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THE SERRE DUALITY THEOREM FOR A NON-COMPACT WEIGHTED CR MANIFOLD

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Abstract. It is proved that the Hodge decomposition and Serre duality hold on a non-compact weighted CR manifold with negligible boundary. A complete CR manifold has negligible boundary. Some examples of complete CR manifolds are presented.

1. Introduction

Let M be a strongly pseudo-convex CR manifold, an *s.p.c. CR manifold* for short, without boundary. A *weighted CR manifold* is an s.p.c. CR manifold endowed with a measure μ, which has a smooth positive density η with respect to the volume form of the CR structure. Then the space (M, μ) has a natural weighted Kohn Laplacian $\Box_\mu$, which we call the *Witten-Kohn Laplacian*.

In this article, we are interested in Serre duality and Hodge decomposition on a non-compact weighted CR manifold. The Serre duality of a compact s.p.c. CR manifold was proved by Tanaka [15] for the case of a trivial line bundle, and recently, the first and the third named authors generalized it to any holomorphic vector bundle E [8]. On the other hand, Kohn's Hodge decomposition for a compact s.p.c. CR manifold was extended to a general s.p.c. CR manifold with *negligible boundary* (Definition 2.4) when E is a trivial line bundle by the second author [13].

The aim of the present article is to extend these results to an arbitrary holomorphic vector bundle E over a general weighted CR manifold with negligible boundary, and to relate them to each other. Namely, by denoting $\mathbb{H}^{p,q}(E)$ the space of E-valued L^2-harmonic forms of (p,q)-type, we will show

Main Theorem. *Let M be a $(2n-1)$-dimensional weighted CR manifold with negligible boundary, and let E be a holomorphic vector bundle over M. Then the L^2-Hodge decomposition*

$$L^2\left(\Omega^{\bullet,q}(E)\right) = \mathbb{H}^{\bullet,q} \oplus \overline{\operatorname{range}\left(\overline{\partial}^{q-1}\right)}^{L^2} \oplus \overline{\operatorname{range}\left(\delta_\mu^{q+1}\right)}^{L^2}$$

holds for $0 < q < n-1$, and the Serre duality

$$\sharp_\mu : \mathbb{H}^{p,q}(E) \cong \mathbb{H}^{n-p,n-(q+1)}(E^*)$$

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holds for every $0 \leq p, q \leq n-1$, *where* E^* *is the dual bundle of* E *and* $\sharp_\mu$ *is the complex-conjugate weighted Hodge star operator. In addition, it follows that*

$$\ker\left(\overline{\partial}^q\right) \Big/ \overline{\mathrm{range}\left(\overline{\partial}^{q-1}\right)}^{L^2} \cong \mathbb{H}^{p,q}(E) \overset{\sharp_\mu}{\cong} \mathbb{H}^{n-p,n-(q+1)}(E^*), \ \textit{for } 0 < q < n-1.$$

We say that M *is complete* and M *is Riemannian complete* if it is complete with respect to the Carnot-Carathéodory distance d_{CC} and Riemannian distance d_g associated to the CR structure, respectively (see Section 2). Then we have

Theorem (Proposition 4.1).

(i) *If* M *is Riemannian complete, then it is complete*

(ii) *If* M *is complete, then* M *has negligible boundary.*

Therefore, the Main Theorem is applicable, for example, to the Heisenberg group, Sasakian space forms, spherical orbits, and unbranched covering over a compact s.p.c. CR manifold with any weight. These are very important s.p.c. CR manifolds, but they have been excluded from the literature because of their non-compactness (see Section 4). Two main points in the proof are: identification of the weak solution of the Laplace equation $\Box_\mu \alpha = 0$ with the harmonic form (Corollary 2.7); explicit expressions for $\overline{\partial}$ and δ_μ in terms of $\overline{\partial}_*$ and δ_{μ_*} via $\sharp_\mu$ (Lemma 3.3).

We organize the article in the following manner: In Section 2, we recall some necessary notions which we will use in the article. Some new results are given, including the self-adjointness of the E-valued Witten-Kohn Laplacian. In Section 3, we will prove the Serre duality, and finally, in Section 4, we present the examples.

2. Strongly pseudo-convex CR manifolds

This section contains preliminary results. We recall some definitions related to a strongly pseudo-convex CR manifold M, focusing on the construction of the space $\Omega^{p,q}(E)$ of E-valued (p,q)-forms, its holomorphic structure $\overline{\partial}^q$, and the Witten-Kohn Laplacian $\Box_\mu$. For a thorough discussion on a geometrical analysis of an s.p.c. CR manifold, we refer the reader to [5] and [15]. We also establish the essential self-adjointness of $\Box_\mu$ and characterize the harmonic forms as the solutions of the Laplace equation with respect to $\Box_\mu$. These results are important steps when we extend our knowledge of a compact manifold to a non-compact one.

A $(2n-1)$-dimensional *strongly pseudo-convex CR manifold* M, we call it simply an *s.p.c. CR manifold*, is an oriented smooth manifold which carries a structure (P, J, θ), that is: $P = \ker(\theta) \subset TM$ is an $(n-1)$-dimensional real subbundle with an almost complex structure $J : P \to P$ satisfying:

$$[X,Y] - [JX,JY] - J[JX,Y] - J[X,JY] \in \Gamma(TM/P), \ \text{for } X, Y \in \Gamma(P),$$

and a contact form $\theta \in \Gamma((TM/P)^*)$ whose *Levi-form* $L(X,Y) = -d\theta(JX,Y)$, for $X, Y \in P$, is positive definite.

Consider the complexification of J and its eigenspace $S = \{X - \sqrt{-1}JX : X \in P\} \subset \mathbb{C}TM$. Then $S \cap \overline{S} = (0)$ and $[\Gamma(S), \Gamma(S)] \subset \Gamma(S)$, where $\overline{S}$ is the complex conjugation of S. With the assumption of the strong convexity of M, there exist the following implications:

- a Riemannian metric $g = -d\theta + \theta \otimes \theta$;
- a volume form $dv = (n-1)!\, \theta \wedge (d\theta)^{n-1}$;

- a distance d_{CC} on M.

Indeed, since the Levi form L is positive definite, it follows for $0 \neq X \in P_x$ that

$$0 \neq 2L(X,X) = -\left(JX(\theta(X)) - X(\theta(JX)) - \theta([JX,X])\right) = \theta([JX,X]).$$

This shows that $[JX,X] \notin P_x$, and hence, P satisfies the Hörmander condition [7]. Due to the Chow theorem [4], P implies a non-degenerate distance d_{CC} on M defined as

$$\mathrm{d}_{CC}(x,y) := \sup\{u(x) - u(y) : u \in C^\infty(M),\ \|\pi\nabla u\|_{L^\infty} \leq 1\}, \tag{2.1}$$

where $\pi : TM \to P$ is the projection with respect to g. We say M *is complete* if the distance space (M, d_{CC}) is complete.

Definition 2.1. A complex vector bundle E over M is called *holomorphic* if it admits a linear differential operator $\overline{\partial}_E : \Gamma(E) \to \Gamma(E \otimes \overline{S}^*)$ satisfying:

(i) $\overline{\partial}_{\overline{X}}(fu) = f\,\overline{\partial}_{\overline{X}}u + (\overline{X}f)\,u$;

(ii) $\overline{\partial}_{\overline{X}}(\overline{\partial}_{\overline{Y}}u) - \overline{\partial}_{\overline{Y}}(\overline{\partial}_{\overline{X}}u) - \overline{\partial}_{\overline{[X,Y]}}u = 0$;

here $f \in C^\infty(M)$, $u \in \Gamma(E)$, and $X, Y \in \Gamma(S)$, where $\overline{\partial}_{\overline{X}}u := \overline{\partial}_E u\left(\overline{X}\right)$.

Example 2.2 (E.g. [15])**.** Let M be a boundary of a strongly pseudoconvex complex manifold and E be a holomorphic vector bundle on the neighbourhood of M. Then $E|_M$ is holomorphic in the above sense.

Hereafter, E stands for a holomorphic vector bundle over M. Consider the vector bundle $\hat{T}M = \mathbb{C}TM/\overline{S}$, which is holomorphic (e.g. p. 15 in [15]) together with the operator $\overline{\partial} = \overline{\partial}_M$:

$$\overline{\partial}_{\overline{X}}u = \varpi([\overline{X}, Z]),$$

for $u \in \Gamma(\hat{T}M)$ with $Z \in \Gamma(\mathbb{C}TM)$ such that $\varpi(Z) = u$ and $X \in \Gamma(S)$. Here $\varpi : \mathbb{C}TM \to \hat{T}M$ is the canonical projection. The distinguished vector bundle $E \otimes \bigwedge^p(\hat{T}M)^*$ with $0 \leq p \leq n-1$ carries a holomorphic structure:

$$\overline{\partial}_E \otimes id_{\wedge^p} + id_E \otimes \overline{\partial}_{\wedge^p}, \tag{2.2}$$

where id is the identity operator on the indicated space, and $\bigwedge^p = \bigwedge^p \hat{T}M$. Hereafter we assume additionally that $(E, \overline{\partial}_E)$ is furnished with a smooth Hermitian fiber metric $\langle\cdot,\cdot\rangle_E$. The bundle which we will study is

$$\Omega^{p,q}(E) = \Omega^{p,q}(M;E) = \Gamma(M; E \otimes \bigwedge^p \hat{T}M^* \otimes \bigwedge^q \overline{S}^*),$$

with the holomorphic structure $\overline{\partial}^q : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ defined as

$$\begin{aligned}(\overline{\partial}^q\alpha)(\overline{X}_1, \cdots, \overline{X}_{q+1}) := &\sum (-1)^i \overline{\partial}_{\overline{X}_i}\left(\alpha\left(\overline{X}_1, \cdots, \hat{\overline{X}}_i, \cdots, \overline{X}_{q+1}\right)\right) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha\left([\overline{X}_i, \overline{X}_j], \overline{X}_1, \cdots, \hat{\overline{X}}_i, \cdots, \hat{\overline{X}}_j, \cdots, \overline{X}_{q+1}\right),\end{aligned}$$

where $\overline{\partial}$ is the holomorphic structure of $E \otimes \bigwedge^p \hat{T}M^*$, $\alpha \in \Omega^{p,q}(E)$, and $X_1, \cdots, X_{q+1}$ belong to $\Gamma(S)$. If E is the trivial line bundle, we simply denote $\Omega^{p,q}(M) = \Omega^{p,q}(M;\mathbb{C})$. Set

$$\Omega^{\bullet,q}(E) = \bigoplus_q \Omega^{p,q}(E),\ \Omega(E) := \bigoplus_q \Omega^{\bullet,q}(E);$$

$$\Omega_0^{p,q}(E) = \{\alpha \in \Omega^{p,q}(E) \mid \alpha \text{ has compact support}\}.$$

Let η be the *weight*, which is a positive smooth function on M, and consider the measure $d\mu = \eta dv$. The associated inner product (α, β) of $\alpha, \beta \in \Omega_0^{p,q}(E)$ is

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle(x)\, d\mu(x),$$

where $\langle \alpha, \beta\rangle(x)$ is the pointwise inner product of α and β at $x \in M$. Denote by $\|\alpha\|_2$ the norm $\sqrt{(\alpha, \alpha)}$, and by $L^2(\Omega^{p,q}(E)) = L^2(\Omega^{p,q}(E), \mu)$ the set of square integrable E-valued measurable (p, q)-forms, which coincides with the completion of $\Omega_0^{p,q}(E)$ with respect to $\|\cdot\|_2$.

Let $\delta_\mu^q : \Omega^{\bullet,q+1}(E) \to \Omega^{\bullet,q}(E)$ be the formal adjoint of $\overline{\partial}^q$ in $L^2(\Omega(E))$. The *Witten-Kohn Laplacian* $\Box_\mu^q : \Omega^{\bullet,q}(E) \to \Omega^{\bullet,q}(E)$ is defined by

$$\Box_\mu^q := \overline{\partial}^{q-1}\delta_\mu^{q-1} + \delta_\mu^q \overline{\partial}^q.$$

In abbreviation, we remove the super index q when the operator is acting on the space of mixed degree forms. The operator $\Box_\mu^q$ is called *subelliptic* (e.g. [5], [15]) if there are positive numbers σ and C_σ such that

$$\|\alpha\|_{(\sigma)}^2 \le C_\sigma \left((\Box_\mu^q \alpha, \alpha) + \|\alpha\|_2^2\right), \text{ for every } \alpha \in \Omega_0^{\bullet,q}(E),$$

where $\|\cdot\|_{(\sigma)}$ is the Sobolev norm of order σ.

Proposition 2.3. *The Witten-Kohn Laplacian $\Box_\mu^q$ is subelliptic for $0 < q < n-1$.*

Proof. By Lemma 3.3, the Kohn Laplacian $\Box = \Box_{dv}$, which is the Witten-Kohn Laplacian with $\eta \equiv 1$, has the form:

$$\Box = \sharp^* \overline{\partial}_* \delta_* \sharp + \sharp^* \delta_* \overline{\partial}_* \sharp, \tag{2.3}$$

where $\sharp = \sharp_{dv}$ (see Definition 3.1), $\overline{\partial}_*$ is the holomorphic structure of $\Omega(E^*)$ and δ_* is its adjoint operator in $L^2(\Omega(E^*), dv)$. Again by Lemma 3.3 and (2.3),

$$\begin{aligned}\Box_\mu \alpha &= \sharp^* \overline{\partial}_* \sharp \sharp^* \delta_* \sharp \alpha + \eta^{-1} \sharp^* (\overline{\partial}_* \eta \wedge \sharp\sharp^* \delta_* \sharp \alpha) \\ &\quad + \sharp^* \delta_* \sharp\sharp^* \overline{\partial}_* \sharp \alpha + \sharp^* \delta_* \sharp (\eta^{-1} \sharp^* (\overline{\partial}_* \eta \wedge \sharp \alpha)) \\ &= \Box \alpha + \eta^{-1} \sharp^* (\overline{\partial}_* \eta \wedge \delta_* \sharp \alpha) + \sharp^* \delta_* (\eta^{-1} (\overline{\partial}_* \eta \wedge \sharp \alpha)),\end{aligned}$$

for $\alpha \in \Omega(E)$. Therefore, $\Box_\mu$ and $\Box$ have the same principal symbols. Since $\Box$ is subelliptic [10], we can draw this conclusion. $\Box$

We consider the following domains:

$$D(\overline{\partial}^q) = \{\alpha \in \Omega^{\bullet,q} : \alpha \text{ and } \overline{\partial}^q \alpha \text{ are square integrable}\};$$

$$D(\delta_\mu^q) = \{\alpha \in \Omega^{\bullet,q+1} : \alpha \text{ and } \delta_\mu^q \alpha \text{ are square integrable}\};$$

$$D(\Box_\mu^q) = \{\alpha \in D(\overline{\partial}^q) \cap D(\delta_\mu^{q+1}) : \overline{\partial}\alpha \in D(\delta_\mu^q) \text{ and } \delta_\mu^{q-1}\alpha \in D(\overline{\partial}^{q-1})\}.$$

We need the following assumption so that $\Box_\mu$ is symmetric:

Definition 2.4. We say M has *negligible boundary* if

$$(\overline{\partial}\alpha, \beta) = (\alpha, \delta_\mu \beta), \text{ for every } \alpha \in D(\overline{\partial}) \text{ and } \beta \in D(\delta_\mu).$$

We say $\Box_\mu^q$ is *essentially self-adjoint* if its L^2-closure is self-adjoint, and $\Box_\mu^q$ is *hypoelliptic* if, whenever the distribution $\Box_\mu^q \alpha$ is smooth, then α is smooth.

It is proved in [11] that

Lemma 2.5. *A subelliptic operator is hypoelliptic.*

The assumption such that M has negligible boundary implies a stronger property to $\Box_\mu$:

Proposition 2.6 (e.g. [13]). *If M has negligible boundary, then $\Box_\mu^q$ is essentially self-adjoint in $L^2(\Omega^{\bullet,q}(E))$ with $0 < q < n-1$.*

Outline of the proof. Set $\alpha_\epsilon = e^{-\Box_\mu \epsilon}\alpha$ for $\alpha \in D(\overline{\delta_\mu}^{L^2})$. By Proposition 2.3 and Lemma 2.5, α_ϵ is smooth for every $\epsilon > 0$ (here we need the assumption: $0 < q < n-1$). Therefore, since $\overline{\delta_\mu}^{L^2}\alpha_\epsilon = \overline{\partial}^*\alpha_\epsilon \to \overline{\partial}^*\alpha$ as $\epsilon \to 0$, we deduce that $\overline{\delta_\mu}^{L^2} \subset \overline{\partial}^*$. Since M has negligible boundary, $\overline{\delta_\mu}^{L^2} = \overline{\partial}^*$, and by von Neumann's theorem (e.g. [14]), $\overline{\partial}^*\delta_\mu^*$ is self-adjoint. Moreover, it follows that

$$\overline{\Box_\mu}^{L^2} = \overline{\partial}^*\delta_\mu^* + \delta_\mu^*\overline{\partial}^*,$$

where the right-hand side is self-adjoint. $\square$

We say α *is harmonic* if $\overline{\partial}\alpha = 0$ and $\delta_\mu\alpha = 0$ in the weak sense. A harmonic form always solves the Laplace equation $\Box_\mu\alpha = 0$, but in general, the converse does not need to be true. However, it follows that

Corollary 2.7. *If M has negligible boundary, then the following conditions are equivalent:*

(i) $\overline{\partial}\alpha = 0$ *and* $\delta_\mu\alpha = 0$ *pointwise;*

(ii) α *is harmonic;*

(iii) α *solves the Laplace equation;*

here, $\alpha \in L^2(C^{\bullet,q}(E))$ *and* $0 < q < n-1$.

Proof. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii) If α is harmonic, then, $(\alpha, \delta_\mu\beta) = 0$ and $(\alpha, \overline{\partial}\gamma) = 0$ for every $\beta \in D(\delta_\mu)$ and $\gamma \in D(\overline{\partial})$. This implies $(\alpha, \Box_\mu\beta) = 0$ for every $\beta \in D(\Box_\mu)$; that is, α is the solution of the Laplace equation.

(iii) $\Rightarrow$ (i) Let α be a solution of the Laplace equation. By Proposition 2.6, there exists a sequence $\alpha_l \in D(\Box_\mu)$ such that

$$\alpha_l \to \alpha \text{ and } \Box_\mu\alpha_l \to 0, \text{ as } l \to \infty.$$

Due to the fact that M has negligible boundary,

$$\|\overline{\partial}\alpha_l\|_2^2 + \|\delta_\mu\alpha_l\|_2^2 = (\Box_\mu\alpha_l, \alpha_l) \to 0, \text{ as } l \to \infty.$$

This shows that $\alpha \in D(\overline{\partial}) \cap D(\delta_\mu)$, and $\overline{\partial}\alpha = \delta_\mu\alpha = 0$ μ-a.e. Due to the hypoellipticity of $\Box_\mu$, α is smooth, and hence, $\overline{\partial}\alpha = \delta_\mu\alpha = 0$ pointwise. $\square$

A consequence of the celebrated Kohn's harmonic theory [10] is the Hodge decomposition of a vector bundle over a *compact* s.p.c. CR manifold. The corresponding result on a non-compact manifold, which is a consequence of Corollary 2.7, is the L^2-Hodge decomposition in the Main Theorem. Since the proof is similar to the case where $\eta \equiv 1$ and E is trivial (e.g. [13]), we will omit the proof here.

3. Serre duality

In this section, we study Serre duality and complete the proof of the Main Theorem. Our method is to relate the operators on E to those on E^* via the *weighted complex-conjugate Hodge star operator* $\sharp_\mu$ (see e.g. [8], [6]). Together with results from the previous section, we obtain the Main Theorem.

We start from the construction of $\sharp_\mu$. Let $* : \bigwedge^k T^*M \to \bigwedge^{2n-1-k} T^*M$ be the Hodge star operator of M with respect to g, which is uniquely determined by $g(*\alpha, \beta)dv = (n-1)!\, \alpha \wedge \beta$, for $\alpha \in \bigwedge^k T^*M$ and $\beta \in \bigwedge^{2n-1-k} T^*M$. $*$ is isometric and involutive, i.e. $g(*\alpha, *\beta) = g(\alpha, \beta)$ and $*^2 = id$, because M is odd-dimensional. As the complexification of $*$ exchanges the set of holomorphic forms and the set of anti-holomorphic forms, the linear map $\sharp = \sharp_M := {}^- \circ *$ satisfies (e.g. Lemma 7.1 [15]):

$$\sharp\,(\Omega^{p,q}(M)) = \Omega^{n-p,n-(q+1)}(M). \tag{3.1}$$

We extend (3.1) to

Definition 3.1. Define

$$\sharp_\mu : \Omega^{p,q}(E) \to \Omega^{n-p,n-(q+1)}(E^*)$$

by

$$\sharp_\mu \alpha := \sum_{1 \le i,j \le r} \eta a_{ji} (\sharp \alpha^i) \otimes s^j, \text{ for } \alpha \in \Omega^{p,q}(E),$$

where $\alpha = \sum \alpha^i \otimes s_i$, $\{s_i\}_{1 \le i \le r}$ is a local frame of E, $\{s^i\}$ is its dual frame of E^*, and $a_{ij} = \langle s_i, s_j \rangle_E$. Moreover, define $\sharp_\mu^* : \Omega^{n-p,n-(q+1)}(E^*) \to \Omega^{p,q}(E)$ by

$$\sharp_\mu^* \phi := \sum \eta^{-1} \overline{\alpha}^{ij}\, (\sharp \phi_j) \otimes s_i,$$

where $\phi = \sum \phi_j \otimes s^j$ and $a^{ij} = \langle s^i, s^j \rangle_{E^*}$, which is the entry of the inverse-matrix of (a_{ij}). Here E^* is furnished with the Hermitian fiber metric induced from E:

$$(\phi, \psi)_{E^*} = (\phi, \psi)_{(E^*, d\mu^-)} = \frac{1}{(n-1)!} \int \sum \phi_i \overline{\psi}_j a^{ij}\, d\mu^-,$$

where $\phi = \sum \phi_j \otimes s^j$, $\psi = \sum \psi_i \otimes s^i$, and $d\mu^- = \eta^{-1} dv$. For $\alpha \in \Omega^{p,q}(E)$ and $\phi \in \Omega^{s,t}(E^*)$, the product $\alpha \wedge \phi$ is defined by

$$\alpha \wedge \phi := \alpha^i \wedge \phi_i \in \Omega^{p+s,q+t}(M), \tag{3.2}$$

where $\alpha = \sum \alpha^i \otimes s_i$ and $\phi = \sum \phi_j \otimes s^j$. The definition is well defined; i.e. it is independent of the choice of the frames. Similar to the Hodge star operator, it follows that

$$(\alpha, \beta)_E = \frac{1}{(n-1)!} \int \alpha \wedge \sharp_\mu \beta, \text{ for } \alpha, \beta \in \Omega(E), \tag{3.3}$$

and

$$(\phi, \psi)_{E^*} = \frac{1}{(n-1)!} \int \phi \wedge \sharp_\mu^* \psi, \text{ for } \phi, \psi \in \Omega(E^*). \tag{3.3*}$$

The operators $\sharp_\mu$ and $\sharp_\mu^*$ satisfy the following properties.

Proposition 3.2. *It follows that*

(i) $\sharp_\mu^* \sharp_\mu = id_{\Omega(E)}$ *and* $\sharp_\mu \sharp_\mu^* = id_{\Omega(E^*)}$*;*
(ii) $(\alpha, \sharp_\mu^* \phi)_E = (\sharp_\mu \alpha, \phi)_{E^*}$*, for every* $\alpha \in L^2(\Omega(E))$ *and* $\phi \in L^2(\Omega(E^*))$*.*

Proof. (i) $\sharp_\mu^* \sharp_\mu \alpha = \sharp_\mu^* \left(\eta \overline{\alpha}_{ji} \overline{*\alpha^i} \otimes s^j \right) = \overline{\alpha}^{kj} \overline{* \left(a_{ji} * \overline{\alpha}^i \right)} \otimes s_k = \alpha.$

(ii) $\left(\alpha, \sharp_\mu^* \phi\right)_E = (\alpha, \sharp_{dv}^* \phi)_{(E,dv)} = (\sharp_{dv} \alpha, \phi)_{(E^*,dv)} = (\sharp_\mu \alpha, \phi)_{E^*}.$ $\square$

We denote by $\overline{\partial}_*$ and δ_{μ_*} the holomorphic structure of $L^2\,(\Omega(E^*))$ and its formal adjoint, respectively.

Lemma 3.3. *It follows that*

$$\left(\alpha, \sharp_\mu^* \overline{\partial}_* \sharp_\mu \beta\right)_E = \left(\overline{\partial}\alpha, (-1)^{p+q+1}\beta\right)_E, \tag{3.4}$$

for $\alpha \in \Omega^{p,q}(E)$ *and* $\beta \in \Omega_0^{p,q+1}(E)$, *and*

$$(-1)^{(p+q+1)} \left(\psi, \delta_{\mu_*}\phi\right)_{E^*} = \left(\psi, \sharp_\mu \overline{\partial} \sharp_\mu^* \phi\right)_{E^*} \tag{3.4*}$$

for $\phi \in \Omega^{p,q}(E^*)$ *and* $\psi \in \Omega_0^{p,q-1}(E^*)$.

Proof. Recall that the holomorphic structure $\overline{\partial}_{\wedge^n}$ coincides with the tangential Cauchy-Riemann operator $(-1)^n d''$ (e.g. Proposition 1.1 in [15]) defined as:

$$d''f := df|_{\overline{S}}, \text{ for } f \in C(M).$$

Therefore, since $\sharp_\mu \beta \in \Omega^{n-p,n-(q+2)}(E^*)$ and $\alpha \wedge \sharp_\mu\beta \in \Omega_0^{n,n-2}(M)$,

$$\begin{aligned} 0 = (-1)^n \int d(\alpha \wedge \sharp_\mu \beta) &= (-1)^n \int d''(\alpha \wedge \sharp_\mu\beta) \\ &= \int \overline{\partial}_{\wedge^n}(\alpha \wedge \sharp_\mu \beta) \\ &= \int \left(\overline{\partial}\alpha \wedge \sharp_\mu \beta + (-1)^{p+q} \alpha \wedge \overline{\partial}_* \sharp_\mu \beta\right) \\ &= \left(\overline{\partial}\alpha, \beta\right)_E + \left(\alpha, (-1)^{p+q} \sharp_\mu^* \overline{\partial}_* \sharp_\mu \beta\right)_E. \end{aligned}$$

We used (3.3) and Proposition 3.2 for the last step. We have (3.4).

Next, by taking into account that $\int d(\phi \wedge \sharp_\mu^* \psi) = 0$, we can prove (3.4*) in a similar way. □

We are now in a position to show:

Theorem 3.4 (Serre duality)**.** *It follows for every* $0 \leq p, q \leq n-1$ *that*

$$\sharp_\mu : \mathbb{H}^{p,q}(E) \cong \mathbb{H}^{n-p,n-(q+1)}(E^*).$$

Proof. Take $\alpha \in \mathbb{H}^{p,q}(E)$. Then $\alpha \in D(\overline{\partial}) \cap D(\delta)$, $\overline{\partial}\alpha = 0$ and $\delta_\mu \alpha = 0$. By Proposition 3.2 and (3.4), for $\beta \in \Omega_0(E)$ we have that

$$\begin{aligned} \left(\delta_{\mu_*} \sharp_\mu \alpha, \sharp_\mu \beta\right)_{E^*} &= \left(\sharp_\mu \alpha, \overline{\partial}_* \sharp_\mu \beta\right)_{E^*} \\ &= \left(\alpha, \sharp_\mu^* \overline{\partial}_* \sharp_\mu \beta\right)_E = \left(\overline{\partial}\alpha, (-1)^{p+q}\beta\right)_E = 0. \end{aligned}$$

Since $\{\sharp_\mu \beta : \beta \in \Omega_0(E)\}$ is dense in $L^2(E^*)$ by Proposition 3.2, we have $\delta_{\mu_*}\sharp_\mu \alpha = 0$. On the other hand, for $\phi \in \Omega_0(E^*)$, we have by Proposition 3.2 and (3.4*) that

$$\begin{aligned} (-1)^{p+q+1} \left(\overline{\partial}_* \sharp_\mu \alpha, \phi\right)_{E^*} &= (-1)^{p+q+1} \left(\sharp_\mu \alpha, \delta_{\mu_*}\phi\right)_{E^*} \\ &= \left(\sharp_\mu \alpha, \sharp_\mu \overline{\partial} \sharp_\mu^* \phi\right)_{E^*} \\ &= \left(\alpha, \overline{\partial} \sharp_\mu^* \phi\right)_E \\ &= \left(\delta_\mu \alpha, \sharp_\mu^* \phi\right)_E = 0. \end{aligned}$$

Therefore, $\overline{\partial}_* \sharp_\mu \alpha = 0$, and we deduce that $\sharp_\mu \alpha \in \mathbb{H}^{n-p,n-(q+1)}(E^*)$.

The reverse implication can be shown by running the argumentation above from the bottom to the top. Now we obtain

$$\sharp_\mu : \mathbb{H}^{p,q}(E) \cong \mathbb{H}^{n-p,n-(q+1)}(E^*),$$

where $\sharp_\mu$ is a complex conjugate linear isomorphism. □

4. Examples

In this section, we present examples of non-compact s.p.c. CR manifolds with negligible boundary. Our argumentation relies on the following proposition.

Proposition 4.1.

(i) *If M is Riemannian complete, then it is complete with respect to d_{CC}.*

(ii) *If M is complete, then M has negligible boundary.*

Proof. (i) Assume that M is Riemannian complete, and let $\{x_n\}_{n>0} \subset M$ be a Cauchy sequence with respect to d_{CC}. Since d_g has the alternative expression [12]

$$\mathrm{d}_g(x,y) = \sup\{u(x) - u(y) : u \in C^\infty(M),\ \|\nabla u\|_{L^\infty} \leq 1\},$$

it follows that $\mathrm{d}_g \leq \mathrm{d}_{CC}$ by Equation (2.1). Hence, $\{x_n\}_{n>0}$ is a Cauchy sequence with respect to d_g, and thus, the limit belongs to M, by the assumption. Since the topologies generated by d_g and d_{CC} are the same, we have the assertion.

(ii) Due to the fact that M is complete, there exists a sequence $\{\chi_l\}_{l>0}$ of smooth functions with compact support such that $0 \leq \chi_l \leq 1$, $\chi_l \to 1$, and $\overline{\partial}\chi_l \to 0$ as $l \to \infty$ ([1], [12]). For $\alpha \in D(\overline{\partial})$, set $\alpha_l = \chi_l \alpha$. Since α_l has compact support for every $l > 0$,

$$(\overline{\partial}\alpha_l, \beta) = (\alpha_l, \delta_\mu \beta) \to (\alpha, \delta_\mu\beta), \text{ as } l \to \infty \text{ for every } \beta \in D(\delta_\mu),$$

where the left-hand side tends to $(\overline{\partial}\alpha, \beta)$ as $l \to \infty$. ☐

The most fundamental example is

Example 4.2 (Heisenberg group)**.** The Heisenberg group $\mathcal{H}(n)$ is

$$\mathcal{H}(n) = \{(w,z) \in \mathbb{C}^n \times \mathbb{C} : \mathrm{Im} z = \|w\|^2\}$$

with the group structure

$$(w,z) \circ (w',z') = (w + w', z + z' + 2\sqrt{-1} w \cdot w').$$

It is a quadratic CR submanifold (see e.g. [2]), whose defining function is

$$f(w,z) = \mathrm{Im} z - \|w\|^2.$$

Consider the following CR manifold which is CR-equivalent to $\mathcal{H}(n)$: $\mathbb{C}^n \times \mathbb{R}$ with the contact form

$$\theta = dt + 2\sum(x_i dy_i - y_i dx_i).$$

Then since the orthonormal frame $\{X_i, Y_i\}$ of P and the characteristic direction ξ are given by

$$X_i = \frac{1}{2}\frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial t},\ Y_i = \frac{1}{2}\frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial t},\ \text{and } \xi = \frac{\partial}{\partial t},$$

no geodesics with respect to $g = -d\theta + \theta \otimes \theta$ reach ∞ in finite time. Due to the Hopf-Rinow theorem, $\mathcal{H}(n)$ is Riemannian complete, and we conclude by Proposition 5.1 that $\mathcal{H}(n)$ is complete. Moreover, it was proved in [13] that $\mathbb{H}^{p,q}(E) = 0$ for $0 < q < n-1$ when E is the trivial bundle over $\mathcal{H}(n)$.

Remark 4.3 ([13]). If M has negligible boundary and additionally either

(1) the Ricci operator is positive on $\Omega^{p,q}$ or

(2) the Ricci operator is non-negative on $\Omega^{p,q}$ and M has infinite volume,

then $\mathbb{H}^{p,q}(E) = 0$ for $0 < q < n-1$ when E is the trivial bundle over M.

Example 4.4 (Sasakian space forms)**.** There exist exactly three types of Riemannian complete simply connected Sasakian space forms: S^{2n+1}, $\mathbb{R}^{2n+1}$, and $D^n \times \mathbb{R}$, where $D \subset \mathbb{C}^n$ is a simply connected bounded domain with Kähler form $d\omega$ (e.g. [3]). The latter two space forms $\mathbb{R}^{2n+1}$ and $D^n \times \mathbb{R}$ are non-compact and they have the contact form $dt - \sum_i y_i dx_i$ and $\omega + dt$, respectively. They have negligible boundary.

Example 4.5 (Spherical orbits)**.** Let O be the orbit of an n^2-dimensional automorphism in an n-dimensional non-homogeneous hyperbolic manifold. If O is spherical, i.e. each point of O has a neighbourhood which is CR-equivalent to an open set of S^{2n-1}, then O is CR-equivalent to one of the following hypersurfaces (e.g. [9]):

(1) A lens space $S^{2n-1}/\mathbb{Z}_m$;
(2) $\sigma = \{(w,z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Re}z = \|w\|^2\}$;
(3) $\sigma' = \{(w,z) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z| = \exp \|w\|^2\}$;
(4) $\omega = \{(w,z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|w\|^2 + \exp(\mathrm{Re}z) = 1\}$;
(5) $\omega_\alpha = \{(w,z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|w\|^2 + |z|^\alpha = 1, z \neq 0\}$, for some $\alpha > 0$.

We are interested in σ and σ' because they are non-compact. First we show that σ is complete. Consider the map $\Phi : \mathcal{H}(n) \to \sigma$ defined as

$$\Phi(w,z) := (w, -\sqrt{-1}z).$$

Clearly, Φ preserves the holomorphic structure. Moreover, since

$$f_\sigma \Phi(w,z) = f_H(w,z),$$

where $f_\sigma = \mathrm{Re}z - \|w\|^2$ is the defining function of σ, the contact structure is also preserved. Since $\mathcal{H}(n)$ is complete, so is σ.

Next we proceed to show that σ' is complete. The differential $df_{\sigma'}$ of the σ''s defining function $f_{\sigma'} = |z|^2 - \exp \|w\|^2$ is

$$df_{\sigma'} = \frac{1}{2}\left(\frac{\overline{z}}{|z|}dz + \frac{z}{|z|}d\overline{z}\right) - \exp \|w\|^2 \left(\sum \left(\overline{w}_i dw_i + w_i d\overline{w}_i\right)\right),$$

and its pull-back $\Psi^* df_{\sigma'}$ by the covering map $\Psi : \sigma \to \sigma'$, defined as $\Psi(w,z) = (w, \exp z)$, is

$$\begin{aligned} &\frac{1}{2}\exp(\mathrm{Re}z)^{-1}\left(\exp \overline{z}\Psi^* dz + \exp z \Psi^* d\overline{z}\right) - \exp\|w\|^2 \left(\sum \left(\overline{w}_i dw_i + w_i d\overline{w}_i\right)\right) \\ &= \frac{1}{2}\exp(\mathrm{Re}z)^{-1}\exp(2\mathrm{Re}z)\left(dz + d\overline{z}\right) - \exp\|w\|^2 \left(\sum \left(\overline{w}_i dw_i + w_i d\overline{w}_i\right)\right) \\ &= \exp\|w\|^2 df_\sigma, \end{aligned}$$

where we have used the fact that $\mathrm{Re}z = \|w\|^2$ for the last step. This shows that $\ker(\theta_\sigma) = \ker(\theta_{\sigma'})$ via Ψ. Moreover, since

$$\Psi^* d\theta_{\sigma'} = d\exp\|w\|^2 \wedge \theta_\sigma + \exp\|w\|^2 d\theta_\sigma,$$

where the first term on the right-hand side vanishes on P and $\exp \|w\|^2 \geq 1$, it follows that the distance associated to $\Psi^* d\theta_{\sigma'}$ is not less than the one associated to $d\theta_\sigma$.

Thus, since these two distances generate the same topology and σ is complete with respect to $d\theta_\sigma$, we may conclude that the distance associated to $\Psi^* d\theta_{\sigma'}$ is complete by the same reason as in the proof of Proposition 4.1.

We can also show that ω is complete, where the proof will appear in a forthcoming paper.

Example 4.6. If M is a compact s.p.c. CR manifold and $M' \to M$ is an unbranched covering, then M' has negligible boundary.

Remark 4.7. Since the distance structure of M is independent of the choice of the weight, all of those examples have negligible boundary with an arbitrary weight.

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