(6) \to D$ stand for addition of components. Then we have

$$\begin{aligned} & \hat{\varphi}_3(\mathsf{f})(\gamma \circ \rho) \circ +_{D^3} \\ = \; & \mathsf{f}(\gamma \circ \rho \circ +_{D^3}) \quad [\text{by the definition of } \hat{\varphi}_3] \\ = \; & \mathsf{f}(\gamma \circ +_{D(6)} \circ \chi) \\ = \; & \mathsf{f}_{D(6)}(\gamma \circ +_{D(6)}) \circ \chi \quad [\text{by Theorem 11}] \\ = \; & \hat{\varphi}_1(\pi_{3,1}(\mathsf{f}))(\gamma) \circ +_{D(6)} \circ \chi \quad [\text{by Proposition 21}] \\ = \; & \pi_{3,1}(\hat{\varphi}_3(\mathsf{f})) \circ +_{D(6)} \circ \chi \quad [\text{by Proposition 20}] \\ = \; & \pi_{3,1}(\hat{\varphi}_3(\mathsf{f})) \circ \rho \circ +_{D^3} \end{aligned}$$

which implies by Proposition 2 that $\hat{\varphi}_3(\mathsf{f})(\gamma \circ \rho) = \pi_{3,1}(\hat{\varphi}_3(\mathsf{f})) \circ \rho$. ■

Thus the mapping $\hat{\varphi}_n : \widehat{\mathbf{J}}^n(M, x; N, y) \to \widehat{\mathbb{J}}^n(M, x; N, y)$ is naturally restricted to a mapping $\varphi_n : \mathbf{J}^n(M, x; N, y) \to \mathbb{J}^n(M, x; N, y)$.

6 The Relationship between the Two Kinds of Tangentials with Coordinates

The principal objective in this section is to show that the mapping $\varphi_n : \mathbf{J}^n(M,x;N,y) \to \mathbb{J}^n(M,x;N,y)$ is bijective for any natural number n in case that coordinates are available. We will assume that M and N are formal manifolds of dimensions p and q respectively. Since our considerations to follow are always infinitesimal, this means that we can assume without any loss of generality that $M = \mathbb{R}^p$ and $N = \mathbb{R}^q$. We will let i with or without subscripts range over natural numbers between 1 and p (including endpoints), while we will let j with or without subscripts range over natural numbers between 1 and q (including endpoints). Let $x = (x^i)$ and $y = (y^j)$. For any natural number n, we denote by $\mathcal{J}^n(p,q)$ the totality of $(\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 ... i_n}^j)$'s of $pq + p^2 q + \ ... \ + p^n q$ elements of $\mathbb{R}$ such that $\alpha_{i_1 ... i_k}^j$'s are symmetric with respect to subscripts, i.e., $\alpha_{i_{\sigma(1)} ... i_{\sigma(k)}}^j = \alpha_{i_1 ... i_k}^j$ for any $\sigma \in \mathfrak{S}_k$ $(2 \leq k \leq n)$. Therefore the number of independent components in $(\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 ... i_n}^j) \in \mathcal{J}^n(p,q)$ is $q \Sigma_{k=0}^n \binom{p+k-1}{p-1} - q = q\binom{p+n}{n} - q$. The canonical projection

$$\begin{aligned} (\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 ... i_n}^j, \alpha_{i_1 ... i_{n+1}}^j) \ &\in \ \mathcal{J}^{n+1}(p,q) \longmapsto \\ (\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 ... i_n}^j) \ &\in \ \mathcal{J}^n(p,q) \end{aligned}$$

is denoted by $\pi_{n+1,n}$. We will use Einstein's summation convention to suppress Σ.

In our previous paper [13] we have defined a natural mapping $\overline{\theta}_n : \mathcal{J}^n(p,q) \to \mathbf{J}^n(M,x;N,y)$, which was shown to be bijective (cf. Theorem 3.7 of [13]). We denote the composition $\overline{\theta}_n \circ \varphi_n$ by $\overline{\omega}_n : \mathcal{J}^n(p,q) \to \mathbb{J}^n(M,x;N,y)$. It is of the form

$$\begin{aligned} &\overline{\omega}_n((\alpha_i^j, \alpha_{i_1 i_2}^j, ..., \alpha_{i_1 i_2 ... i_n}^j))(d \in D_n \longmapsto (x^i) + \Sigma_{r=1}^n \frac{d^r}{r!}(a_{\mathbf{r}}^i) \in \mathbb{R}^p) \\ =&d \in D_n \longmapsto \\ &\Sigma_{r=1}^n \frac{d^r}{r!} \Sigma_{1 \leq r_1 \leq ... \leq r_k \leq n} (\frac{r!}{r_1! ... r_k!} \alpha_{i_1 ... i_k}^j a_{\mathbf{r}_1}^{i_1} ... a_{\mathbf{r}_k}^{i_k}) \in \mathbb{R}^q, \end{aligned}$$

where the last Σ is taken over all partitions of the positive integer r into positive integers $r_1, ..., r_k$ (so that $r = r_1 + ... + r_k$) with $r_1 \leq ... \leq r_k$.

Now we are going to define mappings $\underline{\omega}_n : \mathbb{J}^n(M, x; N, y) \to \mathcal{J}^n(p, q)$ by induction on n such that the diagram

$$\begin{array}{ccc}
\mathbb{J}^{n+1}(M, x; N, y) & \xrightarrow{\underline{\omega}_{n+1}} & \mathcal{J}^{n+1}(p, q) \\
\pi_{n+1,n} \downarrow & & \downarrow \pi_{n+1,n} \\
\mathbb{J}^n(M, x; N, y) & \xrightarrow[\underline{\omega}_n]{} & \mathcal{J}^n(p, q)
\end{array}$$

is commutative. The mapping $\underline{\omega}_0 : \mathbb{J}^0(M, x; N, y) \to \mathcal{J}^0(p, q)$ shall be the trivial one. Assuming that $\underline{\omega}_n : \mathbb{J}^n(M, x; N, y) \to \mathcal{J}^n(p, q)$ is defined, we are going to define $\underline{\omega}_{n+1} : \mathbb{J}^{n+1}(M, x; N, y) \to \mathcal{J}^{n+1}(p, q)$, for which it suffices, by the required commutativity of the above diagram, only to give $\alpha^j_{i_1...i_{n+1}}$'s to each $\mathfrak{f} \in \mathbb{J}^{n+1}(M, x; N, y)$. Let $\mathbf{e}_i$ denote $(0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}^p$, where 1 is inserted at the i-th position, while the other $p - 1$ elements are fixed zero. By the general Kock axiom (cf. Lavendhomme [5, Subsection 2.1.3]), $\mathfrak{f}(d \in D_{n+1} \longmapsto (x^i) + d(d_1\mathbf{e}_{i_1} + ... + d_{n+1}\mathbf{e}_{i_{n+1}}) \in M)$ should be a polynomial of $d, d_1, ..., d_{n+1}$, in which the coefficient of $d^{n+1}d_1...d_{n+1}$ should be of the form $m_1!...m_p!(\alpha^1_{i_1...i_{n+1}}, ..., \alpha^q_{i_1...i_{n+1}}) \in \mathbb{R}^q$, where m_i is the number of i_k's with $i = i_k$. We choose these $\alpha^j_{i_1...i_{n+1}}$'s as our desired $\alpha^j_{i_1...i_{n+1}}$'s. Obviously we have

Proposition 23 *For any $\mathfrak{f} \in \mathbb{J}^n(M, x; N, y)$, we have $\underline{\omega}_n(\mathfrak{f}) \in \mathcal{J}^n(p, q)$.*

It is easy to see that

Proposition 24 *The composition $\underline{\omega}_n \circ \overline{\omega}_n$ is the identity mapping of $\mathcal{J}^n(p, q)$.*

Proof. Using the commutative diagram

$$\begin{array}{ccccc}
\mathcal{J}^{n+1}(p, q) & \xrightarrow{\overline{\omega}_{n+1}} & \mathbb{J}^{n+1}(M, x; N, y) & \xrightarrow{\underline{\omega}_{n+1}} & \mathcal{J}^{n+1}(p, q) \\
\pi_{n+1,n} \downarrow & & \downarrow \pi_{n+1,n} & & \downarrow \pi_{n+1,n} \\
\mathcal{J}^n(p, q) & \xrightarrow[\overline{\omega}_n]{} & \mathbb{J}^n(M, x; N, y) & \xrightarrow[\underline{\omega}_n]{} & \mathcal{J}^n(p, q)
\end{array}$$

we can easily establish the desired result by induction on n. ■

This means in particular that the mapping $\varphi_n : \mathbf{J}^n(M,x;N,y) \rightarrow \mathbb{J}^n(M,x;N,y)$ is injective. To show its surjectivity, simple dimension counting will suffice by dint of the above proposition. Let's note the following plain proposition, which may belong to the folklore.

Proposition 25 *Any* $\mathsf{f} \in \hat{\mathbb{J}}^n(M,x;N,y)$ *is of the form*

$$\begin{aligned}\mathsf{f}(d &\in D_n \longmapsto (x^i) + \Sigma_{r=1}^n \frac{d^r}{r!}(a^i_{\mathbf{r}}) \in \mathbb{R}^p) \\ &= d \in D_n \longmapsto \Sigma_{r=1}^n \frac{d^r}{r!}\Sigma_{1\leq r_1\leq \ldots \leq r_k \leq n}(\Omega^r_{r_1,\ldots,r_k}((a^{i_1}_{\mathbf{r}_1}),\ldots,(a^{i_k}_{\mathbf{r}_k}))) \\ &\in \mathbb{R}^q,\end{aligned}$$

where $\Omega^r_{r_1,\ldots,r_k} : (\mathbb{R}^p)^k \rightarrow \mathbb{R}^q$ *is a symmetric* k*-linear mapping, and the last* Σ *is taken over all partitions of the positive integer* r *into positive integers* $r_1,\ldots,r_k$ *(so that* $r = r_1 + \ldots + r_k$*) with* $r_1 \leq \ldots \leq r_k$

Proof. By the same token as in the proof of Proposition 11 of Lavendhomme [5, Section 1.2]. ■

Proposition 26 *The dimension of* $\mathbb{J}^n(M,x;N,y)$ *is less than or equal to that* $\mathbf{J}^n(M,x;N,y)$*.*

Proof. The dimension of $\hat{\mathbb{J}}^n(M,x;N,y)$, which can be calculated easily by the above proposition, is larger than that of $\mathbf{J}^n(M,x;N,y)$. By way of example, the dimension of $\hat{\mathbb{J}}^2(M,x;N,y)$ is $q\binom{p+2}{2} + (p-1)q$, while that of $\mathbf{J}^2(M,x;N,y)$ is $q\binom{p+2}{2} - q$. In order to reduce the dimension of $\hat{\mathbb{J}}^n(M,x;N,y)$ to that of $\mathbb{J}^n(M,x;N,y)$, which is expected to coincide with that of $\mathbf{J}^n(M,x;N,y)$, we have to take the condition (9) into consideration. In case of $n = 2$, it suffices to consider the condition for $d \in D_2 \longmapsto d^2 \in D$. In case of $n = 3$, it suffices to consider the condition for $d \in D_3 \longmapsto d^2 \in D$, $d \in D_3 \longmapsto d^3 \in D$ and $d \in D_3 \longmapsto d + d^2 \in D_3$. The general case is safely left to the reader. ■

Theorem 27 *The mapping* $\varphi_n : \mathbf{J}^n(M,x;N,y) \rightarrow \mathbb{J}^n(M,x;N,y)$ *is bijective.*

Proof. This follows simply from Propositions 24 and 26. ■

References

[1] Goerss, P. G. and Jardine, J. F.: Simplicial Homotopy Theory, Progress in Mathematics, **174**, Birkhäuser, Basel, 1999.

[2] Kock, A.: A simple axiomatics for differentiation, Math. Scand., **40** (1977), 183-193.

[3] Kock, A.: Taylor series calculus for ring objects of line type, Journal of Pure and Applied Algebra, **12** (1978), 271-293.

[4] Kock, A.: Synthetic Differential Geometry, London Mathematical Society Lecture Note Series, **51**, Cambridge University Press, Cambridge, 1981.

[5] Lavendhomme, R.: Basic Concepts of Synthetic Differential Geometry, Kluwer, Dordrecht, 1996.

[6] Libermann, P.: Sur les prolongements de fibrés principaux et des groupoïdes différentiables banachiques, Analyse Globale, 7-108, Presses Univ. Montréal, Montréal, Quebec, 1971.

[7] Moerdijk, I. and Reyes, G. E.: Models for Smooth Infinitesimal Analysis, Springer-Verlag, New York, 1991.

[8] Navarro, J. A. and Sancho de Salas, J. B.: C^∞-Differentiable Spaces, Lecture Notes in Mathematics, **1824**, Springer-Verlag, Berlin and Heidelberg, 2003.

[9] Nishimura, H.: Theory of microcubes, International Journal of Theoretical Physics, **36** (1997), 1099-1131.

[10] Nishimura, H.: Synthetic differential geometry of jet bundles, Bull. Belg. Math. Soc. Simon Stevin, **8** (2001), 639-650.

[11] Nishimura, H.: Corrigenda to "Synthetic differential geometry of jet bundles", Bull. Belg. Math. Soc. Simon Stevin, **9** (2002), 473.

[12] Nishimura, H.: Holonomicity in sythetic differential geometry of jet bundles, Beiträge zur Algebra und Geometrie, **44** (2003), 471-481.

[13] Nishimura, H.: Higher-order preconnections in synthetic differential geometry of jet bundles, Beiträge zur Algebra und Geometrie, **45** (2004), 677-696.

[14] Saunders, D. J.: The Geometry of Jet Bundles, London Mathematical Society Lecture Note Series, **142**, Cambridge University Press, Cambridge, 1989.

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