

A NOTE ON STRONG HOMOLOGY OF INVERSE SYSTEMS

By

Sibe MARDEŠIĆ

1. Introduction.

Ju. T. Lisica and the author have defined in [4] strong homology groups $H_p(X; G)$ of inverse systems of spaces $X = (X_\lambda, p_{\lambda\lambda'}, A)$ over directed cofinite sets A (every element $\lambda \in A$ has only finitely many predecessors). It was shown in [5] that these groups are functors on the coherent prohomotopy category CPHTop , introduced in [2] and [3]. The notion of strong or Steenrod homology $H_p^s(X; G)$ of an arbitrary space X was then defined [1], [6] and shown to be a functor on the strong shape category SSh [2], [3]. The procedure consisted in choosing a cofinite ANR-resolution $p: X \rightarrow \mathbf{X}$ of X ([7], [8], [9]) and of defining $H_p^s(X; G)$ as $H_p(\mathbf{X}; G)$. That the group $H_p^s(X; G)$ does not depend on the choice of the resolution is a consequence of the following factorization theorem ([3], Theorem II.2.3). If $p: X \rightarrow \mathbf{X}$ is a resolution and $f: X \rightarrow Y$ is a coherent map into a cofinite ANR-system, then there exists a unique coherent homotopy class of coherent maps $g: \mathbf{X} \rightarrow Y$ such that gp and f are coherently homotopic.

The definition of composition in CPHTop and the proof of the factorization theorem essentially used the assumption that the index sets A be cofinite. On the other hand, the construction of the homology groups $H_p(X; G)$ did not require this assumption. Therefore, it remained unclear whether one can use also non-cofinite ANR-resolutions to determine the homology groups $H_p^s(X; G)$ of the space X . To prove that this is indeed the case is the main purpose of this paper. Such an information can prove useful in situations where a non-cofinite ANR-resolution naturally arises.

The main idea of the proof is to replace a given ANR-resolution $p: X \rightarrow \mathbf{X}$ by a cofinite ANR-resolution $p^*: X \rightarrow \mathbf{X}^*$ using the "trick" described in ([9], Theorem I, 1.2). What remains to be done is to exhibit a natural isomorphism $u_*: H_p(\mathbf{X}; G) \rightarrow H_p(\mathbf{X}^*; G)$. The correct formula for u_* is easily found. However, the formula for the inverse v_* of u_* is less obvious. Even more complicated is the verification of the two equalities $u_*v_* = 1$, $v_*u_* = 1$.

In order to simplify notations throughout the paper we omit the coefficient groups G , although all results hold for an arbitrary G .

2. Coherent maps and homotopy of systems.

Let $X=(X_\lambda, p_{\lambda\lambda'}, A)$ and $Y=(Y_\mu, q_{\mu\mu'}, M)$ be inverse systems of spaces over directed sets A and M respectively, i. e., X and Y are assumed to be objects of the category pro-Top (see I. 1 of [9]). In all but the last section we will assume that A and M are antisymmetric, i. e. $\lambda \leq \lambda'$ and $\lambda' \leq \lambda$ implies $\lambda = \lambda'$.

We denote by A^n , $n \geq 0$, the set of all increasing sequences $\mathbf{l}=(\lambda_0, \dots, \lambda_n)$ in A of length n , $\lambda_0 \leq \dots \leq \lambda_n$. If $0 \leq j \leq n$, $\mathbf{l}_j=(\lambda_0, \dots, \hat{\lambda}_j, \dots, \lambda_n)$ denotes the sequence of length $n-1$ obtained from \mathbf{l} by deleting λ_j . Dually, $\mathbf{l}^j=(\lambda_0, \dots, \lambda_j, \lambda_j, \dots, \lambda_n)$ is obtained from \mathbf{l} by repeating λ_j . The standard n -simplex is denoted by Δ^n and $\partial_j^n: \Delta^{n-1} \rightarrow \Delta^n$, $\sigma_j^n: \Delta^{n+1} \rightarrow \Delta^n$ denote the face and degeneracy operators.

According to [2], [3], a coherent map $f: X \rightarrow Y$ consists of the following:

(i) a function $\varphi: \bigcup_{n \geq 0} M^n \rightarrow A$ such that

$$(1) \quad \varphi(\mathbf{m}) \geq \varphi(\mathbf{m}_j), \quad 0 \leq j \leq n, \quad n > 0;$$

(ii) maps $f_m: \Delta^n \times X_{\varphi(\mathbf{m})} \rightarrow Y_{\mu_0}$, $\mathbf{m} \in M^n$, $n \geq 0$, such that

$$(2) \quad f_m(\partial_j^n t, x) = \begin{cases} q_{\mu_0 \mu_1} f_{m_0}(t, p_{\varphi(\mathbf{m}_0) \varphi(\mathbf{m})}(x)), & j=0, \\ f_{m_j}(t, p_{\varphi(\mathbf{m}_j) \varphi(\mathbf{m})}(x)), & 0 < j \leq n. \end{cases}$$

$$(3) \quad f_m(\sigma_j^n t, p_{\varphi(\mathbf{m}) \varphi(\mathbf{m}^j)}(x)) = f_{m^j}(t, x), \quad 0 \leq j \leq n.$$

A coherent homotopy $F: I \times X \rightarrow Y$, connecting coherent maps $f, f': X \rightarrow Y$, is a coherent map, given by a function Φ and by maps $F_m: \Delta^n \times I \times X_{\Phi(\mathbf{m})} \rightarrow Y_{\mu_0}$, such that

$$(4) \quad \Phi(\mathbf{m}) \geq \varphi(\mathbf{m}), \quad \varphi'(\mathbf{m}),$$

$$(5) \quad F_m(t, 0, x) = f_m(t, p_{\varphi(\mathbf{m}) \Phi(\mathbf{m})}(x)),$$

$$(6) \quad F_m(t, 1, x) = f'_m(t, p_{\varphi'(\mathbf{m}) \Phi(\mathbf{m})}(x)).$$

In [3] a coherent map $f: X \rightarrow Y$ was called *special* provided there existed an increasing function $\chi: M \rightarrow A$ satisfying

$$(7) \quad \varphi(\mathbf{m}) = \chi(\mu_n), \quad \mathbf{m} = (\mu_0, \dots, \mu_n).$$

If M is cofinite, then every coherent map $f: X \rightarrow Y$ admits a special coherent map $f': X \rightarrow Y$ such that f and f' are coherently homotopic, i. e. $f \cong f'$. Moreover, if $f', f'': X \rightarrow Y$ are special coherent maps and $f' \cong f''$, then there exists a special coherent homotopy connecting f' and f'' (see [3], Lemmas I.6.5 and I.6.6). In [2] and [3] composition of special coherent maps $f: X \rightarrow Y$, $g: Y \rightarrow Z$ was defined. In the case of cofinite systems, it induces a composition of co-

herently homotopic classes of coherent maps, yielding thus the category CPHTop.

3. Strong homology groups of systems.

Let $X=(X_\lambda, p_{\lambda\lambda'}, A)$ be an inverse system over a directed set A . Following [4], we define a chain complex $C_\#(X)$ as follows. For $p \geq 0$, a strong p -chain of X is a function x which assigns to every $I \in A^n$ a singular $(p+n)$ -chain x_I of X_{λ_0} , $x_I \in C_{p+n}(X_{\lambda_0})$. The group $C_p(X)$ consists of all strong p -chains of X . The boundary operator $d : C_{p+1}(X) \rightarrow C_p(X)$ is given by

$$(1) \quad (dx)_{\lambda_0} = \partial(x_{\lambda_0}),$$

$$(2) \quad (-1)^n(dx)_I = \partial(x_I) - p_{\lambda_0\lambda_1\#}x_{I_0} - \sum_{j=1}^n (-1)^j x_{I_j}, \quad n \geq 1.$$

Strong homology groups $H_p(X)$ of a system X are defined as the homology groups of the chain complex $C_\#(X)$.

If $f : X \rightarrow Y$ is a special coherent map, given by an increasing function $\varphi : M \rightarrow A$ and by maps $f_m : \Delta^n \times X_{\varphi(\mu_n)} \rightarrow Y_{\mu_0}$, then f induces a chain mapping $f_\# : C_\#(X) \rightarrow C_\#(Y)$, defined in (4.1) of [5] by

$$(3) \quad (f_\#x)_{\mu_0 \dots \mu_n} = \sum_{i=0}^n f_{\mu_0 \dots \mu_i \#}(\Delta^i \times x_{\varphi(\mu_i) \dots \varphi(\mu_n)}).$$

Special coherently homotopic maps $f, f' : X \rightarrow Y$ induce chain homotopic chain maps $f_\# \cong f'_\#$. Moreover, $(gf)_\# \cong g_\#f_\#$ (see [5]). Consequently, special coherent maps $f : X \rightarrow Y$ induce homomorphisms $f_* : H_p(X) \rightarrow H_p(Y)$ satisfying $(gf)_* = g_*f_*$. By definition, f_* is the homomorphism induced by the chain mapping $f_\# : C_\#(X) \rightarrow C_\#(Y)$ and depends only on the coherent homotopy class of f . How to define f_* for coherent maps, which are not special, will be shown in 8.

4. Associated cofinite systems.

With every inverse system $X=(X_\lambda, p_{\lambda\lambda'}, A)$ we will now associate a new system $X^*=(X_\alpha^*, p_{\alpha\alpha'}^*, A^*)$, defined as follows (see [9], the proof I, 1, Theorem 2). The set A^* consists of all finite subsets $\alpha \subseteq A$ having a maximum. Since we have assumed that A is antisymmetric, the maximum of α is unique and we denote it shortly by $\bar{\alpha}$. We order A^* by inclusion \subseteq . Clearly, A^* is directed, antisymmetric and cofinite. We now put

$$(1) \quad X_\alpha^* = X_{\bar{\alpha}},$$

$$(2) \quad p_{\alpha\alpha'}^* = p_{\bar{\alpha}\bar{\alpha}'}.$$

We refer to X^* as to the *cofinite system associated with X* .

We now define a map of systems $u = u_X : X \rightarrow X^*$ (in the sense of [4]). It is given by the increasing function $\alpha \mapsto \bar{\alpha}$, $\alpha \in \mathcal{A}^*$, and by the maps $u_\alpha = id : X_{\bar{\alpha}} \rightarrow X_{\bar{\alpha}}^* = X_{\bar{\alpha}}$. It generates a chain mapping $u_\# = u_{X\#} : C_\#(X) \rightarrow C_\#(X^*)$ (§ 3. of [4]), given by

$$(3) \quad (u_\# x)_{\alpha_0 \dots \alpha_n} = x_{\bar{\alpha}_0 \dots \bar{\alpha}_n}, \quad (\alpha_0, \dots, \alpha_n) \in \mathcal{A}^{*n}.$$

If $x \in C_p(X)$, then $x_{\bar{\alpha}_0 \dots \bar{\alpha}_n} \in C_{p+n}(X_{\bar{\alpha}_0}) = C_{p+n}(X_{\alpha_0}^*)$ as desired.

REMARK 1. If one first associates with u a special coherent map u' as in (2.8) of [5], then the induced chain mapping $u'_\#$ is chain homotopic to $u_\#$ and is given by a formula more complicated than (3).

Let $Y^* = (Y_{\bar{\beta}}, q_{\bar{\beta}\beta'}, M^*)$ be the cofinite system associated with $Y = (Y_\mu, q_{\mu\mu'}, M)$. With every coherent map $f : X \rightarrow Y$ we now associate a coherent map $f^* : X^* \rightarrow Y^*$. If f is given by φ and $f_{\beta_0 \dots \beta_n}$, we define f^* by φ^* and $f_{\bar{\beta}_0 \dots \bar{\beta}_n}^* : \Delta^n \times X_{\varphi^*(\beta_0, \dots, \beta_n)}^* \rightarrow Y_{\bar{\beta}_0}^*$, where

$$(4) \quad \varphi^*(\beta_0, \dots, \beta_n) = \{\varphi(\bar{\beta}_{i_0}, \dots, \bar{\beta}_{i_k}) : 0 \leq i_0 < \dots < i_k \leq n, 0 \leq k \leq n\},$$

$$(5) \quad f_{\bar{\beta}_0 \dots \bar{\beta}_n}^* = f_{\bar{\beta}_0 \dots \bar{\beta}_n}.$$

Note that $\beta_0 \sqsubseteq \dots \sqsubseteq \beta_n$ implies $\bar{\beta}_{i_0} \sqsubseteq \dots \sqsubseteq \bar{\beta}_{i_k}$ so that $\varphi^*(\beta_0, \dots, \beta_n)$ is a well-defined finite subset of \mathcal{A} . It belongs to \mathcal{A}^* because (2.1) implies that $\varphi(\bar{\beta}_0, \dots, \bar{\beta}_n)$ is the maximum of $\varphi^*(\beta_0, \dots, \beta_n)$, i. e.,

$$(6) \quad \overline{\varphi^*(\beta_0, \dots, \beta_n)} = \varphi(\bar{\beta}_0, \dots, \bar{\beta}_n)$$

That φ^* and $f_{\bar{\beta}_0 \dots \bar{\beta}_n}^*$ satisfy 2.(2)-2.(3) is immediate. We refer to f^* as to the *coherent map associated with f* .

If $f : X \rightarrow Y$ is a special coherent map, given by an increasing function $\varphi : M \rightarrow \mathcal{A}$ and by maps $f_m : \Delta^n \times X_{\varphi(m)} \rightarrow Y_{\mu_0}$, then one can associate with f a special coherent map $f^+ : X^* \rightarrow Y^*$, given by $\varphi^+ : M^* \rightarrow \mathcal{A}^*$ and $f_{\bar{\beta}_0 \dots \bar{\beta}_n}^+ : \Delta^n \times X_{\varphi^+(\beta_0, \dots, \beta_n)}^* \rightarrow Y_{\bar{\beta}_0}^*$, defined as follows.

$$(7) \quad \varphi^+(\beta) = \varphi(\beta), \quad \beta \in M^*,$$

$$(8) \quad f_{\bar{\beta}_0 \dots \bar{\beta}_n}^+ = f_{\bar{\beta}_0 \dots \bar{\beta}_n}.$$

Note that $\varphi(\beta)$ is a finite subset of \mathcal{A} , because β is a finite subset of M . Moreover, $\varphi(\bar{\beta})$ is the maximum of $\varphi(\beta)$, i. e.,

$$(9) \quad \varphi(\bar{\beta}) = \overline{\varphi^+(\beta)}, \quad \beta \in M^*,$$

which shows that indeed $\varphi^+(\beta) \in \mathcal{A}^*$. Furthermore, $\beta_0 \sqsubseteq \beta_1$ implies $\varphi(\beta_0) \sqsubseteq \varphi(\beta_1)$, which shows that φ^+ is an increasing function.

Also note that (9) implies $X_{\varphi+(\beta_n)}^* = \overline{X_{\varphi+(\beta_n)}} = X_{\varphi(\bar{\beta}_n)}$, $Y_{\beta_0}^* = Y_{\bar{\beta}_0}$, so that (8) defines a map $f_{\bar{\beta}_0 \dots \bar{\beta}_n}^+ : \Delta^n \times X_{\varphi+(\beta_n)}^* \rightarrow Y_{\bar{\beta}_0}^*$. A straightforward verification shows that f^+ satisfies (2.2) and (2.3) (put $\varphi^+(\beta_0, \dots, \beta_n) = \varphi^+(\beta_n)$).

REMARK 2. If f is a special coherent map, then the coherent maps f^* , $f^+ : X^* \rightarrow Y^*$ are coherently homotopic. Indeed, if we put $\Phi(\beta_0, \dots, \beta_n) = \varphi(\beta_n)$, then

$$(10) \quad \varphi^*(\beta_0, \dots, \beta_n) \subseteq \Phi(\beta_0, \dots, \beta_n).$$

Indeed, when f is a special coherent map, $\varphi(\bar{\beta}_{i_0}, \dots, \bar{\beta}_{i_k}) = \varphi(\bar{\beta}_{i_k})$ and therefore (4) becomes $\varphi^*(\beta_0, \dots, \beta_n) = \{\varphi(\bar{\beta}_0), \dots, \varphi(\bar{\beta}_n)\}$. However, $\bar{\beta}_i \in \beta_i \subseteq \beta_n$ and so $\varphi(\bar{\beta}_i) \in \varphi(\beta_n) = \Phi(\beta_0, \dots, \beta_n)$, $0 \leq i \leq n$. Also, $\varphi^+(\beta_0, \dots, \beta_n) = \varphi(\beta_n) = \Phi(\beta_0, \dots, \beta_n)$. Furthermore, note that

$$(11) \quad \begin{aligned} f_{\bar{\beta}_0 \dots \bar{\beta}_n}^* (\mathcal{P}_{\varphi^*(\beta_0, \dots, \beta_n)}^* \varphi(\beta_0, \dots, \beta_n)(x), t) \\ = f_{\bar{\beta}_0 \dots \bar{\beta}_n}^+ (\mathcal{P}_{\varphi^+(\beta_0, \dots, \beta_n)}^+ \varphi(\beta_0, \dots, \beta_n)(x), t). \end{aligned}$$

Indeed, since f is special (φ increases), (6) yields $\overline{\varphi^*(\beta_0, \dots, \beta_n)} = \varphi(\bar{\beta}_0, \dots, \bar{\beta}_n) = \varphi(\bar{\beta}_n) = \overline{\varphi(\beta_n)}$. Since f^+ is special, formula (7) yields $\overline{\varphi^+(\beta_0, \dots, \beta_n)} = \overline{\varphi^+(\beta_n)} = \overline{\varphi(\beta_n)}$. By definition, $\overline{\Phi(\beta_0, \dots, \beta_n)} = \overline{\varphi(\beta_n)}$. All this shows that the constant homotopies $F_{\bar{\beta}_0 \dots \bar{\beta}_n}$ yield a coherent homotopy between f^* and f^+ .

REMARK 3. If f is a special coherent map, then the induced chain mappings $f_{\#}$ and $f_{\#}^+$ satisfy

$$(12) \quad f_{\#}^+ u_{X\#} = u_{Y\#} f_{\#},$$

i.e. the following diagram of chain mappings commutes

$$(13) \quad \begin{array}{ccc} C_{\#}(X^*) & \xleftarrow{u_{X\#}} & C_{\#}(X) \\ f_{\#}^+ \downarrow & & \downarrow f_{\#} \\ C_{\#}(Y^*) & \xleftarrow{u_{Y\#}} & C_{\#}(Y) \end{array} .$$

Indeed,

$$(14) \quad \begin{aligned} (f_{\#}^+ u_{X\#} x)_{\bar{\beta}_0 \dots \bar{\beta}_n} \\ = \sum_{i=0}^n f_{\bar{\beta}_0 \dots \bar{\beta}_i \#}^+ (\Delta^i \times (u_{X\#} x)_{\varphi+(\beta_i) \dots \varphi+(\beta_n)}) \\ = \sum_{i=0}^n f_{\bar{\beta}_0 \dots \bar{\beta}_i \#} (\Delta^i \times x_{\varphi(\bar{\beta}_i) \dots \varphi(\bar{\beta}_n)}) \\ = (f_{\#} x)_{\bar{\beta}_0 \dots \bar{\beta}_n} = (u_{Y\#} f_{\#} x)_{\bar{\beta}_0 \dots \bar{\beta}_n} . \end{aligned}$$

Passing to homology and the induced homomorphisms, we conclude that the following diagrams commute for $p \geq 0$:

$$(15) \quad \begin{array}{ccc} H_p(X^*) & \xleftarrow{u_{X^*}} & H_p(X) \\ f_*^+ \downarrow & & \downarrow f_* \\ H_p(Y^*) & \xleftarrow{u_{Y^*}} & H_p(Y) . \end{array}$$

The main result of this paper is the following theorem proved in 5-7.

THEOREM 1. $u_{X^*}: C_*(X) \rightarrow C_*(X^*)$ is a chain equivalence and therefore, $u_{X^*}: H_p(X) \rightarrow H_p(X^*)$ is an isomorphism.

5. The homotopy inverse v of u_* .

For $u_* = u_{X^*}: C_*(X) \rightarrow C_*(X^*)$ we will now define a chain homotopy inverse $v = v_X: C_*(X^*) \rightarrow C_*(X)$.

We first introduce some notation. $P(n)$, $n \geq 0$, will denote the group of all permutations of the set $\{0, 1, \dots, n\}$. If $\mathbf{l} = (\lambda_0, \dots, \lambda_n) \in A^n$ and $\pi \in P(n)$, we put

$$(1) \quad \mathbf{l}\pi = (\lambda_{\pi(0)}, \dots, \lambda_{\pi(n)}),$$

$$(2) \quad [\mathbf{l}\pi] = (\{\lambda_{\pi(0)}\}, \{\lambda_{\pi(0)}, \lambda_{\pi(1)}\}, \dots, \{\lambda_{\pi(0)}, \dots, \lambda_{\pi(n)}\}).$$

$\mathbf{l}\pi$ is a sequence in A of length n (which need not be increasing).

Each $\{\lambda_{\pi(0)}, \dots, \lambda_{\pi(i)}\} \subseteq \{\lambda_0, \dots, \lambda_n\}$, $0 \leq i \leq n$, is an element of A^* , because it is a finite totally ordered subset of A . Moreover, $[\mathbf{l}\pi]$ is an increasing sequence in A^* of length n , so that $[\mathbf{l}\pi] \in (A^*)^n$.

For $y \in C_p(X^*)$, we now define $vy \in C_p(X)$ as follows. If $n \geq 0$ and $\mathbf{l} = (\lambda_0, \dots, \lambda_n) \in A^n$, we put

$$(3) \quad (vy)_\mathbf{l} = \sum_{\pi \in P(n)} \text{sgn } \pi p_\pi y_{[\mathbf{l}\pi]},$$

where

$$(4) \quad p_\pi = p_{\lambda_0 \lambda_{\pi(0)} \#}.$$

Clearly, $y_{[\mathbf{l}\pi]} \in C_{p+n}(X_{\lambda_{\pi(0)}}^*) = C_{p+n}(X_{\lambda_{\pi(0)}})$ so that $(vy)_\mathbf{l} \in C_{p+n}(X_{\lambda_0})$ as desired.

$v: C_p(X^*) \rightarrow C_p(X)$ is a homomorphism for each $p \geq 0$. Moreover, we have the following assertion.

LEMMA 1. $v: C_*(X^*) \rightarrow C_*(X)$ is a chain mapping, i. e.

$$(5) \quad (vdy)_\mathbf{l} = (dvy)_\mathbf{l}, \quad \mathbf{l} \in A^n.$$

PROOF. Let $l=(\lambda_0, \dots, \lambda_n)$. If $n=0$, then

$$(dvy)_{\lambda_0}=\partial(vy)_{\lambda_0}=\partial y_{l_{\lambda_0}}=(dy)_{l_{\lambda_0}}=(vdy)_{\lambda_0}.$$

We will therefore assume that $n \geq 1$.

By (3) and 3.(2), we see that

$$(6) \quad \begin{aligned} (vdy)_l &= (-1)^n \sum_{\pi \in P(n)} \operatorname{sgn} \pi p_\pi \partial(y_{[l\pi]}) \\ &\quad + (-1)^{n-1} \sum_{\pi \in P(n)} \operatorname{sgn} \pi p_\pi p_\pi^* y_{[l\pi]_0} \\ &\quad + (-1)^{n-1} \sum_{j=1}^{n-1} \sum_{\pi \in P(n)} (-1)^j \operatorname{sgn} \pi p_\pi y_{[l\pi]_j} \\ &\quad - \sum_{\pi \in P(n)} \operatorname{sgn} \pi p_\pi y_{[l\pi]}, \end{aligned}$$

where we have put

$$(7) \quad p_\pi^* = p_{\nu_0(\pi)\nu_1(\pi)\#}$$

$$(8) \quad \nu_0(\pi) = \{\lambda_{\pi(0)}\}, \quad \nu_1(\pi) = \{\lambda_{\pi(0)}, \lambda_{\pi(1)}\}.$$

We will now show that

$$(9) \quad \sum_{\pi \in P(n)} \operatorname{sgn} p_\pi p_\pi^* y_{[l\pi]_0} = 0,$$

$$(10) \quad \sum_{\pi \in P(n)} \operatorname{sgn} \pi p_\pi y_{[l\pi]_j} = 0, \quad 1 \leq j \leq n-1,$$

so that (6) reduces to the first and the last sum.

Indeed, put

$$(11) \quad P_j(n) = \{\pi \in P(n) : \pi(j) < \pi(j+1)\}, \quad 0 \leq j \leq n-1,$$

For any $\pi \in P_j(n)$, $0 \leq j \leq n-1$, define $\pi' \in P(n) \setminus P_j(n)$ by

$$(12) \quad \pi'(i) = \begin{cases} \pi(i), & i \neq j, j+1, \\ \pi(j+1), & i = j, \\ \pi(j), & i = j+1. \end{cases}$$

Clearly, in $[l\pi]$ and $[l\pi']$ only the j^{th} terms differ so that

$$(13) \quad [l\pi]_j = [l\pi']_j, \quad \pi \in P_j(n), \quad 0 \leq j \leq n-1.$$

Furthermore, for $1 \leq j \leq n-1$, $\pi'(0) = \pi(0)$ so that

$$(14) \quad p_\pi = p_{\pi'}, \quad \pi \in P_j(n), \quad 1 \leq j \leq n-1.$$

By (4) and (7), for any $\pi \in P(n)$, we have $p_\pi p_\pi^* = p_{\lambda_0 \nu_1(\pi)\#}$ because $\overline{\nu_0(\pi)} = \lambda_{\pi(0)}$. However, for $\pi \in P_0(n)$, $\nu_1(\pi) = \{\lambda_{\pi(0)}, \lambda_{\pi(1)}\} = \nu_1(\pi')$, so that

$$(15) \quad p_\pi p_\pi^* = p_{\pi'} p_{\pi'}^*, \quad \pi \in P_0(n).$$

Since $\text{sgn } \pi = -\text{sgn } \pi'$ and the permutations of $P(n)$ come in pairs π, π' , where $\pi \in P_j$, we obtain (9) and (10).

We now consider $(dvy)_i$. By 3.(2), we have

$$(16) \quad (dvy)_i = (-1)^n \partial(dvy)_i + (-1)^{n-1} p_{\lambda_0 \lambda_1} (vy)_{i_0} + \sum_{j=1}^n (-1)^{n+j-1} (vy)_{i_j}.$$

We will now examine $(vy)_{i_j}$ for $0 \leq j \leq n$. We first define $\lambda'_0, \dots, \lambda'_{n-1}$ by

$$(17) \quad \lambda'_i = \begin{cases} \lambda_i, & i \leq j-1 \\ \lambda_{i+1}, & i \geq j. \end{cases}$$

Clearly, $\mathbf{l}_j = (\lambda'_0, \dots, \lambda'_{n-1})$. With every $\pi \in P(n-1)$ and $0 \leq j \leq n$, we associate a permutation $\pi'_j \in P(n)$ by putting

$$(18) \quad \pi'_j(i) = \begin{cases} \pi(i), & \text{if } 0 \leq i \leq n-1 \text{ and } \pi(i) \leq j-1, \\ \pi(i)+1, & \text{if } 0 \leq i \leq n-1 \text{ and } \pi(i) \geq j, \\ j, & \text{if } i=n. \end{cases}$$

Note that π'_j belongs to

$$(19) \quad Q_j(n) = \{\pi \in P(n) : \pi(n) = j\}, \quad 0 \leq j \leq n,$$

$$(20) \quad \text{sgn } \pi'_j = (-1)^{n-j} \text{sgn } \pi, \quad 0 \leq j \leq n,$$

$$(21) \quad \lambda'_{\pi'(i)} = \lambda'_{\pi'_j(i)}, \quad 0 \leq i \leq n-1, \quad 0 \leq j \leq n.$$

This shows that the sequence $\mathbf{l}_j \pi = (\lambda'_{\pi(0)}, \dots, \lambda'_{\pi(n-1)})$ (of length $n-1$) is obtained from the sequence $\mathbf{l} \pi'_j = (\lambda'_{\pi'_j(0)}, \dots, \lambda'_{\pi'_j(n)})$ (of length n) by omitting the last term. Therefore,

$$(22) \quad [\mathbf{l}_j \pi] = [\mathbf{l} \pi'_j]_n, \quad 0 \leq j \leq n.$$

Also note that

$$(23) \quad p_{\pi'_j} = \begin{cases} p_{\lambda_0 \lambda_1} p_\pi, & j=0, \\ p_\pi, & 1 \leq j \leq n. \end{cases}$$

Indeed, by (21), for $0 \leq j \leq n$, $p_{\pi'_j} = p_{\lambda_0 \lambda_1 \pi'_j(0)} = p_{\lambda_0 \lambda_1 \pi(0)}$. If $j \geq 1$ the first term of \mathbf{l}_j is λ_0 and the first term of $\mathbf{l}_j \pi$ is $\lambda'_{\pi(0)}$. Therefore, $p_\pi = p_{\lambda_0 \lambda_1 \pi(0)}$ and we have $p_{\pi'_j} = p_\pi$. If $j=0$, then the first term of \mathbf{l}_0 is λ_1 and the first term of $\mathbf{l}_0 \pi$ is $\lambda'_{\pi(0)}$ so that $p_\pi = p_{\lambda_1 \lambda'_{\pi(0)}}$ and $p_{\lambda_0 \lambda_1} p_\pi = p_{\lambda_0 \lambda_1 \pi(0)} = p_{\pi'_0}$.

We therefore have, by (3),

$$(24) \quad (-1)^{n-1} p_{\lambda_0 \lambda_1} (vy)_{i_0} = - \sum_{\pi' \in Q_0(n)} \text{sgn } \pi' p_{\pi'} y_{[\mathbf{l} \pi']_n},$$

$$(25) \quad (-1)^{n+j-1}(vy)_{I_j} = - \sum_{\pi' \in Q_j(n)} \text{sgn } \pi' \hat{p}_{\pi'} y_{[I_{\pi'}]_n}, \quad 1 \leq j \leq n.$$

Now notice that $P(n) = \bigcup_{j=0}^n Q_j(n)$ is a decomposition of $P(n)$ in disjoint sets. Therefore, by (16),

$$(26) \quad (dvy)_I = (-1)^n \partial(vy)_I - \sum_{\pi \in P(n)} \text{sgn } \pi \hat{p}_\pi y_{[I_\pi]_n}.$$

Since, by (3), $(-1)^n \partial(vy)_I$ equals the first sum in (6), we obtain the desired conclusion (5).

6. The homotopy $u_*v \cong 1$.

In this section we will define a chain homotopy D on $C_*(X^*)$ such that

$$(1) \quad (dDx)_a + (Ddx)_a = u_a - (u_*vx)_a$$

for every $\mathbf{a} = (\alpha_0, \dots, \alpha_n) \in A^{*n}$ and $x \in C_p(X^*)$.

Note that

$$(2) \quad (u_*vx)_a = (vx)_a = \sum_{\pi \in P(n)} \text{sgn } \pi \bar{p}_\pi x_{[\mathbf{a}\pi]},$$

where $\bar{\mathbf{a}} = (\bar{\alpha}_0, \dots, \bar{\alpha}_n)$,

$$(3) \quad \bar{p}_\pi = \hat{p}_{\alpha_0 \bar{\alpha}_{\pi(0)}^*}.$$

In order to define D we need more notation. Let $n \geq 0$, $\mathbf{a} = (\alpha_0, \dots, \alpha_n) \in A^{*n}$, $0 \leq k \leq n$, $\pi \in P(k)$. Then we put

$$(4) \quad \bar{\mathbf{a}}\pi = (\overline{\alpha_{\pi(0)}}, \dots, \overline{\alpha_{\pi(k)}}),$$

$$(5) \quad [\bar{\mathbf{a}}\pi] = (\{\overline{\alpha_{\pi(0)}}\}, \dots, \{\overline{\alpha_{\pi(0)}}, \dots, \overline{\alpha_{\pi(k)}}\}).$$

$$(6) \quad \mathbf{a}(k) = (\alpha_k, \dots, \alpha_n).$$

Since $\alpha_0 \subseteq \dots \subseteq \alpha_n$, we have $\bar{\alpha}_0 \subseteq \dots \subseteq \bar{\alpha}_n$ and therefore $\{\overline{\alpha_{\pi(0)}}, \dots, \overline{\alpha_{\pi(i)}}\}$, $0 \leq i \leq k$, is a totally ordered finite set, hence, an element of A^* . Consequently, $[\bar{\mathbf{a}}\pi] \in A^{*k}$. Moreover, $\mathbf{a}(k) \in A^{*n-k}$ and $[\bar{\mathbf{a}}\pi]\mathbf{a}(k) \in A^{*n+1}$, because $\{\overline{\alpha_{\pi(0)}}, \dots, \overline{\alpha_{\pi(k)}}\} = \{\bar{\alpha}_0, \dots, \bar{\alpha}_k\} \subseteq \alpha_k$. E. g., if $\mathbf{a} = (\alpha_0, \alpha_1, \alpha_2)$, $k=1$ and π permutes 0 and 1, then $[\bar{\mathbf{a}}\pi]\mathbf{a}(k) = (\{\bar{\alpha}_1\}, \{\bar{\alpha}_0, \bar{\alpha}_1\}, \alpha_1, \alpha_2)$. Note that $\bar{\mathbf{a}}\pi$ and $[\bar{\mathbf{a}}\pi]$ can be interpreted as 5.(1) and 5.(2) for $\mathbf{l} = (\bar{\alpha}_0, \dots, \bar{\alpha}_k)$. For $x \in C_p(X^*)$ we now put

$$(7) \quad (Dx)_a = (-1)^n \sum_{k=0}^n \sum_{\pi \in P(k)} (-1)^k \text{sgn } \pi \bar{p}_\pi x_{[\bar{\mathbf{a}}\pi]\mathbf{a}(k)}.$$

LEMMA 2. D is a chain homotopy connecting identity with uv , i. e., D satisfies (1).

Note that $x_{[\bar{a}\pi]a(k)} \in C_{p+n+1}(X_{\{\overline{\alpha_\pi(0)}\}}^*) = C_{p+n+1}(X_{\overline{\alpha_\pi(0)}})$ so that $\bar{p}_\pi x_{[\bar{a}\pi]a(k)} \in C_{p+n+1}(X_{\bar{a}_0}) = C_{p+n+1}(X_{\alpha_0}^*)$ as desired.

In the verification of formula (1) we omit the easier cases $n=0$ and $n=1$ and concentrate on $n \geq 2$. By 3.(2) and (7), we have

$$(8) \quad (Ddx)_a = S_1 + S_2 + S_3,$$

where

$$(9) \quad S_1 = - \sum_{k=0}^n \sum_{\pi \in P(k)} (-1)^k \operatorname{sgn} \pi \bar{p}_\pi \partial(x_{[\bar{a}\pi]a(k)}),$$

$$(10) \quad S_2 = \sum_{k=0}^n \sum_{\pi \in P(k)} (-1)^k \operatorname{sgn} \pi \bar{p}_\pi \bar{p}_\pi^* x_{([\bar{a}\pi]a(k))_0},$$

$$(11) \quad S_3 = \sum_{k=0}^n \sum_{j=1}^{n+1} \sum_{\pi \in P(k)} (-1)^{k+j} \operatorname{sgn} \pi \bar{p}_\pi x_{([\bar{a}\pi]a(k))_j},$$

$$(12) \quad \bar{p}_\pi^* = \bar{p}_{\nu_0(\pi)\nu_1(\pi)}^*,$$

$$(13) \quad \nu_0(\pi) = \{\overline{\alpha_\pi(0)}\}, \quad \nu_1(\pi) = \begin{cases} \{\overline{\alpha_\pi(0)}, \overline{\alpha_\pi(1)}\}, & 1 \leq k \leq n, \\ \alpha_0, & k=0 \end{cases}$$

(cf. with 5.(7) and 5.(8)).

For $k=0$ the only permutation of $\{0\}$ is the identity so that $[\bar{a}\pi] = [\bar{a}] = \{\bar{a}_0\}$ and thus $([\bar{a}\pi]a(0))_0 = a(0) = a$. Moreover, $\bar{p}_\pi = id$, $\bar{p}_\pi^* = id$. Therefore, the first term in S_2 equals x_a . The sum of all the remaining terms of S_2 equals 0, because we will see that

$$(14) \quad \sum_{\pi \in P(k)} \operatorname{sgn} \pi \bar{p}_\pi \bar{p}_\pi^* x_{([\bar{a}\pi]a(k))_0} = 0, \quad 1 \leq k \leq n.$$

This will prove that

$$(15) \quad S_2 = x_a.$$

Similarly, we will show that a part of the triple sum S_3 , vanishes, because

$$(16) \quad \sum_{\pi \in P(k)} \operatorname{sgn} \pi \bar{p}_\pi x_{([\bar{a}\pi]a(k))_j} = 0, \quad 1 \leq j \leq k-1, 2 \leq k \leq n.$$

In order to prove (14) and (16), we use some arguments from 5. In particular, since

$$(17) \quad ([\bar{a}\pi]a(k))_j = [\bar{a}\pi]_j a(k), \quad 0 \leq j \leq k-1, \pi \in P(k),$$

5.(13) for $l = (\bar{a}_0, \dots, \bar{a}_k)$ implies

$$(18) \quad ([\bar{a}\pi]a(k))_j = ([\bar{a}\pi']a(k))_j, \quad \pi \in P_j(k), 0 \leq j \leq k-1.$$

Furthermore, 5.(14) and 5.(15) imply

$$(19) \quad \bar{p}_\pi = \bar{p}_{\pi'}, \quad \pi \in P_j(k), \quad 1 \leq j \leq k-1,$$

$$(20) \quad \bar{p}_\pi \bar{p}_\pi^* = \bar{p}_{\pi'} \bar{p}_{\pi'}^*, \quad \pi \in P_0(k).$$

Since $\text{sgn } \pi' = -\text{sgn } \pi$ and the permutations of $P(k)$ come in pairs π, π' , where $\pi \in P_j(k)$, we conclude that (14) and (16) hold indeed.

The summation in S_3 is over the set $\{(k, j) : 0 \leq k \leq n, 1 \leq j \leq n+1\}$, which decomposes in the following subsets of $\mathbf{Z} \times \mathbf{Z}$:

$$\begin{aligned} U_1 &= \{(k, j) : 2 \leq k \leq n, 1 \leq j \leq k-1\}, \\ U_2 &= \{(k, j) : 1 \leq k \leq n, j = k\}, \\ U_3 &= \{(k, j) : 0 \leq k \leq n-1, j = k+1\}, \\ U_4 &= \{(n, n+1)\}, \\ U_5 &= \{(k, j) : 0 \leq k \leq n-1, k+2 \leq j \leq n+1\}, \end{aligned}$$

Denote the part of S_3 corresponding to U_i by S_3^i . Then

$$(21) \quad S_3 = \sum_{j=1}^5 S_3^j.$$

Now (16) implies

$$(22) \quad S_3^1 = 0.$$

The terms of S_3^4 equal

$$(23) \quad -\text{sgn } \pi \bar{p}_\pi \chi_{[\mathbf{a}\pi]}, \quad \pi \in P(n),$$

because

$$(24) \quad ([\bar{\mathbf{a}}\pi]a(n))_{n+1} = ([\bar{\mathbf{a}}\pi]\alpha_n)_{n+1} = [\bar{\mathbf{a}}\pi].$$

Hence, by (2),

$$(25) \quad S_3^4 = -(\mathbf{u}_\# \mathbf{v} \chi)_\mathbf{a}.$$

We will now consider $S_3^2 + S_3^3$. If we replace in the expression for S_3^3 the summation index k by $k+1$, we obtain

$$(26) \quad S_3^2 = \sum_{k=0}^{n-1} \sum_{\pi \in P(k+1)} \text{sgn } \pi \bar{p}_\pi \chi_{([\mathbf{a}\pi]\mathbf{a}(k+1))_{k+1}}.$$

On the other hand,

$$(27) \quad S_3^3 = - \sum_{k=0}^{n-1} \sum_{\pi \in P(k)} \text{sgn } \pi \bar{p}_\pi \chi_{([\mathbf{a}\pi]\mathbf{a}(k))_{k+1}}.$$

With every permutation $\pi \in P(k)$ we now associate a permutation $\pi' \in Q_{k+1}(k+1)$ such that $\pi'(i) = \pi(i)$ for $0 \leq i \leq k$ and $\pi'(k+1) = k+1$ (see 5.(18)). Note that

$$(28) \quad [\bar{\mathbf{a}}\pi']\mathbf{a}(k+1) = [\bar{\mathbf{a}}\pi], \{ \overline{\alpha_{\pi(0)}}, \dots, \overline{\alpha_{\pi(k)}}, \overline{\alpha_{k+1}} \}, \alpha_{k+1}, \dots, \alpha_n,$$

$$(29) \quad [\bar{a}\pi]a(k)=[\bar{a}\alpha], \alpha_k, \alpha_{k+1}, \dots, \alpha_n,$$

so that

$$(30) \quad ([\bar{a}\pi']a(k+1))_{k+1}=([\bar{a}\pi]a(k))_{k+1}.$$

We also have $\text{sgn } \pi'=\text{sgn } \pi$ and $\bar{p}_{\pi'}=\bar{p}_\pi$, because $\pi'(0)=\pi(0)$. Since $\pi \mapsto \pi'$ is a bijection $P(k) \rightarrow Q_{k+1}(k+1) \subseteq P(k+1)$ and $P(k+1)=\bigcup_{j=0}^{k+1} Q_j(k+1)$, we conclude that

$$(31) \quad S_3^2+S_3^3=\sum_{k=0}^{n-1} \sum_{j=0}^k \sum_{\pi \in Q_j(k+1)} \text{sgn } \pi \bar{p}_\pi \chi_{([\bar{a}\pi]a(k+1))_{k+1}}.$$

We will now show that

$$(32) \quad S_3^5=-\sum_{k=0}^{n-1} \sum_{j=k+1}^n \sum_{\pi \in P(k)} (-1)^{k+j} \text{sgn } \pi \bar{p}_\pi \chi_{([\bar{a}_j\pi]a_j(k))}.$$

Let $0 \leq k \leq n$, $\pi \in P(k)$, and let $k+1 \leq j \leq n+1$. Then

$$(33) \quad ([\bar{a}\pi]a(k))_j=[\bar{a}\pi](a(k))_{j-k-1}.$$

On the other hand, if $0 \leq k \leq n-1$, $\pi \in P(k)$ and $k+1 \leq j \leq n$, then

$$(34) \quad [\bar{a}_j\pi]a_j(k)=[\bar{a}\pi](a(k))_{j-k},$$

because $[\bar{a}_j\pi]=[\bar{a}\pi]$. Consequently, (33) and (34) imply

$$(35) \quad ([\bar{a}\pi]a(k))_j=[\bar{a}_{j-1}\pi]a_{j-1}(k), \quad 0 \leq k \leq n-1, \quad k+2 \leq j \leq n+1,$$

and (32) follows.

We will now compute $(dDx)_a$. By 3.(2) and (7) we see that

$$(36) \quad (dDx)_a=T_1+T_2+T_3,$$

where

$$(37) \quad T_1=\sum_{k=0}^n \sum_{\pi \in P(k)} (-1)^k \text{sgn } \pi \bar{p}_\pi \partial(x_{([\bar{a}\pi]a(k))}),$$

$$(38) \quad T_2=\sum_{k=1}^{n-1} \sum_{\pi \in P(k)} (-1)^k \text{sgn } \pi p_{\bar{a}_0\bar{a}_1} \bar{p}_\pi \chi_{([\bar{a}_0\pi]a_0(k))},$$

$$(39) \quad T_3=\sum_{k=0}^{n-1} \sum_{j=1}^n \sum_{\pi \in P(k)} (-1)^{k+j} \text{sgn } \pi \bar{p}_\pi \chi_{([\bar{a}_j\pi]a_j(k))}.$$

We see, by (9) and (37), that

$$(40) \quad S_1+T_1=0.$$

We now decompose T_3 in two summands T_3^1, T_3^2 corresponding to the decomposition of the set $V=\{(k, j): 0 \leq k \leq n-1, 1 \leq j \leq n\}$ in sets

$$V_1=\{(k, j): 1 \leq k \leq n-1, 1 \leq j \leq k\},$$

$$V_2 = \{(k, j) : 0 \leq k \leq n-1, k+1 \leq j \leq n\}.$$

It follows from (32) that

$$(41) \quad S_3^2 + T_3^2 = 0.$$

Taking into account (15), (22), (25), (40) and (41), in order to prove (1), it remains to show that

$$(42) \quad S_3^2 + S_3^1 = -(T_2 + T_3^1).$$

We now analyze T_2 and T_3^1 . Let $0 \leq k \leq n-1$. With every $\pi \in P(k)$ and $0 \leq j \leq k$ we associate a permutation $\pi'_j \in Q_j(k+1)$, defined by 5.(18) (with $k+1$ in place of n). Then (see 5.(20), 5.(22))

$$(43) \quad \text{sgn } \pi'_j = (-1)^{k+1-j} \text{sgn } \pi, \quad 0 \leq j \leq k,$$

$$(44) \quad [\bar{a}_j \pi] = [\bar{a} \pi'_j]_{k+1}, \quad 0 \leq j \leq k.$$

Moreover, $\mathbf{a}_j(k) = \mathbf{a}(k+1)$ for $j \leq k$ and (since $[\bar{a} \pi'_j]$ is of length $k+1$)

$$[\bar{a} \pi'_j]_{k+1} \mathbf{a}(k+1) = ([\bar{a} \pi'_j] \mathbf{a}(k+1))_{k+1}.$$

We thus obtain

$$(45) \quad [\bar{a}_j \pi] \mathbf{a}_j(k) = ([\bar{a} \pi'_j] \mathbf{a}(k+1))_{k+1}, \quad 0 \leq j \leq k.$$

Also note (see 5.(23)) that

$$(46) \quad \bar{p}_{\pi'_j} = \begin{cases} p_{\bar{a}_0 \bar{a}_1} \bar{p}_{\pi}, & j=0, \\ \bar{p}_{\pi}, & 1 \leq j \leq k. \end{cases}$$

Finally, observe that $\pi \rightarrow \pi'_j$ is a bijection $P(k) \rightarrow Q_j(k+1)$. Therefore,

$$(47) \quad T_2 = - \sum_{k=0}^{n-1} \sum_{\pi' \in Q_0(k+1)} \text{sgn } \pi' \bar{p}_{\pi'} \chi_{([\bar{a} \pi'] \mathbf{a}(k+1))_{k+1}},$$

$$(48) \quad T_3^1 = - \sum_{k=1}^{n-1} \sum_{j=1}^k \sum_{\pi' \in Q_j(k+1)} \text{sgn } \pi' \bar{p}_{\pi'} \chi_{[\mathbf{a} \pi'] \mathbf{a}_j(k)}.$$

The summation in (31) is over the set $\{(k, j) : 0 \leq k \leq n-1, 0 \leq j \leq k\}$, which decomposes in $\{(k, j) : 0 \leq k \leq n-1, j=0\}$ and $\{(k, j) : 1 \leq k \leq n-1, 1 \leq j \leq k\}$. Therefore, (31), (47) and (48) show that indeed (42) holds. This completes the proof of Lemma 2.

7. The homotopy $vu_{\#} \cong 1$.

In §7 of [4] with every inverse system X a reduced chain complex $\hat{C}_{\#}(X)$ was defined. It was the restriction of $C_{\#}(X)$ to non-degenerate sequences $\mathbf{l} = (\lambda_0, \dots, \lambda_n) \in \hat{A}^n$. These are sequences such that $\lambda_i \leq \lambda_{i+1}$ and $\lambda_i \neq \lambda_{i+1}, i=0, \dots, n-1$.

There is a chain mapping $i: C_{\#}(X) \rightarrow \hat{C}_{\#}(X)$ defined by

$$(1) \quad (i(x))_l = x_l, \quad l \in \hat{A}^n.$$

Also in §7 of [4], a chain mapping $r: \hat{C}_{\#}(X) \rightarrow C_{\#}(X)$ was defined by

$$(2) \quad (r(y))_l = \begin{cases} y_l, & l \in \hat{A}^n, \\ 0, & l \in A^n \setminus \hat{A}^n, \end{cases}$$

and it was shown that $ir=1$, $ri \cong 1$. The latter relation means that there exists a chain homotopy E such that

$$(3) \quad dEx + Edx = x - rix, \quad x \in C_{\#}(X).$$

LEMMA 3. *The chain mapping $vu_{\#}: C_{\#}(X) \rightarrow C_{\#}(X)$ satisfies*

$$(4) \quad vu_{\#}ry = ry, \quad y \in \hat{C}_{\#}(X).$$

PROOF. It suffices to show that for and $x \in C_p(X)$ and any $l = (\lambda_0, \dots, \lambda_n) \in A^n$ the chain $(vu_{\#}x - x)_l$ is a finite sum of terms of the form $\pm p_{\lambda_0 \lambda'_0 \#} x_{l'}$, where $\lambda_0 \leq \lambda'_0$, $l' = (\lambda'_0, \dots, \lambda'_n) \in A^n \setminus \hat{A}^n$. Indeed, if this is the case, then for $x = ry$, $y \in \hat{C}_{\#}(X)$, one can express $(vu_{\#}ry - ry)_l$, $l \in A^n$, as a finite sum of terms of the form $\pm p_{\lambda_0 \lambda'_0 \#}(ry)_{l'}$. Since, by (2), each of these terms vanishes, we conclude that (4) holds.

By 5.(3), we have

$$(5) \quad (vu_{\#}x)_l = \sum_{\pi \in P(n)} \text{sgn } \pi p_{\pi}(u_{\#}x)_{[l\pi]}, \quad l \in A^n.$$

If π is the identity map, then $l\pi = l$, so that

$$(6) \quad [l\pi] = (\{\lambda_0\}, \dots, \{\lambda_0, \dots, \lambda_n\}), \quad \pi = id.$$

Since $\max\{\lambda_0, \dots, \lambda_i\} = \lambda_i$, we conclude that

$$(7) \quad \text{sgn } \pi p_{\pi}(u_{\#}x)_{[l\pi]} = x_l, \quad \pi = id \in P(n).$$

Therefore, $(vu_{\#}x - x)_l$ is a finite sum of terms of $\pm p_{\lambda_0 \lambda_{\pi(0)} \#}(u_{\#}x)_{[l\pi]}$, where $\pi \in P(n)$ and $\pi \neq id$. For any such π there exist indexes j , $0 \leq j < n$, such that $\pi(j+1) < \pi(j)$. For the smallest such j we have $\pi(0) < \dots < \pi(j)$, $\pi(j+1) < \pi(j)$ so that

$$(8) \quad \max\{\lambda_{\pi(0)}, \dots, \lambda_{\pi(j)}\} = \lambda_{\pi(j)} = \max\{\lambda_{\pi(0)}, \dots, \lambda_{\pi(j)}, \lambda_{\pi(j+1)}\}.$$

Since $[l\pi]$ is of the form (ν_0, \dots, ν_n) with $\nu_i = \{\lambda_{\pi(0)}, \dots, \lambda_{\pi(i)}\}$, we see that $\bar{\nu}_j = \bar{\nu}_{j+1}$, so that $l' = (\bar{\nu}_0, \dots, \bar{\nu}_n) \in A^n \setminus \hat{A}^n$. However, $(u_{\#}x)_{[l\pi]} = (u_{\#}x)_{\nu_0 \dots \nu_n} = x_{\bar{\nu}_0 \dots \bar{\nu}_n} = x_{l'}$ as desired.

LEMMA 4. *Let E be a homotopy satisfying (3). Then*

$$(9) \quad D = vuE - E$$

is a chain homotopy satisfying

$$(10) \quad dDx + Ddx = vux - x, \quad x \in C_p(X).$$

PROOF. By (3) and (4), we have

$$\begin{aligned} dDx + Ddx &= dvuEx + vuEdx \\ &\quad - dEx - Edx = vu(dEx + Edx) - (dEx + Edx) \\ &= vu(x - rix) - (x - rix) \\ &= (vux - eix) - (x - rix) = vux - x. \end{aligned}$$

Lemmas 1, 2 and 4 complete the proof of Theorem 1.

8. Homomorphisms induced by arbitrary coherent maps.

Let $f: X \rightarrow Y$ be an arbitrary coherent map. In order to define the induced homomorphisms $f_*: H_p(X) \rightarrow H_p(Y)$ one proceeds as follows. Consider the cofinite systems X^* and Y^* associated with X and Y respectively and consider the chain mappings $u_{X*}: C_*(X) \rightarrow C_*(X^*)$ and $u_{Y*}: C_*(Y) \rightarrow C_*(Y^*)$. Let $v_{Y*}: C_*(Y^*) \rightarrow C_*(X^*)$ be the homotopy inverse of u_{Y*} . Let $f^*: X^* \rightarrow Y^*$ be the coherent mapping associated with f . Since Y^* is cofinite, there exists a special coherent map $f^+: X^* \rightarrow Y^*$, which is coherently homotopic to f^* (see Lemma 6.5 of [5]). We now take for f_* the homomorphism induced by the chain mapping

$$(1) \quad f_* = v_{Y*} f^+_{*} u_{X*}.$$

The homomorphism f_* is independent of the choice of f^+ . Indeed, for another choice f^+_{\dagger} , one has $f^+ \cong f \cong f^+_{\dagger}$, so that $f_* \cong v_{Y*} f^+_{\dagger} u_{X*}$.

REMARK 4. If $f: X \rightarrow Y$ is a special coherent map, one can choose $f^+ \cong f^*$ as in Remark 2. Then, by Remark 3, $f^+_{*} u_{X*} = u_{Y*} f_*$, and therefore

$$(2) \quad v_{Y*} f^+_{*} u_{X*} \cong f_*.$$

This shows that for special f the new definition of f_* agrees with the previous one, given in 3.

REMARK 5. If $f, g: X \rightarrow Y$ are coherently homotopic coherent maps, then $f_*, g_*: C_*(X) \rightarrow C_*(Y)$ are chain homotopic chain mappings and therefore $f_* = g_*: H_p(X) \rightarrow H_p(Y)$.

In order to establish this assertion, it suffices to show that the associated coherent mappings $f^*, g^*: X^* \rightarrow Y^*$ (defined in §4) are coherently homotopic. Let g be given by ψ and g_m and let $F: I \times X \rightarrow Y$ be a coherent homotopy from f to g , given by Φ and F_m . We associate with F the coherent map $F^*: I \times X^* \rightarrow Y^*$ as in §4. It is given by Φ^* and $F_{\beta_0 \dots \beta_n}^*$. It is then straightforward to verify that F^* is a coherent homotopy from f^* to g^* .

REMARK 6. Let $f: X \rightarrow Y$ be a coherent map. If Y is cofinite, there exists a special coherent map $f_1: X \rightarrow Y$ such that $f \cong f_1$. It is a consequence of Remark 5 that $f_* = f_{1*}$. However, by Remark 4, f_{1*} can be obtained directly using the induced chain mapping f_{1*} of the special coherent map f_1 .

REMARK 7. The definition of f_* for an arbitrary coherent map $f: X \rightarrow Y$ shows that also in this case the diagram 4.(15) commutes. Moreover, by Remark 6, one can replace in this diagram f_*^\dagger by f_*^* . This shows the naturality of the isomorphisms u_{X^*} .

9. Homology of spaces using arbitrary ANR-resolutions.

Let $X = (X_\lambda, p_{\lambda\lambda'}, A)$ be an inverse system and let $p: X \rightarrow X$ be a morphism of pro-Top, i. e. a collection of maps $p_\lambda: X \rightarrow X_\lambda$ such that $p_{\lambda\lambda'} p_{\lambda'} = p_\lambda$ for $\lambda \leq \lambda'$. We say that p is a resolution of X (see [7], (8) and [9]) provided the following two conditions are satisfied:

(R1) Let P be an ANR (for metric spaces), let \mathcal{C} be an open covering of P and let $f: X \rightarrow P$ be a map. Then there exist a $\lambda \in A$ and a map $g: X_\lambda \rightarrow P$ such that gp_λ and f are \mathcal{C} -near maps.

(R2) Let P be an ANR and \mathcal{C} an open covering of P . Then there exists an open covering \mathcal{C}' of P such that whenever $\lambda \in A$ and $g, g': X_\lambda \rightarrow P$ are maps such that gp_λ and $g'p_\lambda$ are \mathcal{C}' -near maps, then there exists a $\lambda' \geq \lambda$ such that $gp_{\lambda\lambda'}$ and $g'p_{\lambda\lambda'}$ are \mathcal{C} -near maps.

If all X_λ are ANR's we say that $p: X \rightarrow X$ is an ANR-resolution.

Let $X^* = (X_\alpha^*, p_{\alpha\alpha'}^*, A^*)$ be the cofinite system associated with X described in 4. We define $p_\alpha^*: X \rightarrow X_\alpha^*$, $\alpha \in A^*$, by $p_\alpha^* = p_{\bar{\alpha}}: X \rightarrow X_{\bar{\alpha}} = X_\alpha^*$. Note that the maps p_α^* , $\alpha \in A^*$, define a morphism $p^*: X \rightarrow X^*$ of pro-Top, because

$$p_{\alpha\alpha'}^* p_\alpha^* = p_\alpha^* \quad \text{for } \alpha \leq \alpha'.$$

THEOREM 2. If $p: X \rightarrow X$ is an arbitrary resolution (ANR-resolution) of the space X , then $p^*: X \rightarrow X^*$ is also a resolution (ANR-resolution) of X and

$$(3) \quad u_X p = p^*.$$

PROOF. (1) is an immediate consequence of the definitions.

In order to verify (R1), consider an ANR P , an open covering $\mathcal{C}\mathcal{V}$ of P and a map $f: X \rightarrow P$. Choose $\lambda \in \mathcal{A}$ and $g: X_\lambda \rightarrow P$ as in (R1) for p . If we put $\alpha = \{\lambda\} \in \mathcal{A}^*$, then $X_\alpha^* = X_\lambda$, $p_\alpha^* = p_\lambda$ and $gp_\alpha^* = gp_\lambda$ is $\mathcal{C}\mathcal{V}$ -near f .

In order to verify (R2), consider $P \in \text{ANR}$ and an open covering $\mathcal{C}\mathcal{V}$ of P . Choose $\mathcal{C}\mathcal{V}'$ as in (R2) for p . We claim that $\mathcal{C}\mathcal{V}'$ also satisfies (R2) for p^* .

Indeed, let $\alpha \in \mathcal{A}^*$ and let $g, g': X_\alpha^* \rightarrow P$ be maps such that gp_α^* and $g'p_\alpha^*$ are $\mathcal{C}\mathcal{V}'$ -near. Since $gp_\alpha^* = gp_{\bar{\alpha}}$, $g'p_\alpha^* = g'p_{\bar{\alpha}}$, we conclude that there is a $\lambda' \in \mathcal{A}$, $\lambda' \geq \bar{\alpha}$, such that $gp_{\bar{\alpha}\lambda'}$ and $g'p_{\bar{\alpha}\lambda'}$ are $\mathcal{C}\mathcal{V}$ -near maps. Put $\alpha' = \alpha \cup \{\lambda'\}$. Clearly, $\alpha' \in \mathcal{A}^*$ and $\bar{\alpha}' = \lambda'$. Since $p_{\bar{\alpha}\alpha'}^* = p_{\bar{\alpha}\lambda'} = p_{\bar{\alpha}\lambda'}$, we conclude that $gp_{\alpha'}^*$ and $g'p_{\alpha'}^*$ are $\mathcal{C}\mathcal{V}$ -near maps.

REMARK 8. Let X be an arbitrary space and let $p: X \rightarrow X$ be an ANR-resolution of X . By definition [1], [6], the homology group $H_p^S(X)$ of the space X can be identified with the homology group $H_p(X^*)$ of the cofinite ANR-resolution X^* . However, by Theorem 1, u_{X^*} establishes a natural isomorphism $H_p(X) \rightarrow H_p(X^*)$. Therefore, $H_p^S(X)$ can also be identified with the homology group $H_p(X)$, where $p: X \rightarrow X$ is an arbitrary (non-cofinite) ANR-resolution of the space X .

10. Eliminating the assumption of anti-symmetry.

In this section we assume that \mathcal{A} is a directed set, which need not be anti-symmetric. If $\lambda_0 \leq \lambda_1$ and $\lambda_1 \leq \lambda_0$, we put $\lambda_0 \sim \lambda_1$. Clearly, \sim is an equivalence relation. Let $\mathcal{A}' \subseteq \mathcal{A}$ be a subset of \mathcal{A} which contains precisely one element from every equivalence class of \mathcal{A} with respect to \sim . The set \mathcal{A}' is directed and antisymmetric

With every system $X = (X_\lambda, p_{\lambda_0\lambda_1}, \mathcal{A})$ we now associate its restriction $X' = (X_\lambda, p_{\lambda_0\lambda_1}, \mathcal{A}')$ to $\mathcal{A}' \subseteq \mathcal{A}$. We then define a map of systems $s = s_X: X \rightarrow X'$ by the inclusion map $\mathcal{A}' \rightarrow \mathcal{A}$ and by the identity maps $s_\lambda = id: X_\lambda \rightarrow X_\lambda$. The induced chain mappings $s_\# : C_\#(X) \rightarrow C_\#(X')$ is given by

$$(1) \quad (s_\# x)_{\lambda_0 \dots \lambda_n} = x_{\lambda_0 \dots \lambda_n}, \quad (\lambda_0, \dots, \lambda_n) \in \mathcal{A}'^n.$$

With every coherent map $f: X \rightarrow Y$ we associate a coherent map $f': X' \rightarrow Y'$. If f is given by φ and $f_{\mu_0 \dots \mu_n}$, then f' is given by φ' and $f'_{\mu_0 \dots \mu_n}$ defined as follows: $\varphi'(\mathbf{m}), \mathbf{m} = (\mu_0, \dots, \mu_n) \in M^n$, is the only element of \mathcal{A}' such that $\varphi'(\mathbf{m}) \sim \varphi(\mathbf{m})$. The mapping $f'_\# : \Delta^n \times X_{\varphi'(\mathbf{m})} \rightarrow Y_{\mu_0}$ is given by

$$(2) \quad f'_\#(t, x) = f_\#(t, p_{\varphi(\mathbf{m})\varphi'(\mathbf{m})}(x)).$$

REMARK 9. If $f: X \rightarrow Y$ is special, then also $f': X' \rightarrow Y'$ is special and the induced chain mappings $f_#, f'_#$ satisfy the naturality condition

$$(3) \quad f'_# s_{X\#} \cong s_{Y\#} f_#.$$

To see this first notice that $\varphi'(m) \sim \varphi(m) = \varphi(\mu_n) \sim \varphi'(\mu_n)$, so that $\varphi'(m) = \varphi'(\mu_n)$. Furthermore, if $\mu_0 \leq \mu_1$, then $\varphi'(\mu_0) \sim \varphi(\mu_0) \leq \varphi(\mu_1) \sim \varphi'(\mu_1)$, so that φ' increases. This proves that f' is also a special coherent map.

To verify (3) we first consider the special coherent map $f_1: X \rightarrow Y'$, given by φ and $f_m, m \in M'^n$. The induced chain map $f_{1\#}: C_*(X) \rightarrow C_*(Y')$ satisfies

$$(4) \quad f_{1\#} = s_{Y\#} f_#.$$

We then consider the special coherent map $f_2: X \rightarrow Y'$, given by φ' and (2), and observe that

$$(5) \quad f_{2\#} = f'_# s_{X\#}.$$

Finally, since $\varphi \leq \varphi'$, f_1 and f_2 are coherently homotopic (even congruent in the sense of §5 of [5]) so that (see §3),

$$(6) \quad f_{1\#} \cong f_{2\#}.$$

THEOREM 3. $s_{X\#}: C_*(X) \rightarrow C_*(X')$ is a chain equivalence and, therefore, $s_{X*}: H_p(X) \rightarrow H_p(X')$ is an isomorphism.

PROOF. We define the inverse chain mapping $w = w_X: C_*(X') \rightarrow C_*(X)$ by

$$(7) \quad (wy)_{\lambda_0 \dots \lambda_n} = \rho_{\lambda_0 \lambda'_0 \#} \gamma_{\lambda'_0 \dots \lambda'_n},$$

where λ' is the only element of A' such that $\lambda \sim \lambda'$. Clearly,

$$(8) \quad (s_{\#} wy)_{\lambda_0 \dots \lambda_n} = (wy)_{\lambda_0 \dots \lambda_n} = \gamma_{\lambda_0 \dots \lambda_n},$$

when $\lambda_0, \dots, \lambda_n \in A'$. Therefore, $s_{\#} w = 1$.

On the other hand,

$$(9) \quad (ws_{\#} x)_{\lambda_0 \dots \lambda_n} = \rho_{\lambda_0 \lambda'_0 \#} x_{\lambda'_0 \dots \lambda'_n},$$

so that $sw_{\#}$ can differ from 1. However,

$$(10) \quad ws_{\#} \cong 1.$$

To establish (10), we put

$$(11) \quad (-1)^n (Dx)_{\lambda_0 \dots \lambda_n} = \sum_{k=0}^n (-1)^k x_{\lambda_0 \dots \lambda_k \lambda'_k \dots \lambda'_n}.$$

We will now verify

$$(12) \quad (dDx + Ddx)_{\lambda_0 \dots \lambda_n} = (ws_{\#} x - x)_{\lambda_0 \dots \lambda_n}.$$

We concentrate on the case $n \geq 2$. If $l = (\lambda_0, \dots, \lambda_n)$, $n \geq 2$, and $0 \leq k \leq n$, we have

$$(13) \quad (Ddx)_{\lambda_0 \dots \lambda_n} = \sum_{i=0}^9 A_i,$$

where

$$(14) \quad \begin{aligned} A_1 &= - \sum_{k=0}^n (-1)^k \partial(x_{\lambda_0 \dots \lambda_k \lambda'_k \dots \lambda'_n}), \\ A_2 &= p_{\lambda_0 \lambda'_0 \#} x_{\lambda_0 \dots \lambda'_n} = (wsx)_{\lambda_0 \dots \lambda_n}, \\ A_3 &= \sum_{k=1}^n (-1)^k p_{\lambda_0 \lambda_1 \#} x_{\lambda_1 \dots \lambda_k \lambda'_k \dots \lambda'_n}, \\ A_4 &= \sum_{k=2}^n \sum_{j=1}^{k-1} (-1)^{k+j} x_{\lambda_0 \dots \lambda_j \dots \lambda_k \lambda'_k \dots \lambda'_n}, \\ A_5 &= \sum_{k=1}^n x_{\lambda_0 \dots \lambda_{k-1} \lambda'_k \dots \lambda'_n}, \\ A_6 &= - \sum_{k=0}^{n-1} x_{\lambda_0 \dots \lambda_k \lambda'_{k+1} \dots \lambda'_n}, \\ A_7 &= \sum_{k=0}^{n-1} \sum_{j=k+1}^{n-1} (-1)^{k+j+1} x_{\lambda_0 \dots \lambda_k \lambda'_k \dots \lambda'_j \dots \lambda'_n}, \\ A_8 &= \sum_{k=0}^{n-1} (-1)^{k+n+1} x_{\lambda_0 \dots \lambda_k \lambda'_k \dots \lambda'_{n-1}}, \\ (15) \quad A_9 &= -x_{\lambda_0 \dots \lambda_n}. \end{aligned}$$

We see immediately that

$$(16) \quad A_5 + A_6 = 0.$$

On the other hand, we have

$$(17) \quad (dDx)_{\lambda_0 \dots \lambda_n} = \sum_{i=1}^5 B_i,$$

where

$$\begin{aligned} B_1 &= \sum_{k=0}^n (-1)^k \partial(x_{\lambda_0 \dots \lambda_k \lambda'_k \dots \lambda'_n}), \\ B_2 &= p_{\lambda_0 \lambda_1 \#} \sum_{k=1}^n (-1)^{k+1} x_{\lambda_1 \dots \lambda_k \lambda'_k \dots \lambda'_n}, \\ B_3 &= \sum_{k=1}^{n-1} \sum_{j=1}^k (-1)^{k+j} x_{\lambda_0 \dots \lambda_j \dots \lambda_{k+1} \lambda'_{k+1} \dots \lambda'_n}, \\ B_4 &= \sum_{k=0}^{n-1} \sum_{j=k+1}^{n-1} (-1)^{k+j} x_{\lambda_0 \dots \lambda_k \lambda'_k \dots \lambda'_j \dots \lambda'_n}, \end{aligned}$$

$$B_5 = \sum_{k=0}^{n-1} (-1)^{k+n} x_{\lambda_0 \dots \lambda_k \lambda'_k \dots \lambda'_{n-1}}.$$

We see that

$$A_1 + B_1 = 0, \quad A_3 + B_2 = 0, \quad A_4 + B_3 = 0, \quad A_7 + B_4 = 0, \quad A_8 + B_5 = 0.$$

Therefore, (12) follows from (13)-(17).

REMARK 10. We can now define the induced homomorphism $f_*: H_p(\mathbf{X}) \rightarrow H_p(\mathbf{Y})$ of an arbitrary coherent map $f: \mathbf{X} \rightarrow \mathbf{Y}$ between systems, which need not be antisymmetric. By definition, f_* is the homomorphism induced by the chain mapping $f_\# = w_{Y\#} f'_\# s_{X\#}: C_\#(\mathbf{X}) \rightarrow C_\#(\mathbf{Y})$. Clearly, f_* satisfies the naturality condition

$$(18) \quad s_{Y\#} f_* = f'_\# s_{X\#}.$$

Moreover, $f \cong f_1$ implies $f_* = f_{1*}$. Also note that whenever f is special, f_* can be obtained directly from the chain mapping induced by f as in 3. This is a consequence of Remark 9.

REMARK 11. If $p: X \rightarrow \mathbf{X}$ is a resolution (ANR-resolution) of the space X , then we define a morphism $p': X \rightarrow \mathbf{X}'$ of pro-Top by putting $p'_\lambda = p_\lambda$ for $\lambda \in A'$. That p' is also a resolution (ANR-resolution) of X is obvious. Moreover, $s_X p = p'$. Since $s_{X\#}$ is a natural isomorphism, one can identify $H_p(\mathbf{X}')$ with $H_p(\mathbf{X})$. This and Remark 8 show that the homology group $H_p^S(X)$ of the space X can be identified with $H_p(\mathbf{X})$, where $p: X \rightarrow \mathbf{X}$ is an arbitrary ANR-resolution of X (A need not even be anti-symmetric).

References

- [1] Ju. T. Lisica and S. Mardešić, Steenrod-Sitnikov homology for arbitrary spaces, *Bull. Amer. Math. Soc.*, **9** (1983), 207-210.
- [2] ———, Coherent prohomotopy and a strong shape category of topological spaces, in: *Topology, Proc. Intern. Topological Conference (Leningrad, 1982)*, Lecture Notes in Math. 1060, Springer-Verlag, Berlin 1984, pp. 164-173.
- [3] ———, Coherent prohomotopy and strong shape theory, *Glasnik Mat.*, **19** (1984), 335-399.
- [4] ———, Strong homology of inverse systems of spaces I, *Topology and its Appl.*, **19** (1985), 29-43.
- [5] ———, Strong homology of inverse systems of spaces II, *Topology and its Appl.*, **19** (1985), 45-64.
- [6] ———, Steenrod homology, *Proceedings Geometric and Algebraic Topology*, Banach Center Publ. **18** (1986), Warsaw, pp. 333-344.
- [7] S. Mardešić, Approximate polyhedra, resolutions of maps and shape fibrations, *Fund. Math.*, **114** (1981), 53-78.

- [8] ———, Inverse limits and resolutions, Shape Theory and Geom. Top. Proc. (Dubrovnik, 1981), Lecture Notes in Math. 870, Springer-Verlag, Berlin, 1981, 239-252.
- [9] S. Mardešić and J. Segal, Shape theory, North Holland Publ. Co., Amsterdam, 1982.

Department of Mathematics
University of Zagreb
p. o. box 187
41001 Zagreb, Yugoslavia